THE JOURNAL OF FINANCE • VOL. XLIX, NO. 4 • SEPTEMBER 1994

Exploiting the Conditional Density in Estimating the Term Structure: An Application to the Cox, Ingersoll, and Ross Model

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ABSTRACT

We propose an empirical method that utilizes the conditional density of the state variables to estimate and test a term structure model with known price formulae, using data on both discount and coupon bonds. The method is applied to an extension of a two-factor model due to Cox, Ingersoll, and Ross (1985; CIR). Our results show that estimates based on only bills imply unreasonably large price errors for longer maturities. We reject the original CIR model using a likelihood ratio test, and conclude that the extended CIR model also fails to provide a good description of the Treasury market.

THE TERM STRUCTURE IS the function that maps the time to maturity of a discount bond to its current price (or yield to maturity). The study of this functional relationship has long been of interest to economists. In most interest rate models, the term structure at any point in time is fully determined by the values of some exogenously specified state variables. This article presents an empirical method that utilizes the conditional density of the state variables to estimate and test a term structure model with known bond formulae, using data on both discount and coupon bond prices. When the conditional density is not known in a continuous-time model, such as Longstaff and Schwartz (1992) and Sun (1992), the method can be applied to the discrete-time analog. As an application, we estimate and test a two-factor interest rate model due to Cox, Ingersoll, and Ross (1985; CIR).

The CIR model is a general equilibrium model with explicit analytical expressions for the equilibrium interest rate dynamics and bond prices. Being a general equilibrium model, it contains all the elements of the traditional expectations hypotheses. In contrast to the arbitrage models, such as Bren-

*Pearson is from the University of Rochester, and Sun is from Columbia University. This article was previously titled "An Empirical Examination of the Cox, Ingersoll, and Ross Model of the Term Structure of Interest Rates." We are grateful to Wayne Ferson, Hua He, Chi-Fu Huang, John Long, Francis Longstaff, and Suresh Sundaresan for helpful comments. Special thanks go to René Stulz (the editor) and an anonymous referee for many constructive suggestions. All remaining errors are our own.

nan and Schwartz (1979) and Vasicek (1977), an equilibrium approach automatically insures internal consistency and makes no assumptions on the risk premium required by investors for bearing interest rate risk.[1] In addition, an equilibrium model provides information about how underlying variables in the economy affect the term structure. Furthermore, the formulae of the CIR model are potentially useful for practical applications such as pricing options on bonds, hedging bond portfolios, and formulating dynamic trading strategies.

In most continuous-time financial models, the underlying state variables are not observable. It is, therefore, difficult to exploit their conditional joint density in estimation and testing. Specifically, the state variable in one-factor term structure models is the instantaneous interest rate, which is the yield of a discount bond maturing at the next instant. Since such instruments do not exist, the state variable is not observable. This prevents the existing empirical work from employing the conditional density of the state variable. For example, in testing the CIR model, Gibbons and Ramaswamy (1993) use the steady-state density of the interest rate, while Brown and Dybvig (1986) utilize neither the conditional nor the steady-state density. Also, in testing the restrictions imposed by the instantaneous interest rate, Chan, Karolyi, Longstaff, and Sanders (1992) and Longstaff and Schwartz (1992) use the one-month Treasury yield as a proxy for the instantaneous interest rate, which can induce significant measurement errors.

However, the formulae for bond prices in a term structure model enable us to infer the state variables from the bond prices. The key to our approach is to recover the unobservable state variables from the observed market prices and to utilize the conditional density of the state variables in estimation and testing. The conditional density determines the evolution of the term structure through time. We apply the method of maximum likelihood to estimate the parameters of the CIR model, using data on discount and coupon bond prices. For testing, we nest the CIR model within an extended model discussed in CIR and Marsh (1980) and apply the likelihood ratio test. The strength of our approach is its ability to incorporate the time-series information contained in a sample of Treasury bills, notes, and bonds. The use of coupon bonds potentially permits more powerful tests because the prices of long-term bonds are relatively more sensitive to the model parameters.

We apply the method to two bond portfolios constructed from the most liquid Treasury bills, notes, and bonds. To investigate the importance of data selection, we also apply the method to another two data sets: the three-month and six-month bills, and the three-month bill and the long bond. Our results show that the estimates based on two individual issues, especially those based on only bills, imply unreasonably large price errors. Therefore, for practical purposes, one has to be very cautious in using estimates based on only bills. We reject the CIR model by the likelihood ratio test. As for the

[1]As pointed out by CIR (1985, p. 398), inappropriate specification of the risk premium leads to a model that guarantees arbitrage opportunities rather than precludes them.

extended model, we assess the model based on its performance in within-sample prediction. Surprisingly, the predictive performance of the extended CIR model is as poor as a naive "martingale model," which assumes that the expected future yields are the current yields. In addition, the model cannot accommodate the three data sets because the nominal interest rates implied by the three data sets do not "track" each other through time. Hence, the extended model seems unable to provide a good description of Treasury markets.

The rest of the article is organized as follows. In the next section we briefly describe the CIR model and the extension that underlies our empirical investigation. Section II reviews related empirical work. Section III presents the econometric method, and Section IV describes the data. We discuss the parameter estimates in Section V, and assess the performance of the CIR model and its extension in Section VI. Section VII concludes the article.

I. The Extended Two-factor CIR Model

A. CIR Real Discount Bonds

In a single-good continuous-time production economy in which a representative agent has a logarithmic von Neumann-Morgenstern utility function, CIR show that the equilibrium (instantaneous) real interest rate, r, follows a "mean-reverting square-root" process:

$$dr = \kappa_1 (\theta_1 - r)\, dt + \sigma_1 \sqrt{r}\, dZ^{(1)}, \tag{1}$$

where $\kappa_1 (> 0)$ and $\theta_1 (> 0)$ are constants, and $Z^{(1)}$ is a standard (zero drift and unit variance) Brownian motion. The parameter θ_1 is the long-term mean of the real interest rate, and κ_1 determines the adjustment speed of r toward the long-term mean. The real interest rate is expected to move up whenever its realization is below the long-term mean and to move down whenever above the long-term mean. The adjustment speed increases with the parameter κ_1. The local variance of the real interest rate process is $\sigma_1^2 r$.

Let $\hat{P}_t(\tau)$ denote the real price at time t of a real discount bond with maturity τ, which will pay one unit of consumption at time $t + \tau$ and nothing at any other time. The real discount bond price is shown to be

$$\hat{P}_t(\tau) = A(\tau) e^{-B(\tau) r_t}, \tag{2}$$

where

$$A(\tau) \equiv \left[\frac{2\gamma e^{(\gamma + \lambda + \kappa_1)\tau/2}}{(\gamma + \lambda + \kappa_1)(e^{\gamma\tau} - 1) + 2\gamma} \right]^{2\kappa_1\theta_1/\sigma_1^2}$$

$$B(\tau) \equiv \frac{2(e^{\gamma\tau} - 1)}{(\gamma + \lambda + \kappa_1)(e^{\gamma\tau} - 1) + 2\gamma},$$

$$\gamma \equiv \sqrt{(\kappa_1 + \lambda)^2 + 2\sigma_1^2},$$

and λ is a constant determining the factor premium of bonds. Specifically, the expected rate of return on the bond is $r - \lambda r B(\tau)$, and the factor premium is positive if $\lambda < 0$.

B. CIR Nominal Discount Bonds

In addition to the general equilibrium model of the real term structure, Section 7 of CIR extends the theory to a world of inflation in which the exogenous price level has no effect on the real equilibrium. Under different hypotheses on the price level, various pricing formulae for a nominal discount bond are derived. Sun (1987) finds that two of the proposed price level processes perform remarkably well when confronted with price level data measured by the *Consumer Price Index* (CPI). Here we focus on one of them. The dynamics of the price level, p, is assumed to be

$$dp = ypdt + \sigma_p p\sqrt{y}\, dZ^{(2)},$$

where y is a "mean-reverting square-root" process:[2]

$$dy = \kappa_2(\theta_2 - y)\, dt + \sigma_2\sqrt{y}\, dZ^{(3)},$$

and $\kappa_2(>0)$, $\theta_2(>0)$, and $\sigma_p(<1)$ are constants. The two standard Brownian motions, $Z^{(2)}$ and $Z^{(3)}$, are correlated with coefficient ρ, but independent of $Z^{(1)}$.

The interpretations of the three parameters of the state variable, y, are exactly the same as those of the parameters of the real interest rate process given in equation (1). Conditional on y, the price level is lognormally distributed. The parameter σ_p determines how well the inflation rate can be explained by the state variable y. It can be shown that the (instantaneous) expected inflation rate at time t is

$$\lim_{\tau \downarrow 0} \frac{\varepsilon_t\{\ln(p_{t+\tau}/p_t)\}}{\tau} = \left(1 - \sigma_p^2/2\right) y_t, \tag{3}$$

where $\varepsilon_t\{\cdot\}$ is the conditional expectation given all information available up to time t.

