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Source: *The Journal of Finance*, Vol. 49, No. 2 (Jun., 1994), pp. 543-556
Published by: Blackwell Publishing for the American Finance Association
Stable URL: http://www.jstor.org/stable/2329162
Accessed: 25/11/2009 10:36

THE JOURNAL OF FINANCE • VOL. XLIX, NO. 2 • JUNE 1994

On Stock Market Returns and Returns on Investment

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ABSTRACT

This article presents general conditions under which it is possible to obtain asset pricing relations from the intertemporal optimal investment decision of the firm. Under the assumption of linear homogeneous production and adjustment cost functions (the Hayashi (1982) conditions), it is possible to establish, state by state, the equality between the return on investment and the market return of the financial claims issued by the firm. This result proves to be, in essence, robust to the consideration of very general constraints on investment and the inclusion of taxes.

STANDARD WORK IN ASSET pricing relates the returns on financial claims with the intertemporal marginal rate of substitution of economic agents (e.g., Merton (1973), Lucas (1978), Breeden (1979), and Hansen and Singleton (1982)). The intertemporal marginal rate of substitution typically involves a function of consumption growth. The empirical performance of this approach has not been very successful in explaining the relation between financial variables and the business cycle. On one hand, consumption is probably too smooth to explain the behavior of asset returns, and, on the other hand, this variable is not an accurate proxy for economic activity.

Production variables such as investment and output, are relatively more volatile and more characteristic of economic fluctuations than consumption. Furthermore, studies by Fama and Gibbons (1982), Fama and French (1989), and Barro (1989) have documented a significant relation between stock returns and both output and investment. These facts seem to suggest that theoretical models which succeed in relating asset returns with production side variables might be empirically more successful.

Standard general equilibrium models with production (e.g., Brock (1982), Sharatchandra (1990), Braun (1990), as well as Restoy and Rockinger (1991)) assume price-taking managers who maximize the net present discounted value of the firm in a world with complete markets. The first order conditions are stochastic Euler equations, which state that the conditional mean of returns on investment evaluated with state contingent claim prices must be equal to 1. Thus, as any return on a financial claim, investment returns have to satisfy a traditional Euler equation. However, state contingent claim prices

*Restoy is from the Bank of Spain. Rockinger is from Groupe HEC. We are grateful to Gary Chamberlain, Greg Mankiw, Philippe Weil, and an anonymous referee for helpful comments. Rockinger wishes to acknowledge financial support from the Fondation Nationale pour l'Enseignement de la Gestion des Entreprises.

are not obtainable from standard specifications of the technology. It follows that, in general, the first order conditions of the optimization problem involve unobserved terms unless one is willing to assume a particular specification of the relative prices. This requires a particular specification for consumer preferences (as in Sharatchandra (1990) or Braun (1990)).

Research that attempts to identify tradeable portfolios that replicate the physical investment payoffs in a world with no arbitrage opportunities (as in Cochrane (1991)) is probably more promising. This assumption ensures that investment payoffs can be replicated with existing financial claims. Cochrane assumes a particular technology and adjustment cost function that allows him to identify the replicating portfolio with the return on the stock of the firm. In this article we build on Cochrane's (1991) result and obtain a general set of conditions under which one can identify the portfolio that replicates the investment return with the return on the financial claims issued by the firm.

The structure of this article is as follows. In Section I we find that it is possible to obtain strong asset pricing implications when a model similar to the one used in q-theory of investment is set in an uncertain environment. In particular we show that in a world with no arbitrage opportunities, the return on investing in a firm with linear homogeneous production and adjustment cost function (the Hayashi (1982) conditions) is equal *state by state* to the market return of a claim on the stock of the firm. In Section II we extend the analysis to the case where the firm is constrained in its possibilities to raise external funds to finance investment and find a similar result to the one obtained in Section I. The expression of the investment return, though, is modified to incorporate a term that measures the binding character of the constraint. In Section III, we focus on taxation and find that the investment return can be replicated by a portfolio composed by the stock of the firm and a bond portfolio. Section IV concludes the article.