Let $P_t(\tau)$ denote the (nominal) price at time t of a nominal discount bond with maturity τ, which will pay $1/p_{t+\tau}$ units of consumption at time $t+\tau$ and nothing at any other time. Given that the price level and the real equilibrium are independent, the nominal bond price equals the real bond price times the expected depreciation rate of purchasing power. That is,

$$\begin{aligned} P_t(\tau) &= \hat{P}_t(\tau)\varepsilon_t\{p_t/p_{t+\tau}\} \\ &= A(\tau)C(\tau)\exp\{-B(\tau)r_t - D(\tau)y_t\}, \end{aligned} \tag{4}$$

[2] In the CIR model, the state variable, y, is called the expected inflation rate, which is somewhat misleading because it also affects the variance of the inflation rate. The expected inflation rate is equal to y only when the parameter σ_p is zero. In fact, y is a subordinated process which, along with the Brownian motion, $Z^{(2)}$, determines the evolution of the price level.

where

$$C(\tau) \equiv \left[\frac{2\xi e^{(\xi+\kappa_2+\rho\sigma_2\sigma_p)\tau/2}}{(\xi+\kappa_2+\rho\sigma_2\sigma_p)(e^{\xi\tau}-1)+2\xi}\right]^{2\kappa_2\theta_2/\sigma_2^2}$$

$$D(\tau) \equiv \frac{2(e^{\xi\tau}-1)\left(1-\sigma_p^2\right)}{(\xi+\kappa_2+\rho\sigma_2\sigma_p)(e^{\xi\tau}-1)+2\xi},$$

$$\xi \equiv \sqrt{(\kappa_2+\rho\sigma_2\sigma_p)^2+2\sigma_2^2\left(1-\sigma_p^2\right)}.$$

The corresponding (instantaneous) nominal yield is

$$\begin{aligned} R_t &\equiv -\lim_{\tau\downarrow 0}\varepsilon_t\{\ln P_t(\tau)/\tau\} \\ &= r_t + \left(1-\sigma_p^2\right)y_t. \end{aligned}$$

Notice that the nominal bond price is independent of the current price level. This independence allows us to perform an empirical investigation without using the price level (CPI) data, and avoids the time-averaging problem of the CPI and the nonsynchronous data problem between bond prices and the CPI.[3]

C. The Extended Model

For empirical purposes, we nest the CIR model within an extended model. The real interest rate and the expected inflation rate in the CIR model cannot be negative, which is a more stringent requirement than that the nominal interest rate cannot be negative in the presence of a nominally riskless numeraire (e.g., fiat money). We translate the real interest rate process and the state variable, y, so that either can be negative. This translation provides a more flexible specification of the two processes and enables us to test the original CIR model by the likelihood ratio test.

The translation of the real interest rate is discussed in CIR and Marsh (1980), and is justified in an equilibrium setting by Pearson (1990). In the translation, the equilibrium real interest rate, r', is $\bar{r}+r$, where $\bar{r}$ is a constant and r is specified by equation (1). The equilibrium factor premium of real bonds is λr. This extended model can be shown to include Oldfield and Rogalski (1987), Schaefer and Schwartz (1984), and Vasicek (1977) as special cases. Similarly, the state variable, y, is translated to be $y' = \bar{y}+y$, where $\bar{y}$ is a constant. Let $\overline{R} \equiv \bar{r}+\bar{y}$. The price at time t of a nominal bond in the extended model is

$$P_t(\tau) = A(\tau)C(\tau)\exp\left\{-\overline{R}\tau - B(\tau)r_t - D(\tau)y_t\right\}, \tag{5}$$

[3]There are measurement errors in treating CPI as the price level at a point in time because it is compiled with prices that are sampled during a month. There is an extensive literature dealing with this time-averaging problem. For example, Working (1960) is the classical article that points out how the use of an average can introduce correlations not present in the original series.

where $A(\tau)$, $B(\tau)$, $C(\tau)$, and $D(\tau)$ are defined in price formulae (2) and (4). The corresponding nominal yield is

$$R_t = \overline{R} + r_t + \left(1 - \sigma_p^2\right) y_t. \tag{6}$$

This extended model is the base for our empirical investigation. The parameter $\overline{R}$ can be viewed as the lower bound of the nominal yield, which is expected to be nonnegative. We can test the original CIR model by testing if $\overline{R}$ is significantly different from zero.

One remark is in order. Using data on bond prices, we can identify only $\overline{R}$, but not $\bar{r}$ and $\bar{y}$ separately. Identifying the two parameters provides information about the individual performance of the real CIR model and the price level process. However, to do so, we need to include the CPI in the data. We do not pursue the investigation to avoid the time-averaging problem and the nonsynchronous data problem.

II. Related Empirical Work

In this section we briefly review related empirical studies on the term structure model. Stambaugh (1988) tests the implication of the CIR model that conditional expected holding period returns of discount bonds are linearly related to forward rates. If bond prices are described by a k-factor model, conditional expected returns should be explained by k forward rates. This leads to a straightforward test of the model when the data set includes enough discount bonds to compute more than k implied forward rates. Stambaugh finds only weak evidence for more than two factors. The limitations of this approach are: it can be applied only to discount bonds, the underlying state variables are neither identified nor recovered, and the parameters of the CIR model are not estimated.

Brown and Dybvig (1986) estimate certain combinations of the parameters of the one-factor CIR model, interpreted as a model of nominal bond prices. Let $Q_t(\tau)$ and $\tilde{Q}_t(\tau)$ denote the CIR model price and the observed market price at time t. Brown and Dybvig (1986) assume that $\tilde{Q}_t(\tau) = Q_t(\tau) + \epsilon_t$, where ϵ_t is independently and identically normally distributed across the bonds. Then they apply the nonlinear regression to estimate r_t and the parameters for each sample date, t, using a time-series of U.S. Treasury issues. Clearly, the method fails to utilize the theoretical distribution of r implied by the model. The CIR model not only provides bond price formulae cross-sectionally but also the evolution of the term structure through time. Moreover, Brown and Dybvig (1986) cannot estimate all the four parameters of the CIR model because not all of them can be identified in the price formula (2). Specifically, the adjustment speed, κ_1, and the factor premium, λ, always appear in summation, $\kappa_1 + \lambda$. To identify either of them, we need to use the conditional density of the interest rate, which does not depend on λ.

Gibbons and Ramaswamy (1993) apply the generalized method of moments (GMM) developed by Hansen (1982) to estimate and test the one-factor real

CIR model. Assuming that the price level is independent of the real economy, they use the steady-state distribution of the real interest rate to calculate the unconditional means, variances, and covariances of the real yields to maturity of nominal discount bonds. The advantage to this approach is that it permits an empirical investigation of the real model without specifying an explicit price level process. A disadvantage is that this approach uses only the steady-state density of the interest rate. The conditional density is the one that determines the evolution of the term structure through time. Further, their procedure can be applied only to discount bonds. While Treasury notes and bonds are currently issued on a regular cycle, there are not enough coupon bonds available to construct, without interpolation or extrapolation, the implied prices of discount bonds with various times to maturity. In practice, Gibbons and Ramaswamy can use only Treasury bills. This restriction is a serious drawback because, relative to long-term coupon bonds, the Treasury bill prices are less sensitive to the model parameters.

Chan, Karolyi, Longstaff, and Sanders (1992) and Longstaff and Schwartz (1992) also employ GMM to test the restrictions imposed by the instantaneous interest rate. In the test they use the one-month Treasury yield as a proxy for the unobservable instantaneous interest rate. This practice induces measurement errors. The instantaneous interest rate does not depend on the factor premium, while the one-month yield does. The measurement error can be significant if the factor premium is different from zero. For example, Table I compares the instantaneous interest rate with the one-month yield in the CIR model. Given the parameter estimates in Chan, Karolyi, Longstaff, and Sanders (1992, p. 1218): $\kappa_1 = 0.2339$, $\theta_1 = 0.0808$, and $\sigma_1^2 = 0.0073$, the one-month yield is significantly higher than the instantaneous interest rate when the factor premium is large ($\lambda = -1.0$). Further, the measurement error appears to depend on the level of the interest rate.

Table I

Instantaneous Interest Rate versus One-Month Discount Bond Yield In the Cox, Ingersoll, and Ross (CIR) Model

This table compares the difference between the instantaneous interest rate and the one-month discount bond yield implied by the CIR model at different levels of the factor premium. The CIR real interest rate, r, is described by $dr = \kappa_1(\theta_1 - r)dt + \sigma_1\sqrt{r}\,dZ^{(1)}$, where θ_1 is the long-term mean, κ_1, is the adjustment speed, $\sigma^2 r$ is the local variance, and $Z^{(1)}$ is a standard Brownian motion. The real factor premium is λr. The yield for a discount bond with maturity τ is $[rB(\tau) - \ln A(\tau)]/\tau$, where $A(\tau)$ and $B(\tau)$ are defined in price formula (2). The parameter estimates are taken from Chan, Karolyi, Longstaff, and Sanders (1992, p. 1218): $\kappa_1 = 0.2339$, $\theta_1 = 0.0808$, and $\sigma_1^2 = 0.0073$.

	One-Month Discount Bond Yield (%)		
Factor Premium	$r = 8.08\%$	$r = 3.08\%$	$r = 13.08\%$
$\lambda = 0.0$	8.08	3.13	13.03
$\lambda = -0.2$	8.15	3.15	13.14
$\lambda = -1.0$	8.42	3.26	13.87

III. Econometric Method

We first discuss the application of the method of maximum likelihood to a time series of two discount bond prices. The key to our approach is to use the known conditional density of the unobservable state variables through the bond price formula (5). Then we apply the method to data on two bond portfolios to incorporate the cross-sectional information contained in discount and coupon bond prices.

A. Likelihood Function of Two Discount Bonds

In the CIR model, the joint density for r_s and y_s conditional on r_t and y_t, for $s > t$, is

$$f(r_s, y_s | r_t, y_t) = \prod_{j=1}^{2} d_j(t,s) e^{-u_j(t,s) - v_j(t,s)} \left[\frac{v_j(t,s)}{u_j(t,s)} \right]^{q_j/2} \times I_{q_j}\left(2\sqrt{u_j(t,s) v_j(t,s)} \right),$$

where

$$\begin{aligned} d_j(t,s) &\equiv \frac{2\kappa_j}{\sigma_j^2[1 - e^{-\kappa_j(s-t)}]}, \quad j = 1,2, \\ u_1(t,s) &\equiv d_1(t,s) r_t e^{-\kappa_1(s-t)} \\ u_2(t,s) &\equiv d_2(t,s) y_t e^{-\kappa_2(s-t)}, \\ v_1(t,s) &\equiv d_1(t,s) r_s, \\ v_2(t,s) &\equiv d_2(t,s) y_s, \\ q_j &\equiv 2\kappa_j \theta_j / \sigma_j^2 - 1, \quad j = 1,2, \end{aligned}$$

and $I_q(\cdot)$ is the modified Bessel function of the first kind of order q.[4] The conditional density determines the evolution of the term structure through time. However, the method of maximum likelihood cannot be based on unobservable r and y. Instead, we apply the method to two observable bond prices, the conditional density of which can be obtained through the price formula (5).

Consider two discount bonds prices: $P_s(\tau_1)$ and $P_s(\tau_2)$. From equation (5), we can recover the two state variables from the two bond prices:

$$r_s = \frac{D(\tau_2)[E(\tau_1) - \ln P_s(\tau_1)] - D(\tau_1)[E(\tau_2) - \ln P_s(\tau_2)]}{B(\tau_1)D(\tau_2) - D(\tau_1)B(\tau_2)}, \tag{7}$$

$$y_s = \frac{B(\tau_1)[E(\tau_2) - \ln P_s(\tau_2)] - B(\tau_2)[E(\tau_1) - \ln P_s(\tau_1)],}{B(\tau_1)D(\tau_2) - D(\tau_1)B(\tau_2)} \tag{8}$$

[4]See Olver (1965) for the properties of the modified Bessel function.

where

$$E(\tau_i) \equiv \ln\{A(\tau_i)C(\tau_i)\} - \bar{R}\tau_i, \quad i = 1,2.$$

The Jocobian of the transformation is:

$$J \equiv \begin{vmatrix} \dfrac{\partial r_s}{\partial P_s(\tau_1)} & \dfrac{\partial r_s}{\partial P_s(\tau_2)} \\ \dfrac{\partial y_s}{\partial P_s(\tau_1)} & \dfrac{\partial y_s}{\partial P_s(\tau_2)} \end{vmatrix} = \begin{vmatrix} \dfrac{\partial P_s(\tau_1)}{\partial r_s} & \dfrac{\partial P_s(\tau_1)}{\partial y_s} \\ \dfrac{\partial P_s(\tau_2)}{\partial r_s} & \dfrac{\partial P_s(\tau_2)}{\partial y_s} \end{vmatrix}^{-1}$$

$$= \begin{vmatrix} \dfrac{-D(\tau_2)}{[B(\tau_1)D(\tau_2) - D(\tau_1)B(\tau_2)]P_s(\tau_1)} & \dfrac{D(\tau_1)}{[B(\tau_1)D(\tau_2) - D(\tau_1)B(\tau_2)]P_s(\tau_2)} \\ \dfrac{B(\tau_2)}{[B(\tau_1)D(\tau_2) - D(\tau_1)B(\tau_2)]P_s(\tau_1)} & \dfrac{-B(\tau_1)}{[B(\tau_1)D(\tau_1) - D(\tau_1)B(\tau_2)]P_s(\tau_2)} \end{vmatrix}$$

$$= \frac{1}{[B(\tau_1)D(\tau_2) - D(\tau_1)B(\tau_2)]P_s(\tau_1)P_s(\tau_2)}.$$

The joint density for $P_s(\tau_1)$ and $P_s(\tau_2)$ conditional on $P_t(\tau_1)$ and $P_t(\tau_2)$, for $s > t$, is therefore

$$\begin{aligned} &f(P_s(\tau_1), P_s(\tau_2)|P_t(\tau_1), P_t(\tau_2)) \\ &\quad = |J| f(r_s, y_s|r_t, y_t) \\ &\quad = \frac{\prod_{j=1}^{2} d_j(t, s)e^{-u_j(t, s)-v_j(t, s)}\left[\dfrac{v_j(t, s)}{u_j(t, s)}\right]^{q_j/2} I_{q_j}\left(2\sqrt{u_j(t, s)v_j(t, s)}\right)}{|B(\tau_1)D(\tau_2) - D(\tau_1)B(\tau_2)|P_s(\tau_1)P_s(\tau_2)}, \end{aligned} \tag{9}$$

which is a function of the observable $P_s(\tau_1)$ and $P_s(\tau_2)$ through equations (7) and (8).

Suppose that there are N observations of the two bonds prices available at times $t_1, t_2, \ldots, T_N$. Let $\underline{P}_s$ denote the vector $(P_s(\tau_1), P_s(\tau_2))$. The logarithm of the likelihood function is

$$\begin{aligned} &\mathscr{L}\left(\kappa, \sigma^2, \theta_1, \lambda, \kappa_2, \sigma_2, \theta_2, \sigma_p, \rho\right) \\ &\quad \equiv \ln f\left(\underline{P}_{t_2}, P_{t_3}, \ldots, \underline{P}_{t_N}|\underline{P}_{t_1}\right) \\ &\quad = \ln \prod_{i=2}^{N} f\left(\underline{P}_{t_i}|\underline{P}_{t_{i-1}}\right) \\ &\quad = \sum_{i=2}^{N} \ln f\left(\underline{P}_{t_i}|\underline{P}_{t_{i-1}}\right), \end{aligned}$$

where the second equality follows from the fact that r and y are Markov processes. We can now apply the method of maximum likelihood, using a time series of the two bond prices. The advantage of this method is that we can take full advantage of the probability distribution of the state variables without imposing any assumption other than those in the CIR model.

However, using only two discount bond prices fails to capture the average level of yields since it employs only two points on the yield curves at any instant of time. The information contained in the time series of two points is limited and hardly enough for us to have parameter estimates consistent with the entire yield curve. We cannot directly extend this method to three or more individual bonds because there are only two state variables in the price formula (5). The joint distribution of any three nominal discount bonds must be "degenerate" and the conditional density does not exist. One remedy to mitigate this deficiency is to apply the method to two bond portfolios, which enables us to extract more cross-sectional information in the term structure.

B. Two Bond Portfolios

Let $Q_{1,t}$ and $Q_{2,t}$ be the values at time t of two portfolios of discount and coupon bonds. The theoretical portfolio values predicted by the CIR model are

$$Q_{i,t} = \sum_{j=1}^{n_i} c_{i,j} P_t(\tau_{i,j}), \quad i = 1,2,$$

where $c_{i,j}$ is the jth coupon payment of the ith portfolio, and $\tau_{i,j}$, $j = 1, \ldots, n_i$ are the corresponding times to future coupon payment dates. The portfolio can consist of only one issue. For example, suppose that the first portfolio is a three-month bill and the second is a 10 percent thirty-year bond. Then $n_1 = 1$, $c_{1,1} = 100$, $\tau_{1,1} = 0.25$, and $n_2 = 60$, $c_{2,j} = 5$, $j = 1, \ldots, 59$, $c_{2,60} = 105$, $\tau_{2,j} = 0.5j$, $j = 1, \ldots, 60$. (Throughout the article, time intervals are measured in years, and bond prices are expressed as percentages of face values.)