I. The Basic Model

Assume a standard multiperiod production economy under uncertainty. Trading takes place at discrete points of time indexed by $t = 1, 2, \dots$. The resolution of uncertainty can be represented by an event tree generated by the realizations of an exogenous technical shock. There exists a probability measure under which any (normalized) payoff structure must be priced as a conditional expectation to prevent arbitrage (e.g., Harrison and Kreps (1979) and Huang and Litzenberger (1988)). Denote the expectations operator under this alternative probability by E^* and let p_t, x_{t+1} denote the price of an asset and its random payoff. Assume the existence of an infinitely lived and locally riskless security with a strictly positive price at all times. Call b_t the price at time t of such a claim chosen as numeraire. Let $P_t = p_t/b_t$, $X_{t+1} = x_{t+1}/b_{t+1}$ be the normalized prices. Then, the return $R_{t+1} = X_{t+1}/P_t$ has to satisfy the no-arbitrage condition

$$E_t^*\{R_{t+1}\} = 1. \tag{1}$$

Consider now a representative firm under perfect competition. Let k_t, i_t and λ_t stand for the capital stock of the firm, investment, and the random technical shock, which is assumed to be markovian. Define, for simplicity, a single factor production function $y_t = f(k_t)\lambda_t$ and a capital accumulation rule $k_{t+1} = a(k_t, i_t)$. Assume $f(\cdot)$ and $a(\cdot,\cdot)$ are continuously differentiable with strictly positive partial derivatives over their domains. The capital accumulation rule incorporates capital depreciation as well as installation costs. For notational convenience we denote in capital letters the corresponding normalized variables in terms of the price of the infinitely lived security taken as numeraire. Therefore, define as $K_t = k_t/b_t$, $I_t = i_t/b_t$, $Y_t = y_t/b_t$, $F(K_t) = f(K_t b_t)/b_t$ and as $A(K_t, I_t) = a(K_t b_t, I_t b_t)/b_{t+1}$ the normalized capital stock, investment, output, production function and capital accumulation rule. Finally define the normalized cash flow as $D_j = F(K_j)\lambda_j - I_j$.

Equation (1) and a standard transversality condition yield that the (normalized) gross value at time t of a firm can be written as the expected (under the equivalent probabilities) value of the random stream of future (normalized) cash flows $\{D_j\}_{j=t+1}^{+\infty}$. More formally[1]

$$V_t = E_t^* \left\{ \sum_{j=t+1}^{+\infty} D_j \right\}. \tag{2}$$

Define $J_t \equiv V_t - I_t$ as the net value of the firm at period t. Assume producers choose the investment path that maximizes the expected future stream of cash flows subject to a technological constraint. We can then write J_t as

$$J_t = \max_{\{I_t\}} E_t^* \left\{ -I_t + \sum_{j=t+1}^{+\infty} \left[F(K_j)\lambda_j - I_j \right] \right\},$$
$$\text{s.t.} \quad K_{t+1} = A(K_t, I_t). \tag{3}$$

The first order conditions for the producer problem can be written as follows

$$E_t^* \left\{ \left[F_K(K_{t+1})\lambda_{t+1} + \frac{A_K(K_{t+1}, I_{t+1})}{A_I(K_{t+1}, I_{t+1})} \right] A_I(K_t, I_t) \right\} = 1. \tag{4}$$

[1]In a standard complete markets discrete space state setting, the relation between standard and equivalent conditional probabilities and state contingent claim prices at period t is given by the expression:

$$\pi^*(a_{t+1}) = \pi_t(a_{t+1}) m_{t+1}(a_{t+1}) b_{t+1}/b_t = \phi_t(a_{t+1}) b_{t+1}/b_t,$$

where a_t denotes a node of the event tree at period t, m_{t+1} is a stochastic discount factor (e.g., $\beta u'(c_{t+1})/u'(c_t)$), and $\phi_t(a_{t+1})$ is the price at period t of the state contingent claim that pays off one unit of the consumption good at period $t+1$ in state a_{t+1} and zero otherwise (see e.g., Huang and Litzenberger, (1988)). Thus, all the results of this article could be derived using the standard probability measure as in most of the representative-agent asset pricing literature. For example, equation (1) could be written as $E_t\{\sum_{j=t+1}^{\infty}(\prod_{i=t+1}^{j} m_i)d_j\}$, where d_t is the cash flow at period t. However, the use of an equivalent probability measure in this paper is not only formally more elegant but also provides notational simplicity to the dynamic programming analysis that follows.