Conceptually it is as straightforward to apply the method of maximum likelihood to data on two bond portfolios. However there is one practical complexity. When one of the two portfolios has more than one payment, we are unable to derive closed-form expressions for r and y in terms of the two portfolio values. Nonetheless, the implicit function theorem guarantees that we can carry out a numerical search to find r_t and y_t as functions of two given portfolio values $Q_{1,t}$ and $Q_{2,t}$, and generalize the conditional density (equation (10)) to be:

$$f(Q_{1,s}, Q_{2,s} | Q_{1,t}, Q_{2,t}) = \begin{vmatrix} \dfrac{\partial Q_{1,s}}{\partial r} & \dfrac{\partial Q_{1,s}}{\partial y} \\ \dfrac{\partial Q_{2,s}}{\partial r} & \dfrac{\partial Q_{2,s}}{\partial y} \end{vmatrix}^{-1} \prod_{j=1}^{2} d_j(t,s) e^{-u_j(t,s) - v_j(t,s)} \times \left[\frac{v_j(t,s)}{u_j(t,s)} \right]^{q_j/2} I_{q_j}\!\left(2\sqrt{u_j(t,s) v_j(t,s)}\right).$$

The method of maximum likelihood is then based on the likelihood function for N observations of the two portfolio values:

$$\mathscr{L}\left(\kappa, \sigma^2, \theta_1, \lambda, \kappa_2, \sigma_2, \theta_2, \sigma_p, \rho\right) = \sum_{i=2}^{N} \ln f\left(Q_{1,t_i}, Q_{2,t_i} | Q_{1,t_{i-1}}, Q_{2,t_{i-1}}\right). \quad (10)$$

Using data of two bond portfolios has two advantages. First, we can incorporate more cross-sectional information in the term structure and capture the average level of yields. This is especially important for practical applications. We will elaborate this point when discussing the empirical results. Secondly, it can mitigate problems caused by measurement errors such as not using the actual trade data, nonsynchronous quotations, and quotation errors, or market imperfections such as transactions costs and holding costs of maintaining a short position.[5] As long as these "disturbances" are white noise across issues, the portfolio values are less affected by the problems. On the other hand, if there is a priori specification of a statistical model for "disturbances" such as in Brown and Dybvig (1986), the statistical model should be combined with the above likelihood function for estimation and testing.

Before presenting the empirical evidence, we briefly compare our method with Brown and Dybvig (1986) and Gibbons and Ramaswamy (1993). Relative to Brown and Dybvig, by using two bond portfolios we are able to utilize the conditional density of the state variables and to incorporate the time-series information contained in a sample. Besides, we can estimate all of the parameters of the model. Relative to Gibbons and Ramaswamy, by specifying explicitly the price level process, we can use the conditional density instead of the steady-state density. The conditional density is the one that determines the evolution of the term structure through time. In addition, our method can make use of the information in coupon bond prices. Clearly, we cannot test the real CIR model because our test is a joint test for the real CIR model and the specified price level process.

IV. Data Description

We apply the econometric method developed to estimate and test the CIR model using price data of U.S. Treasury bills, notes, and bonds. Our data set consists of monthly observations on the prices of ten different bills, notes, and bonds drawn from Government Bond Master File prepared by the Center for Research in Security Prices. The sample period is from December 1971 to December 1986. Each month the file gives us the prices and other data for the just-auctioned three- and six-month bills, the longest maturity bill available,

[5]The major holding costs of a short position in Treasury markets are reverse spreads, the differences between the lending rates on general collateral and specific issues. The spreads are usually positive (Stigum (1990), p. 596).

the currently issued two-, three-, four-, five-, seven-, and ten-year notes, and the currently issued noncallable, nonconvertible bond without a special tax status.

In the early part of the sample period, not all of the current notes were available. For example, the Treasury first issued ten-year notes in 1976. When the current notes were not available, we select the most recently issued note or bond with the appropriate maturity. Specifically, when a ten-year note was not available, we use the most recently issued noncallable, nonconvertible bond with no special tax status and a remaining maturity of between seven and ten years. Similarly, when a four-year note was not available, we use the most recently issued note with a remaining maturity of between three and four years. We begin the sample with December 1971 because all of the maturities have been continuously available since then.[6] For all issues, the price used is the mean of the bid and the ask (flat) prices on the last business day of the month, plus the accrued interest to the settlement date.

The sample selection procedure produces a sample that covers the entire yield curve and consists predominantly of current, actively traded securities (or the so-called "on-the-run" issues). The sample consists solely of such securities when they are available, and the most recently issued securities when they are not. We restrict our attention to the most recently issued securities because the institutional literature such as Fabozzi and Pollack (1987) and Stigum (1990) indicates that the government securities issued some time in the past are not actively traded, and the quoted prices may not be reasonable approximations of equilibrium prices.

The ten Treasury issues are divided into two portfolios according to their maturities. The first portfolio consists of the five shorter maturities, and the second consists of the five longer maturities.[7] Both portfolios are equally weighted, one-fifth for each issue. Forming portfolios by maturities conforms to the market belief that there are different sectors in the yield curve. To investigate the importance of the data selection, we also apply the method to another two data sets: the three-month and six-month bills, and the three-month bill and the long bond.

In order to check the stability of the parameter estimates, we provide empirical results for the entire sample period and two subperiods: the period between December 1971 and September 1979 and the period between October 1979 and December 1986. The choice of subperiods is dictated by the fact that in October 1979 the Federal Reserve announced changes in how it would implement monetary policy. It is possible that the behavior of interest rates differs in the periods before and after this date. (See, for example, Huizinga and Mishkin (1985) and Stigum (1990).)

[6] Prior to December 1971, there were no noncallable, nonconvertible bonds without a special tax status and a remaining maturity greater than ten years.

[7] We also form the first portfolio with the six shorter maturities and the second with the four longer maturities. The results are not significantly different.

To get a feel for the data, we provide in Figure 1 the surfaces of the Treasury yield curves for the entire sample period, and in Table II the summary statistics for the yields. The yield curve is typically upward sloping, but inverted around 1979 to 1981. The yield volatilities, measured by standard deviations, decrease with maturities. Clearly, interest rates before September 1979 were significantly lower and less volatile.

There is one market practice that requires us to modify slightly the econometric method presented in Section III, which assumes that the bond prices in the data are for cash settlement. Treasury prices are, however, quoted on the basis of settlement in two business days, or "skip-day" settlement. That is, bonds and cash are exchanged two business days after the quotation date. Hence the reported prices are actually not bond prices, but rather the forward prices of very short-term forward contracts.

Fortunately, closed-form expressions for forward prices are available in a term structure model with known price formulae. Recall that $P_t(\tau)$ is the price at time t of a discount bond with maturity τ. Let $F_t(s, \tau)$ denote the price at time t of a forward contract that is written on the discount bond and expires at s, where $t < s < t + \tau$. The forward price is (see, e.g., Cox, Inger-

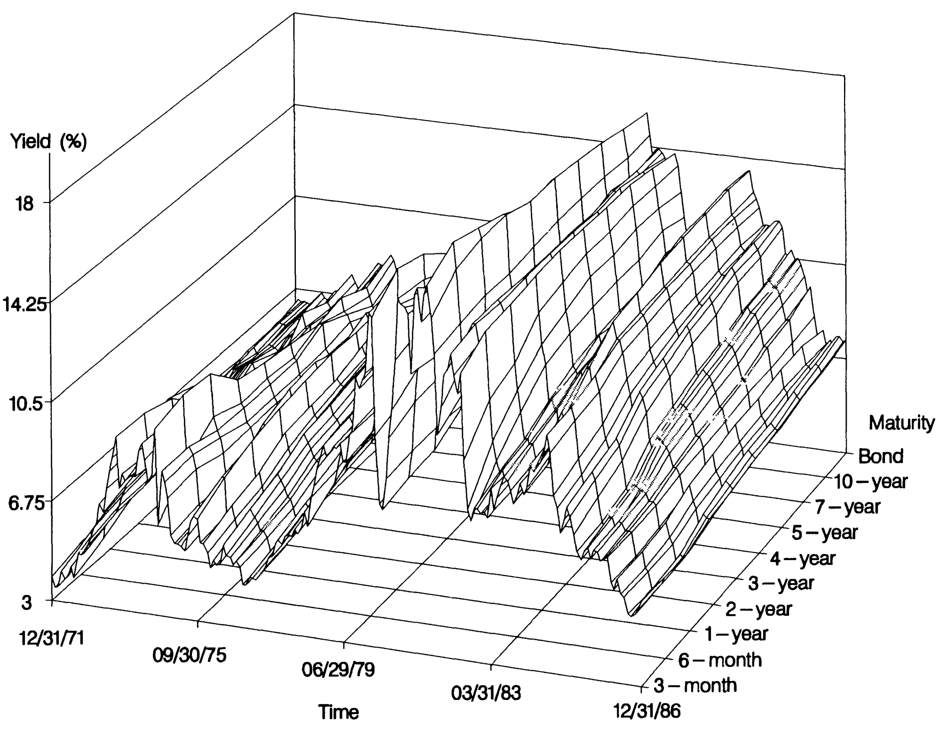

Figure 1. The historical U.S. Treasury yields. This figure presents the historical movement of yields to maturity of ten on-the-run Treasury bills, notes, and bonds. The sample contains 181 monthly observations in the period from December 1971 to December 1986.

Table II

Summary Statistics for Treasury Yields

This table presents the sample means and standard deviations (STD) of the yields to maturity of ten on-the-run Treasury bills, notes, and bonds. The sample contains 181 monthly observations in the period between December 1971 and December 1986. The two subperiods are December 1971 to September 1979 and October 1979 to December 1986, with 94 and 87 observations, respectively.

	Yield to Maturity (%)					
	12/71–12/86		12/71–9/79		10/79–12/86	
Maturity	Mean	STD	Mean	STD	Mean	STD
3-month	8.51	3.12	6.77	1.90	10.39	3.09
6-month	8.78	3.10	7.05	1.82	10.64	3.11
1-year	8.89	2.96	7.19	1.70	10.73	2.94
2-year	8.96	2.78	7.21	1.34	10.85	2.70
3-year	9.06	2.69	7.27	1.13	11.00	2.54
4-year	9.15	2.68	7.30	1.05	11.15	2.45
5-year	9.19	2.65	7.31	1.02	11.22	2.35
7-year	9.37	2.54	7.54	0.91	11.36	2.23
10-year	9.35	2.54	7.46	0.85	11.39	2.13
Bond	9.51	2.40	7.72	0.82	11.45	2.01

soll, and Ross (1981))

$$F_t(s,\tau) = P_t(\tau)/P_t(s-t).$$

Our data actually consist of observations on these forward prices, with t being the quotation date and s the settlement date.

For the extended CIR model, we have from equation (5) that

$$F_t(s,\tau) = \frac{A(\tau)C(\tau)}{A(s-t)C(s-t)} \times \exp\{-\bar{R}(\tau+t-s) - [B(\tau)-B(s-t)]r_t - [D(\tau)-D(s-t)]y_t\}.$$

Then, it is straightforward to modify our econometric method to deal with the "skip-day" settlement. Note that only the existence of a closed-form bond pricing formula enables us to be exactly correct in handling the settlement problem. While the relationship between the bond prices for cash settlement and the forward prices is known, in general we cannot recover the prices for cash settlement via some transformation of the data. This is due to the fact that the prices for cash settlement of bonds maturing on the settlement date are not available in the market.

V. Parameter Estimates

In this section we discuss the parameter estimates. Applying the method of maximum likelihood, we look for the parameter estimates that maximize the

Table III

Maximum Likelihood Parameter Estimates on Three Data Sets

This table reports the maximum likelihood parameter estimates of the extended Cox, Ingersoll, and Ross (CIR) model, based on three data sets. (The numbers in parentheses are the corresponding asymptotics standard errors). The sample period is from December 1971 to December 1986, with 181 monthly prices of ten on-the-run Treasury bills, notes, and bonds. The three different data sets used in estimation are two bond portfolios, the three-month bill and the long bond, and the three- and six-month bills. For the bond portfolios, the first (second) is an equally weighted portfolio of the five on-the-run issues with shorter (longer) maturities. The CIR real interest rate, r, is described by $dr = \kappa_1(\theta_1 - r)dt + \sigma_1\sqrt{r}\,dZ^{(1)}$, where θ_1 is the long-term mean, κ_1 is the adjustment speed, $\sigma^2 r$ is the local variance, and $Z^{(1)}$ is a standard Brownian motion. The real factor premium is λr. The dynamics of the price level, p, is $dp = ypdt + \sigma_p\sqrt{y}\,dZ^{(2)}$, where the state variable y follows $dy = \kappa_2(\theta_2 - y)dt + \sigma_2\sqrt{y}\,dZ^{(3)}$. The two standard Brownian motions, $Z^{(2)}$ and $Z^{(3)}$, are correlated with coefficient ρ. The nominal interest rate, R_t, is $r_t + (1 - \sigma_p^2)y_t$ in the CIR model, and $\bar{R} + r_t + (1 - \sigma_p^2)y_t$ in the extended model. The parameter $\bar{R}$ is the lower bound of the nominal interest rate. The estimate for ρ is not reported because it is not significantly different from -1 in all cases.

	Estimate (Standard Error)		
Parameter	Bond Portfolios	Bill and Bond	Two Bills[a]
κ_1	0.8762 (0.2572)	1.3609 (0.3580)	9.2396
θ_1	0.0311 (0.0085)	0.0322 (0.0059)	10.0301
σ_1	0.1707 (0.0120)	0.1849 (0.0157)	0.0231
λ	−0.1282 (0.0232)	−0.0845 (0.0245)	−0.0068
σ_p	0.6549 (0.1122)	0.7408 (0.0870)	0.3565
κ_2	0.0042 (0.0016)	0.0036 (0.0015)	0.1104
θ_2	0.0684 (0.0242)	0.0845 (0.0323)	0.0600
σ_2	0.0920 (0.0127)	0.1005 (0.0141)	0.1395
$\bar{R}$	0.0258 (0.0036)	0.0263 (0.0017)	−10.0000
Likelihood ratio for the restriction $\bar{R} = 0$:[b]			
	12.7370	20.1250	9.7443

[a]The constraint that $\bar{R} \geq -10$ is binding in the maximization, so the asymptotic standard errors cannot be calculated.

[b]Prob$\{\chi_1^2 \geq 7.88\} = 0.5$ percent.

likelihood function (10).[8] The corresponding asymptotic covariance matrix is the inverse of the Hessian matrix, which consists of the second derivatives of the log-likelihood function with respect to the parameters. Table III provides the parameter estimates and their asymptotic standard errors for the entire sample, based on three sets of data: two bond portfolios, the three-month bill and the long bond, and the three-month and six-month bills. (In the next subsection, we will discuss why the asymptotic standard errors are not available for the estimates based on three-month and six-month bills. The estimate for ρ in the price level process is not reported because it is

[8]We use Subroutine E04UCF from the NAG library to perform the maximization. The function $e^{-z}I_q(z)$ in the conditional density was evaluated using the backward recursion in Section 19.4.2 of Luke (1977), except when either z or q was large. For large z and q, the asymptotic expansion in Olver (1965) was used.

not significantly different from negative one in all cases. We start with the estimates based on the two bills, the third column in the table, to show that estimates based on the short end of the yield curve can be very misleading.

A. Estimates Based on Two Bills

Before discussing the estimates, we need to point out that using only bill prices gives too flat a likelihood surface to identify all the parameters. Specifically, in search for parameter estimates to maximize the likelihood function, the optimization algorithm quickly pushes $\overline{R}$ to about -1. Then the likelihood function increases very slowly as $\overline{R}$ and σ^2 decrease steadily, while θ_1 increases, such that $\theta_1 + \overline{R}$ and $\sigma_1^2\overline{R}$ are approximately constant. The increase in the maximum value of the likelihood function is imperceptible when $\overline{R}$ is less than -10. For example, given the accuracy to which we can evaluate the Bessel function, the value at $\overline{R} = -10$ is almost identical to that at $\overline{R} = -35$, provided that the other parameters are adjusted accordingly. As a practical matter, the maximum value of the likelihood function with the constraint that $\overline{R} \geq -10$ equals that of the unconstrained likelihood function. We hence impose the somewhat arbitrary restriction $\overline{R} \geq -10$ in the maximization for our results. Since $\overline{R}$ is the lower bound of the nominal interest rate, a significantly negative estimate is puzzling.

Due to the identification problem, the standard errors for the estimates would be quite large if the unconstrained maximum could be calculated. In fact, at $\overline{R} = -10$ the Hessian matrix is so nearly singular that it cannot be inverted with accuracy. In addition, the asymptotic covariance matrix is the inverse of the Hessian matrix when the estimates are in the interior of the parameter space. Since the constraint that $\overline{R} \geq -10$ is bonding, the standard errors for the estimates are simply not available.[9]

The parameter estimates based on two bills are strikingly different from the other two sets: the lower bound of the nominal interest rate, $\overline{R}$, the adjustment speeds of r and y, κ_1 and κ_2, and the long-term mean of r, θ_1, are much larger.