Equation 4 states that along an optimal investment path it must be true that expected marginal benefits are equal to marginal costs. Notice that $1/A_I(K_t, I_t)$ is the value (in terms of investment at period t) of a unit of capital installed during the next production period. This marginal unit of capital produces $F_K(K_{t+1})\lambda_{t+1}$ units of the investment good at time $t + 1$ but also depreciates into $A_K(K_{t+1}, I_{t+1})$ units of capital, which are worth $A_K(K_{t+1}, I_{t+1})/A_I(K_{t+1}, I_{t+1})$ units of investment at period $t + 1$.

We can define the *investment return* as

$$R^I_{t+1} \equiv \left[F_K(K_{t+1})\lambda_{t+1} + \frac{A_K(K_{t+1}, I_{t+1})}{A_I(K_{t+1}, I_{t+1})}\right] A_I(K_t, I_t). \tag{5}$$

Therefore, the first order condition for an optimal investment schedule (equation (4)) implies that the investment return should satisfy an Euler equation analogous to equation (1):

$$E^*_t\{R^I_{t+1}\} = 1. \tag{6}$$

This expression states that the investment return has to satisfy, as any other asset return, a set of arbitrage conditions represented generically by equation (1). Equation (6) can be tested if the equivalent probability measure is specified. This approach (used implicitly by Sharatchandra (1990) and Braun (1990)) requires, however, the assumption of a particular preference structure to obtain explicit expressions for the state contingent claim prices that define the new probabilities.

If it is feasible to identify a tradeable portfolio that replicates the investment payoff, then standard arbitrage arguments yield that the investment return should be equal state by state to the return of the replicating portfolio. A natural application of this idea is to relate the investment return of the firm with the return of the claims issued by the firm. Cochrane (1991) deals with a particular case where the return on expanding the capital stock of the firm is equivalent to the market return on the claims issued by the firm.[2] Proposition 1 establishes general conditions under which the investment return is indeed equal to the market return of the stock of the firm. In order to prove that proposition we will make use of the following Lemma[3]:

LEMMA 1: *If the production function and the capital accumulation rule are linear homogeneous, then* $V_t = K_{t+1}/A_I(K_t, I_t)$.

Notice that $A_I(K_t, I_t)$ is the relative price of one unit of installed capital in terms of noninstalled capital. Therefore Lemma 1 states that under a set of standard conditions in investment theory the replacement value of the capital stock of a firm is a stochastic fraction of its market value. Thus Lemma 1 simply extends Hayashi's Q-theory to this general uncertain environment.

[2] Cochrane (1991) obtains this result for a two-factor technology with exogenous marginal productivity of labor and capital.

[3] Proof of this Lemma together with those of Lemma 3 and Lemma 4 can be found in the Appendix.

Now, defining the market return of the firm as $R^M_{t+1} = (D_{t+1} + V_{t+1})/V_t$, we are ready to state the main result of this section:

PROPOSITION 1: *Under the conditions of Lemma* 1, *investment and market returns are equal state by state.*

Proof of Proposition 1: The investment return can be rewritten as

$$R^I_{t+1} = \frac{F_K(K_{t+1})\lambda_{t+1} + \dfrac{A_K(K_{t+1}, I_{t+1})}{A_I(K_{t+1}, I_{t+1})}}{\dfrac{1}{A_I(K_t, I_t)}}.$$

Then, multiplication of the numerator and denominator by K_{t+1}, addition and subtraction of I_{t+1} in the numerator, as well as application of the Euler theorem for linear homogeneous functions yield:

$$R^I_{t+1} = \frac{F(K_{t+1})\lambda_{t+1} - I_{t+1} + K_{t+2}/A_I(K_{t+1}, I_{t+1})}{K_{t+1}/A_I(K_t, I_t)}. \tag{7}$$

Using Lemma 1, equation (7) becomes

$$R^I_{t+1} = \frac{D_{t+1} + V_{t+1}}{V_t} = R^M_{t+1},$$

which concludes the proof.

The result that investment and market returns are equal requires exactly the Hayashi (1982) conditions to make average and marginal valuations of the capital stock equal. A constant returns to scale production function is required to prevent the firm from obtaining quasi-rents beyond the remuneration to inputs. A linear homogeneous capital accumulation rule is required to ensure that the average market valuation of the installed capital is equal to the marginal one. Notice that in general, the stock market is evaluating the proceedings from owning an average unit of capital, while the investment return is related to the marginal unit of invested capital. Thus, this proposition can be seen as a direct consequence of obtaining that, under uncertainty, Hayashi's result, stated in a quasi-difference form, holds in every state of nature.