The estimate for κ_1, the adjustment speed of r, is 9.25, which is similar to that in Gibbons and Ramaswamy (1993) and implies a very fast mean reversion. To get a feel for the adjustment speed, we can use equation (19) in the CIR model (p. 392):

$$\varepsilon\{r_s|r_t\} = r_t e^{-\kappa_1(s-t)} + \theta_1[1 - e^{-\kappa_1(s-t)}], \tag{11}$$

where $\varepsilon\{\cdot|r_t\}$ is the conditional expectation given r_t. For example, the "half-life" of the process, the time when the real interest rate is expected to have a value halfway between the current level and the long-run mean, is $(\ln 2/\kappa_1)$,

[9]The "constrained" standard errors for the other parameters, calculated under the constraint that $\overline{R} = -10$, are misleading because they can give an illusion of precision in the parameter estimation. We thank the referee for this point.

roughly a month given that $\kappa_1 = 9.25$. The adjustment speed seems too quick given the historical behavior of interest rates.

The deficiency of using only bills in estimation is more evident when we investigate the price errors implied by the estimates. The last two columns of Table IV provide the sample mean and standard deviation of the price errors for the ten on-the-run Treasury issues. The price error is the difference between the market price and the model price given the estimates and the recovered r and y. Since we use the three- and six-month bills to recover r and y in estimation, the model price always matches the market price for the two bills. However, the price errors are extremely large in the long end of the yield curve. For example, the mean for the long bond is 3.49 per \$100 face value, the standard deviation is 10.72, and the maximum is 24.40.

We provide in Figure 2 some typical U.S. Treasury yield curves and those implied by the three sets of parameter estimates of the extended CIR model. Panel A presents the yields in January 1971 when interest rates were low. Panel B gives the yields in August 1981 when the rates were high, and Panel C, the yields in December 1986 when the rates were moderate. In addition, Panel D depicts the average yields over the entire sample period. Except in

Table IV

Price Errors Implied by Maximum Likelihood Parameter Estimates

This table provides the summary statistics for the price errors implied by the three sets of maximum likelihood parameter estimates of the extended Cox, Ingersoll, and Ross model. The price error is the difference between the market and the model prices per \$100 face value. The sample period is from December 1971 to December 1986, with 181 monthly prices of ten on-the-run Treasury bills, notes, and bonds. The three different data sets used in estimation are two bond portfolios, the three-month bill and the long bond, and the three- and six-month bills. For the bond portfolios, the first (second) is an equally weighted portfolio of the five on-the-run issues with shorter (longer) maturities.

	Implied Price Errors (in Dollars) per \$100 face value								
	Two Bond Portfolios			Bill and Bond			Two Bills		
Maturity	Mean	STD	MAX	Mean	STD	MAX	Mean	STD	MAX
3-month	−0.03	0.14	0.49	0.00	0.00	0.00	0.00	0.00	0.00
6-month	−0.01	0.16	0.46	0.06	0.13	0.54	0.00	0.00	0.00
1-year	−0.03	0.14	0.49	0.17	0.41	1.49	−0.01	0.19	0.73
2-year	−0.01	0.16	0.46	0.29	0.93	3.29	−0.14	1.06	3.67
3-year	0.04	0.17	0.60	0.43	1.18	3.68	−0.09	1.95	5.08
4-year	−0.02	0.20	0.88	0.60	1.53	4.86	0.04	2.95	8.76
5-year	0.02	0.35	0.93	0.56	1.76	4.94	0.17	3.95	9.62
7-year	0.07	0.65	3.69	0.93	1.41	5.03	0.82	5.25	11.45
10-year	−0.04	0.87	4.22	0.06	1.72	4.95	0.93	7.05	14.85
Bond	0.32	0.51	2.25	0.00	0.00	0.00	3.49	10.72	24.40

Sum of squared price errors:

Two Bond Portfolios	Bill and Bond	Two Bills
864	2638	18953

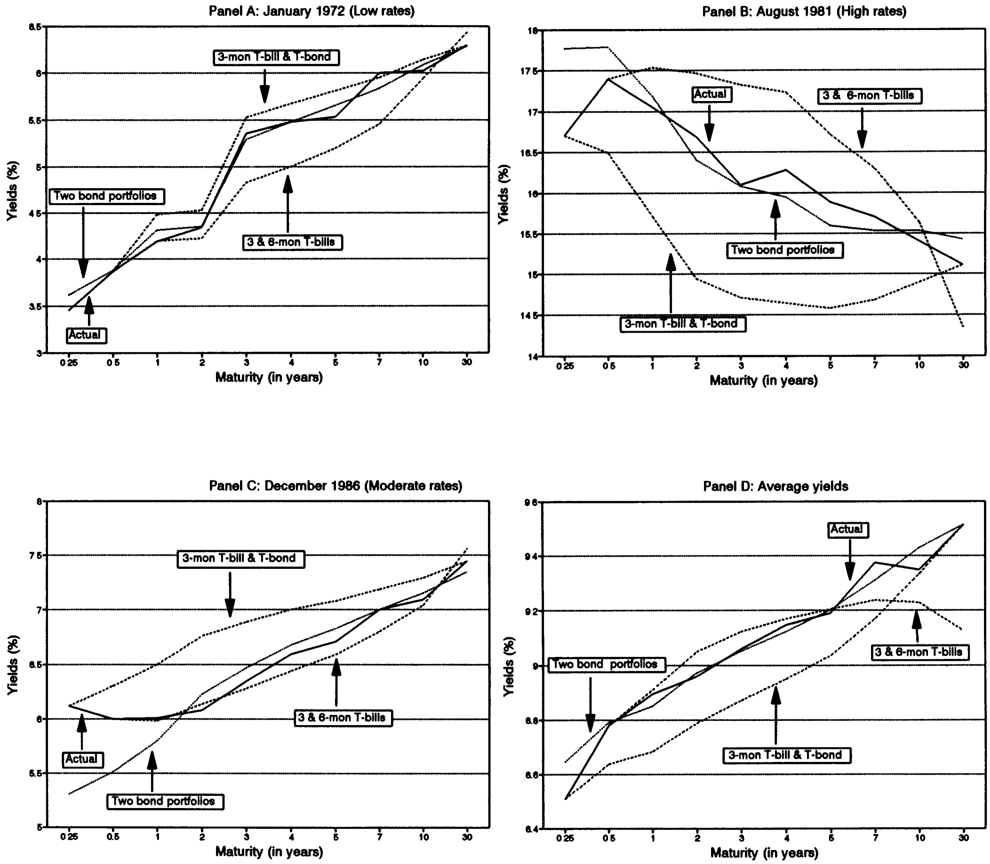

Figure 2. The U.S. Treasury yields and those implied by maximum likelihood parameter estimates. This figure compares the historical yields to maturity of ten on-the-run Treasury issues with those implied by three sets of maximum likelihood parameter estimates of the extended CIR model. Panel A presents the yields in January 1971 when interest rates were low; Panel B presents the yields in August 1981 when the rates were high; Panel C presents the yields in December 1986 when the rates were moderate; and Panel D presents the average yields over the entire sample period. The maximum likelihood parameter estimates of the extended CIR model are based on three different data sets of Treasury prices. The sample period is from December 1971 to December 1986, with 181 monthly observations of ten on-the-run Treasury bills, notes, and bonds. The three different data sets used in estimation are two bond portfolios, the three-month bill and the long bond, and the three- and six-month bills. For the bond portfolios, the first (second) is an equally weighted portfolio of the five on-the-run issues with shorter (longer) maturities.

Panel C, the yield curve implied by the estimates based on two bills deviates considerably from the actual one, especially when interest rates are high and the curve is inverted. Obviously, the estimates based on the two bills imply unreasonably large price errors for longer maturities. This is a serious limitation for practical applications, such as using the model to price options on bonds, to hedge bond portfolios, and to formulate dynamic trading strategies.

B. Estimates Based on One Bill and One Bond

The large price errors in the long end of the term structure clearly indicate that we can include longer-term Treasury issues in estimation to improve fitting the yield curve. Further, employing longer-term bonds in the data can help identify the parameters. The reason is that longer-term bond prices are much more sensitive to changes in the long-term mean of the real interest rate, θ_1. Once θ_1 is identified, so are $\overline{R}$ and σ_1. As a trial, we use the three-month bill and the long bond to estimate the model.

As expected, we no longer have difficulty identifying all the parameters. The estimates (in the middle of Table III) are quite different from those based on two bills and more in line with those based on two bond portfolios. Most importantly, the fitting of the yield curve is much improved. The middle two columns of Table IV show that the implied price errors are such smaller. (In this case the three-month bill and the long bond do not have price errors.) The sum of squared errors is about one-seventh of that implied by the estimates based on the two bills. Though most means of price errors are slightly higher, most standard deviations are much lower.

However, the rather large price errors in the middle maturities show that there is room for improvement. In Panel B and C of Figure 2, the yield curves implied by estimates on the three-month bill and the long bond are very different from the actual ones. Further, the model prices are on average greater than the market prices for all maturities. The average price errors are all positive in Table IV, and the corresponding implied yield curve lies completely below the actual one in Panel D of Figure 2, except at the two end points. This mispricing indicates the need to include middle maturities in estimation and testing.