Proposition 1 shows how asset pricing implications can be derived from those technological conditions in a very general framework under uncertainty. Notice that the extension of this proposition to a multifactor technology is almost immediate. The definition of the investment return remains unchanged if the technology is assumed to include variable inputs together with the fixed input by just allowing the marginal productivity of capital to depend also on the other inputs. Furthermore, the equality between market and investment returns holds as long as it is assumed that firms are price takers in the factor markets and a complete set of state contingent factor prices exist.[4]

In the next sections we investigate if the result stated in this section is modified when including financial constraints and taxation.

II. Constrained Investment

Asymmetric information and capital market imperfections provide explanations for the existence of limited possibilities of raising external funds to finance investment. The empirical investment literature (e.g., Fazzari, Hubbard, and Petersen (1988) and Hall (1991)) has tried to incorporate financing constraints by including proxies of the liquidity status of the firm in a standard q-type of relation. However, this approach has failed to directly model the effects of those restrictions on the optimal investment decisions. Below we show how in our framework this issue can be modeled in a more satisfactory way for very general specifications of the financing constraints.

Assume that, at period t, the firm can only invest a proportion $\zeta_t(\lambda_t, t)$ of its capital stock. This seems to be a natural and sufficiently general way to account for the existence of financial rationing. The producer problem can therefore be written as

$$
\begin{aligned}
J_t = \max_{\{I_t\}} E_t^* & \left\{ -I_t + \sum_{j=t+1}^{+\infty} \left[F(K_j)\lambda_j - I_j \right] \right\} \\
\text{s.t.} \quad & K_{t+1} = A(K_t, I_t) \\
& I_t \leq \zeta_t K_t. \qquad (8)
\end{aligned}
$$

To solve this program, we need to introduce a sequence of Lagrange multipliers γ_t for the set of constraints in equation (8).[5] Those multipliers are set up to be nonnegative and zero if the constraint is not binding, so that $\gamma_t[\zeta_t K_t - I_t] = 0$ for all t.

The recursive form of the optimization problem is

$$J(K_t, \lambda_t) = \max_{\{I_t\}} E_t^* \{ -I_t + F(K_{t+1})\lambda_{t+1} + J(K_{t+1}, \lambda_{t+1}) + \gamma_t[\zeta_t K_t - I_t] \}.$$

[4] For empirical purposes nothing prevents the inclusion of labor L_t paid at wage w_t as long as the production function remains linear homogeneous. In this case, the investment return becomes

$$R_{t+1}^I = \left[F_K(K_{t+1}, L_{t+1})\lambda_{t+1} + \frac{A_K(K_{t+1}, I_{t+1})}{A_I(K_{t+1}, I_{t+1})} \right] A_I(K_t, I_t).$$

Since a complete set of contingent contracts exists, the first order conditions with respect to labor yield $F_L(K_{t+1}, L_{t+1}) - w_{t+1} = 0$ for all t. Adding this new condition allows us to obtain that $R_{t+1}^I = R_{t+1}^M$ as in the single factor case.

[5] This framework is also useful to model the effects of investment irreversibility by writing the inequality constraint as $I_t \geq 0$.

The first order conditions of the maximization problem imply, as previously, the no-arbitrage condition

$$E_t^*\{R_{t+1}^I\} = 1$$

where the investment return R_{t+1}^I is now defined as

$$R_{t+1}^I \equiv \left[F_K(K_{t+1})\lambda_{t+1} + (1+\gamma_{t+1})\frac{A_K(K_{t+1}, I_{t+1})}{A_I(K_{t+1}, I_{t+1})} + \gamma_{t+1}\frac{I_{t+1}}{K_{t+1}}\right] \bigg/ \left[\frac{(1+\gamma_t)}{A_I(K_t, I_t)}\right]. \tag{9}$$

As before, this equation states that along an optimal investment path marginal costs have to be equal to marginal expected benefits. If at period t and period $t+1$ the firm is not liquidity constrained the investment return takes the same expression as seen in Section 2. If the firm is constrained over time, then there is an interplay of various factors that modify the expression of the investment return. First, if the constraint is binding at time t every unit of installed capital at period $t+1$ is more valuable, because feasible investment is below its optimal level at period t. The value of a unit of installed capital is therefore $(1+\gamma_t)/A_I(K_t, I_t)$. Second, at period $t+1$ the marginal unit of capital produces as before $F_K(K_{t+1})\lambda_{t+1}$. Third, each unit of investment at period t increases the capital stock at period $t+1$ and, therefore, relaxes the possibly binding character of the constraint at period $t+1$. This effect is measured by the term $\gamma_{t+1}I_{t+1}/K_{t+1}$. Fourth, every unit of remaining installed capital $A_K(K_{t+1}, I_{t+1})$ has a higher value at $t+2$ if the constraint is binding at $t+1$. This is the explanation of the term $(1+\gamma_{t+1})A_K(K_{t+1}, I_{t+1})/A_I(K_{t+1}, I_{t+1})$.

As in the basic model, under some technological conditions it is possible to link the return on the investment with the returns of a claim on the capital stock of the firm. As a first step, observe that the market value of the firm is $V_t = J_t + I_t$. Next, by using arguments similar to the ones used in the proof of Lemma 1 one can show:

LEMMA 2: *Under the conditions of Lemma* 1:

$$V_t = (1+\gamma_t)\frac{K_{t+1}}{A_I(K_t, I_t)}. \tag{10}$$

Lemma 2 states that the market value of a claim to a capital stock of the firm is still proportional to its replacement cost. If the constraint is not binding, then the Lagrange multiplier is zero and the value of the firm is equal to the unconstrained case. If the constraint is binding, then the Lagrange multiplier is positive. In this case, the desired investment is larger than the feasible investment, and therefore the existing capital stock is more valuable. This gets translated into a greater shadow price of capital. Now we are ready to state the liquidity constraints version of Proposition 1.

PROPOSITION 2: *Under the conditions of Lemma* 1, *the investment return, as defined in equation* (9), *is equal to the market return state by state.*

Proof of Proposition 2: Multiplication of the numerator and denominator by K_{t+1}, using the Euler theorem for homogeneous functions, and Lemma 2 yield

$$R^I_{t+1} = \frac{F(K_{t+1})\lambda_{t+1} + (1+\gamma_{t+1})\dfrac{A_K(K_{t+1}, I_{t+1})}{A_I(K_{t+1}, I_{t+1})}K_{t+1} + \gamma_{t+1}I_{t+1}}{(1+\gamma_t)\dfrac{K_{t+1}}{A_I(K_t, I_t)}}$$

$$= \frac{F(K_{t+1})\lambda_{t+1} - I_{t+1} + V_{t+1}}{V_t} = R^M_{t+1}. \quad (11)$$

One of the limitations of the model[6] outlined in Sections 2 and 3 is the absence of taxes. In the following section we analyze how the relation between market and investment returns varies in the presence of distortionary taxation. In particular, we modify the model to include taxation on corporate profits, credits on investment expenditures, and depreciation allowances of the capital stock.

III. Taxes

Let u_j, w_j and $g(s, j-s)$ be respectively the corporate tax rate, the investment tax credit, and a depreciation allowance for equipment of age s at period j.

Therefore, total depreciation allowances at period j are $da_j = \sum_{s=1}^{+\infty} g(s, j-s)i_{j-s}$. As done in the previous sections, it is convenient to perform a normalization in units of the infinitely lived security by redefining $DA_j = da_j/b_j$ and $G(s, j-s) = g(s, j-s)b_{j-s}/b_j$. Under this renormalization, total depreciation allowances can be written as:

$$DA_j = \sum_{s=1}^{+\infty} G(s, j-s)I_{j-s}. \quad (12)$$

The cash flows of the firm are given by after-tax income minus investment expenditures adjusted for the investment tax credit plus depreciation al-

[6] Naturally, an important concern in the empirical analysis of Proposition 2 is the estimation of the Lagrange multipliers γ_t, since closed form solutions are not available. However, using data on Tobin's q for firms or sectors, Lagrange multipliers can be inferred by using equation (10). This requires the availability of well behaved estimates of the technological parameters. In order to obtain those, one possibility is to estimate the technological parameters using data on firms that are ex ante nonconstrained using Euler equations for investment returns or exploiting Lemma 1 or Proposition 1. A similar method has been successfully suggested and applied by Zeldes (1989) in an analysis of liquidity constraints on consumption.