C. Estimates Based on Two Bond Portfolios

Table IV and Figure 2 confirm that we should use two bond portfolios to incorporate the cross-sectional information contained in the entire yield curve. The estimates imply much smaller price errors and yield curves much more consistent with actual ones. In the following we discuss the parameter estimates based on the two bond portfolios. The extended model requires that all the parameters, except λ, σ_p, and $\overline{R}$, be strictly positive. The requirement is confirmed by the parameter estimates in Table III for the entire sample period. The lower bound of the nominal interest rate, $\overline{R}$, is 2.58 percent, which seems reasonable given that the minimum 3-month yield is 3.12 during the period.

The adjustment speeds of r and y, κ_1 and κ_2, are significantly lower than those based on two bills. The "half-lives" of the two processes are roughly 9 months and 16.5 years. The adjustment speed for y seems unreasonably low, which can be confirmed by the implied steady-state standard deviations of the expected inflation rate, $\bar{y} + (1 - \sigma_p{}^2/2)y_t$ (c.f., equation (3)), and the nominal interest rate, $\overline{R} + r_t + (1 - \sigma_p{}^2)y_t$ (equation (6)). Intuitively, when the adjustment speed is too slow, the variability of the process will be too great. The estimate for σ_2^2 is neither the conditional nor the steady-state variance of y. The steady-state variance is $\sigma_2^2\theta_2/2\kappa_2$. The implied steady-state standard deviation is therefore 15 percent for the expected inflation rate and 17.2 percent for the nominal interest rate, much too high to be compatible with historical data given in Table II. Also, the estimates imply a rather high (9.60 percent) long-term mean of the nominal interest rate ($\bar{R} + \theta_1 + (1 - \sigma_p^2)\theta_2$).

The estimate for the risk premium parameter of the real interest rate, λ, is significantly negative. The instantaneous expected excess return on any real discount bond in the extended model is $-\lambda r B(\tau)$. A negative λ implies positive real term premiums, which is consistent with stylized facts associated with bond returns such as those in Fama (1984).

Finally, Table V reports the parameter estimates for the two subperiods. The results should be interpreted with caution. The asymptotic standard errors for λ, σ_p, κ_2, and $\bar{R}$ in the first subperiod are much larger than those in the whole sample. Further, the algorithm failed to converge for the second

Table V

Maximum Likelihood Parameter Estimates for Subperiods

This table reports the maximum likelihood parameter estimates of the extended Cox, Ingersoll, and Ross (CIR) model for two subperiods. (The numbers in parentheses are the corresponding asymptotic standard errors). The sample period is from December 1971 to December 1986, with 181 monthly prices of ten on-the-run Treasury bills, notes, and bonds. The two subperiods are December 1971 to September 1979 and October 1979 to December 1986, with 94 and 87 observations, respectively. The maximum likelihood parameter estimates are based on monthly observations of two bond portfolios. The first (second) is an equally weighted portfolio of the five on-the-run Treasury issues with shorter (longer) maturities. The CIR real interest rate, r, is described by $dr = \kappa_1(\theta_1 - r)dt + \sigma_1\sqrt{r}\,dZ^{(1)}$, where θ_1 is the long-term mean, κ_1 is the adjustment speed, $\sigma^2 r$ is the local variance, and $Z^{(1)}$ is a standard Brownian motion. The real factor premium is λr. The dynamics of the price level, p, is $dp = ypdt + \sigma_p\sqrt{y}\,dZ^{(2)}$, where the state variable, y, follows $dy = \kappa_2(\theta_2 - y)dt + \sigma_2\sqrt{y}\,dZ^{(3)}$. The two standard Brownian motions, $Z^{(2)}$ and $Z^{(3)}$, are correlated with coefficient ρ. The nominal interest rate R_t is $r_t + (1 - \sigma_p^2)y_t$ in the CIR model, and $\bar{R} - r_t + (1 - \sigma_p^2)y_t$ in the extended model. The parameter $\bar{R}$ is the lower bound of the nominal interest rate. The estimate for ρ is not reported because it is not significantly different from -1 in all cases.

Parameter	Estimate (Standard Error)		
	12/71–12/86	12/71–9/79	10/79–12/86[a]
κ_1	0.8762 (0.2572)	1.1407 (0.3249)	0.7314
θ_1	0.0311 (0.0085)	0.0589 (0.0274)	0.0234
σ_1	0.1707 (0.0120)	0.1092 (0.0278)	0.2080
λ	−0.1282 (0.0232)	−0.1181 (0.1850)	−0.0836
σ_p	0.6549 (0.1122)	0.1652 (0.5777)	0.7295
κ_2	0.0042 (0.0016)	0.0411 (0.0399)	0.0037
θ_2	0.0684 (0.0242)	0.0325 (0.0198)	0.0855
σ_2	0.0920 (0.0127)	0.0468 (0.0141)	0.1121
$\bar{R}$	0.0258 (0.0036)	−0.0169 (0.0311)	0.0270

[a]The constraint that r and y are positive is binding, so the asymptotic standard errors cannot be calculated.

subperiod, and the constraint that r and y are positive is binding so that the asymptotic standard errors cannot be calculated. The parameter estimates seem to support the hypothesis that the behavior of interest rates differs in the two subperiods. The long-term mean of the nominal interest rate is 7.37 percent in the first period and 9.04 percent in the second period. The estimates are consistent with the fact that the Treasury yields were lower and less volatile before the Federal Reserve changed its policy.

VI. Model Assessments

In this section we assess the original and the extended CIR models. The two-factor CIR model is a special case of the extended model with $\overline{R} = 0$. The estimates and the asymptotic standard errors given in Table III indicate that $\overline{R}$ is significantly different from zero. We can also use the likelihood ratio to test the validity of the CIR model.[10] Likelihood ratio tests for the particular restrictions are calculated by taking twice the difference between the maximum values of the log-likelihood function with and without the restrictions. The limit distribution of the likelihood ratio is chi-square with the degree of freedom equal to the number of restrictions. In the last row of Table III, we give the likelihood ratio for the restriction $\overline{R} = 0$.[11] The smallest ratio is 9.7443, which implies a probability value of less than 0.5 percent and rejects the null hypothesis $\overline{R} = 0$, or the validity of the two-factor CIR model.

The above results favor the extended model against the two-factor CIR model. How about the performance of the extended CIR model? First, when discussing the parameter estimates, we mention that the estimates imply certain behaviors not compatible with real observations. For example, the "half-life" of the real interest rate based on the bills is only a month, and the steady-state standard deviations of the expected inflation rate and the nominal interest rate based on the bond portfolios are as high as 15 and 17.2 percent.

Secondly, the parameter estimates are significantly different from one another when we confront the extended CIR model with the three data sets.[12] If the extended CIR model provides a good description of the term structure, similar parameter estimates should be obtained using different data sets. However, the identification problem casts doubts on the validity of the

[10] The asymptotic t-statistics formed from the estimate and standard error of $\overline{R}$ is the square root of the Wald test statistic for the restriction $\overline{R} = 0$. Monte Carlo evidence such as in Gallant (1987, especially Chapter 1 and page 586) indicates that the sampling distributions of Wald test statistics are not as well approximated by their limiting distributions as those of likelihood ratio statistics. Moreover, Gallant (1987) and Gregory and Veall (1985) show that the Wald test is not invariant to a reparameterization that keeps $\overline{R}$ unaffected. The t-statistic could still be changed because the standard error of the estimate for $\overline{R}$ depends upon all the elements of the Hessian.

[11] In the case of two bill prices, we actually test the restriction $\overline{R} = 0$ against $\overline{R} \geq -10$. The maximum value of the unconstrained likelihood function, although it cannot be calculated, is greater than that of the likelihood function with the constraint $\overline{R} \geq -10$. Clearly, this reinforces our rejection of the hypothesis that $\overline{R} = 0$.

Table VI

Correlations of the Nominal Interest Rates Implied by Three Sets of Maximum Likelihood Parameter Estimates

This table presents the correlations among the nominal interest rates implied by three sets of maximum likelihood parameter estimates of the extended Cox, Ingersoll, and Ross (CIR) model. The nominal interest rate at time t is $\overline{R} + r_t + (1 - \sigma_p^2)y_t$ in the extended CIR model, where r_t is the corresponding real interest rate, and y_t is the state variable for the price level. The parameter $\overline{R}$ is the lower bound of the nominal interest rate, and the parameter σ_p determines how well the inflation rate can be explained by the state variable y. The sample period is from December 1971 to December 1986, with 181 monthly prices of ten on-the-run Treasury bills, notes, and bonds. The three different data sets used in estimation are two bond portfolios, the three-month bill and the long bond, and the three- and six-month bills. For the bond portfolios, the first (second) is an equally weighted portfolio of the five on-the-run issues with shorter (longer) maturities.