lowances on existing capital:

$$D_j = [1 - u_j]F(K_j)\lambda_j - [1 - w_j]I_j + u_j DA_j. \tag{13}$$

Therefore the objective function will be:

$$\max_{\{I_t\}} E_t^* \left\{ \sum_{j=t+1}^{+\infty} [1-u_j]F(K_j)\lambda_j - [1-w_j]I_j + u_j \sum_{s=1}^{+\infty} G(s, j-s) I_{j-s} \right\}. \tag{14}$$

Define for further use the following random quantities:

$$z_t \equiv \sum_{j=t+1}^{+\infty} u_j G(j-t, t),$$

$$\alpha_t \equiv \sum_{j=t+1}^{+\infty} z_j I_j,$$

$$\beta_t \equiv \sum_{j=t+1}^{+\infty} \sum_{l=-\infty}^{t-1} u_j G(j-l, l) I_l + z_t I_t = \sum_{l=-\infty}^{t} I_l \sum_{j=t+1}^{+\infty} u_j G(j-l, l),$$

and define also $\hat{z}_t \equiv E_t^*[z_t]$, $\hat{\alpha}_t \equiv E_t^*[\alpha_t]$, and $\hat{\beta}_t \equiv E_t^*[\beta_t]$. It is now possible to split the sum at period t of total depreciation allowances into two parts.[7]

LEMMA 3:

$$\sum_{j=t+1}^{+\infty} u_j \sum_{s=1}^{+\infty} G(s, j-s) I_{j-s} = \alpha_t + \beta_t.$$

The terms α_t and β_t respectively measure the discounted sum of future income from the depreciation allowances associated with future investment and existing capital. From Lemma 3 and equation (14), the first order condition of the optimization problem becomes:

$$E_t^*\{R_{t+1}^I\} = 1 \tag{15}$$

where the expression for investment return is

$$R_{t+1}^I \equiv \left[[1 - u_{t+1}]F_K(K_{t+1})\lambda_{t+1} + (1 - w_{t+1} - \hat{z}_{t+1}) \cdot \frac{A_K(K_{t+1}, I_{t+1})}{A_I(K_{t+1}, I_{t+1})} \right] \Big/ \left[\frac{1 - w_t - \hat{z}_t}{A_I(K_t, I_t)} \right]. \tag{16}$$

This expression is similar to the one given in Section 1. The only modification affects the relative price of invested capital at every period t. This variable now includes both the investment tax credit and the present discounted value

[7]A similar decomposition can be found in Hayashi (1982) and Summers (1981), as well as in Salinger and Summers (1983).

of the income to be obtained from future depreciation allowances associated with this investment.

The next step is to calculate the ex-dividend market value of the firm:

LEMMA 4: *Under the conditions of Lemma* 3:

$$V_t = \frac{1 - w_t - \hat{z}_t}{A_I(K_t, I_t)} K_{t+1} + \hat{\beta}_t.$$

Unlike the previous cases, the ex-dividend market value of the firm is no longer proportional to the replacement cost of the capital stock. This result prevents us from stating that in this economy with depreciation allowances of the capital stock, investment returns are equal to market returns. However, we can still identify a replicating tradeable portfolio for the investment payoff. In order to do that, notice first that $\hat{\beta}_t$ can be seen as the (normalized) value at period t of a portfolio of perpetual coupon bonds whose total (normalized) payoff from period t onwards is β_t. Each bond is indexed by l ($l \leq t$). $u_j g(j - l, l)$ and i_l are, respectively, the coupon paid by bond l at period j ($j > t$) and the amount invested in that bond.[8]

PROPOSITION 3: *Consider the following investment strategy at period t: purchase the capital stock of the firm and short sell the bond portfolio $\{i_l, l \leq t\}$. Then, the return on this strategy between t and $t + 1$ is equal to the investment return, as defined in equation* (16), *state by state.*

Proof of Proposition 3: Notice that $E^*_{t+1}[\beta_t]$ is the cum-dividend price of the bond portfolio at period $t + 1$. Therefore, we just have to show that

$$R^I_{t+1} = \frac{D_{t+1} + V_{t+1} - E^*_{t+1}[\beta_t]}{V_t - \hat{\beta}_t}. \tag{17}$$

Multiplying numerator and denominator of equation (16) by K_{t+1}, we have that, from Lemma 4, the resulting denominator is $V_t - \hat{\beta}_t$.