	Data Set		
	Bond Portfolios	Bill and Bond	Two Bills
	Panel A: Correlations of Rates		
Bond portolios	1.00	0.98	0.89
Bill and Bond		1.00	0.96
	Panel B: First-Order Autocorrelations		
	0.95	0.95	0.90
	Panel C: Correlations of Changes in Rates		
Bond portfolios	1.00	0.84	0.43
Bill and bond		1.00	0.81

comparison of estimates. Instead, we investigate the three time series of the nominal interest rates implied by the three sets of estimates. The three time series should track one another through time if the extended CIR model describes the movement of the Treasury market.

To shed light on this issue, in Table VI we examine the correlations of the three nominal interest rates. Panel A shows that the three rates are highly correlated. The smallest correlation is 0.89 between the bond portfolios and two bills. The 'level' correlations, however, are misleading. Nelson and Plosser (1982) point out that the interest rate is probably approximated by an integrated process, which is evidenced in Panel B. All three first-order correlations are close to one, indicating that the 'level' correlations are likely to be misleading. The correlations of changes in rates do not suffer from this problem. (The autocorrelation of the changes in rates is 0.20, 0.05, and 0.19 respectively.) The correlations of the changes given in Panel C does not

[12] Specifically, given a particular data set, say the bills, we use the likelihood ratio to test if the other two sets of parameter estimates are significantly different from the estimates based on the bills. All the test statistics are considerably greater than the 1 percent critical value for the chi-square distribution with nine degrees of freedom.

support the hypothesis that the three rates track one another through time. The correlation between the bond portfolios and two bills is only 0.43. (The null hypothesis is that they are perfectly correlated.[13]) In other words, the extended CIR model cannot accommodate the three data sets simultaneously, which is evidence against the model.

Last, we investigate the model's performance in within-sample prediction. Since estimates based on two individual issues do not fit the entire yield curve, we focus on the estimates based on the two bond portfolios. At any sample time point t, given the parameter estimates in Table III and the recovered r_t and y_t, equation (11) and the counterpart for y can be used to calculate the expected r_{t+1} and y_{t+1} at the next month and, accordingly, the predicted Treasury price for each maturity. Recall that in estimation the two bond portfolios do not have price errors, but not individual issues. To give the extended CIR model the most chance to succeed, we look at the predicted value versus the actual value of the two portfolios. As a benchmark, the performance of the extended CIR model is compared with that of a naive "martingale model," which assumes that the expected future yields one-month ahead are the current yields. *A priori the extended CIR model is expected to outperform the martingale model.* Findings indicating that the extended CIR model does not perform better are interpreted as evidence against the model.

To assess the predictive performance, we follow Fama and Gibbons (1984) and regress the actual value of the two bond portfolios on the predicted value. Table VII presents the regressions of the extended CIR and the martingale models for the whole sample and two subperiods. Fama and Gibbons (1984) propose three criteria for a good model: a) conditional unbiasedness, that is, an intercept, α, close to zero, and a regression coefficient, β, close to one; b) serially uncorrelated residuals, e; and c) a low residual standard error, *se*.

Surprisingly, the extended CIR model performs as poorly as the martingale model. Half of the first-order residual autocorrelations, denoted as ρ_1, and all the intercepts are significantly different from zero, while the regression coefficients are significantly different from one. The residual standard errors for the two models are very similar in magnitude. Both models perform relatively better in the first subperiod when the interest rate was stable. After the Federal Reserve changed policy in October 1979, the interest rate has been more volatile. This explains why both models perform poorly in the second period. Based on its poor performance in within-sample prediction, we may conclude that the extended model fails to provide a good description of the movement of the on-the-run Treasury prices.

To further our understanding about the poor performance, we calculate the correlations between changes in the interest rates and the prediction errors implied by the extended CIR model. The errors are the differences between the predicted and the actual values of the two bond portfolios. To account for

[13] We cannot test if the sample correlation is significantly different from one, because the asymptotic distribution of a sample correlation does not exist under the null hypothesis of a perfect correlation (Anderson (1984), p. 122).

Table VII

Within-Sample Regressions of Actual Portfolio Values on Predicted Values

This table reports the performance of the extended Cox, Ingersoll, and Ross (CIR) model and a naive martingale model in within-sample prediction for two bond portfolios. The performance is assessed by the following regression:

$$\text{Actual portfolio value} = \alpha + \beta \cdot \text{Predicted portfolio value} + e.$$

The first-order autocorrelation of the residual e is denoted ρ_1 and the standard error, *se*. For the extended CIR model, given the parameter estimates and the current values of the two state variables, r and y, we calculate the expected r and y one month ahead from equation (11) and the counterpart for y, and the predicted value of the two bond portfolios accordingly. The maximum likelihood parameter estimates of the extended model are based on monthly observations of two bond portfolios. The first (second) is an equally weighted portfolio of the five on-the-run Treasury issues with shorter (longer) maturities. The martingale model assumes that the expected future yields one month ahead are the current yields. The sample period is from December 1971 to December 1986, and the two subperiods are December 1971 to September 1979 and October 1979 to December 1986. (The number in the parenthesis is the standard error of the parameter estimate).

	(12/71–12/86)		(12/71–9/79)		(10/79–12/86)	
Parameter	CIR	Martingale	CIR	Martingale	CIR	Martingale
The first portfolio						
α	29.50	22.94	22.46	12.33	45.75	40.89
	(3.89)	(4.21)	(4.70)	(5.01)	(6.13)	(6.68)
β	0.70	0.76	0.77	0.87	0.53	0.57
	(0.04)	(0.04)	(0.05)	(0.05)	(0.06)	(0.07)
R^2	0.63	0.64	0.74	0.76	0.45	0.45
ρ_1	0.26	0.27	0.05[a]	0.07[a]	0.25	0.28
se	0.66	0.72	0.41	0.44	0.77	0.84
The second portfolio						
α	29.01	28.13	9.02	6.56	68.79	67.94
	(3.35)	(3.36)	(2.78)	(2.82)	(5.61)	(5.64)
β	0.71	0.72	0.91	0.94	0.33	0.34
	(0.03)	(0.03)	(0.03)	(0.03)	(0.05)	(0.05)
R^2	0.72	0.73	0.92	0.92	0.31	0.32
ρ_1	0.21	0.19	0.08[a]	0.07[a]	0.03[a]	0.05[a]
se	2.01	2.02	1.10	1.12	1.93	1.94

[a]All the parameter estimates except those marked are greater than two standard errors.

the difference in duration, we use different interest rates for the two portfolios. Specifically, the correlation between changes in the one-year (seven-year) interest rate and the prediction errors of the first (second) portfolio is 0.96 (0.98). That is, the extended model consistently underestimates interest rates (and overestimates the portfolio values) when the rates move up, and overestimates interest rates when the rates move down. In other words, the dynamics of interest rates in the extended CIR model cannot capture the volatility exhibited in the movement of interest rates. This finding and those in Garbade (1986) and Litterman and Scheinkman (1988), which show that there are three factors in the on-the-run yield curve, imply that it is a formidable challenge to use a two-factor interest rate model to describe the Treasury market.

VII. Conclusion

We develop an approach to use the method of maximum likelihood to estimate and test an extension of a two-factor term structure model due to Cox, Ingersoll, and Ross (1985). Our method exploits the conditional density of the unobserved state variables. We estimate the model using data on the prices of both discount and coupon U.S. Treasury securities. Results based on three different groups of bonds covering the entire range of maturities reject the original CIR model. An extended version of the CIR model also fails to describe the bond price data.

In addition, our results show that the estimates based on only the short end of the yield curve imply unreasonably large price errors for longer maturities. Though our estimation procedure is much easier to implement with discount bonds, the implied large price errors may limit the usefulness of using only discount bonds. This is especially important for practical applications, such as using the model to price options on bonds, to hedge bond portfolios, and to formulate dynamic trading strategies.

Since our nominal two-factor model is obtained by appending the price level process to the one-factor model of the real economy, it has implications for the dynamics of the price level. One possible direction for further research is to include price level data in the estimation and testing. We can use data on two bond portfolios and the price level to estimate the parameters of the conditional joint density of p, r, and y. There are, however, two major difficulties. First, one needs to tackle the time-averaging problem of CPI and the nonsynchronous data problem between bond prices and CPI. Second, the conditional joint density of p, r, and y must be known. If the density is not available in the continuous-time framework, a discrete-time approximation has to be used.

Also, we can employ the approach to estimate and test most continuous-time term structure models such as Oldfield and Rogalski (1987), Schaefer and Schwartz (1984), and Vasick (1978). More generally, the approach may be applied to any asset pricing model in which the pricing functions are invertible and the conditional density of the unknown state variables is known. Finally, when there is a *a priori* specification of a statistical model for "disturbances" caused by measurement errors or market imperfections, as in Brown and Dybvig (1986), we can incorporate the statistical model in the empirical method.

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