From equation (13), the linear homogeneity of the capital accumulation rule, and Lemma 4 we can write the numerator as

$$(1 - u_{t+1})F_K(t+1)K_{t+1}\lambda_{t+1} + (1 - w_{t+1} - \hat{z}_{t+1})\frac{A_K(t+1)}{A_I(t+1)}K_{t+1}$$
$$= V_{t+1} + D_{t+1} - \hat{\beta}_{t+1} + \hat{z}_{t+1}I_{t+1} - u_{t+1}DA_{t+1}. \tag{18}$$

Now, from Lemma 3, and the definitions of α_t, β_t, $\hat{\alpha}_t$, and $\hat{\beta}_t$, the amount obtained from depreciation allowances at period $t + 1$ can be written as

$$u_{t+1}DA_{t+1} = E^*_{t+1}[\alpha_t + \beta_t] - \hat{\alpha}_{t+1} - \hat{\beta}_{t+1}$$
$$= \hat{z}_{t+1}I_{t+1} + E^*_{t+1}[\beta_t] - \hat{\beta}_{t+1}. \tag{19}$$

[8] Of course, in the finite horizon case this portfolio of perpetual coupon bonds can be replicated by a portfolio of coupon bonds with different maturities.

Substituting equation (19) in equation (18) we obtain the numerator of equation (17).

Thus, even when investment returns are not equal to market returns, the difference between those two returns can be interpreted as the return on a portfolio of tradeable real bonds.[9]

IV. Conclusion

This article presents general conditions under which it is possible to obtain asset pricing relations from the intertemporal optimal investment decision of the firm. The basic model places firms in an uncertain environment with no arbitrage opportunities. Under the assumption of linear homogeneous production and adjustment cost functions (the Hayashi (1982) conditions), it is possible to establish, state by state, the equality between the return on investment and the market return of the financial claims issued by the firm. This allows us to relate asset prices to technology without explicitly specifying discount factors or assuming the existence of a representative consumer. This result proves to be, in essence, robust to the consideration of very general constraints on investment and the inclusion of taxes. This article places the findings of Cochrane (1991) in a general setup and provides a fertile framework to obtain asset pricing implications from technological and financial characteristics of the firm.

Appendix

Proof of Lemma 1: The problem defined by equation (3) can be written compactly in the following recursive manner:

$$J(K_t, \lambda_t) = \max_{\{I_t\}} E_t^*\{-I_t + F(A(K_t, I_t))\lambda_{t+1} + J(A(K_t, I_t), \lambda_{t+1})\}. \quad (20)$$

The first order condition is

$$E_t^*\{[F_K(K_{t+1})\lambda_{t+1} + J_K(K_{t+1}, \lambda_{t+1})]A_I(K_t, I_t)\} = 1. \quad (21)$$

Combining the first order condition for optimality in equation (21) with the capital accumulation rule yield the envelope condition:

$$J_K(K_{t+1}, \lambda_{t+1}) = \frac{A_K(K_{t+1}, I_{t+1})}{A_I(K_{t+1}, I_{t+1})} \quad (22)$$

[9] For empirical purposes, the value of the bond portfolio can be obtained using data on interest rates at different maturities. Thus, consider, for example a finite investment horizon, a constant corporate tax rate u, and exponential depreciation allowances at a constant rate δ. Then, by standard arbitrage arguments the value of the bond portfolio at period t is $u \sum_{l \le t} \sum_{j=t+1}^{T} i_l \delta^{j-l}/(1 + r_t(j-t))^{j-t}$, where $r_t(s)$ is a s-period interest rate at period t. Similarly, the cum-dividend price of the bond portfolio is just the value of the bond portfolio at period $t+1$ plus the coupons paid at that period (i.e., $u \sum_{l \le t} i_l \delta^{t+1-l}$).

and therefore, the first order condition for optimal investment becomes

$$E_t^*\left\{\left[F_K(K_{t+1})\lambda_{t+1} + \frac{A_K(K_{t+1}, I_{t+1})}{A_I(K_{t+1}, I_{t+1})}\right]A_I(K_t, I_t)\right\} = 1, \tag{23}$$

where the expression between curly brackets is the investment return.

Consider the experiment of multiplying K_t and I_t by some constant μ. Then, equation (22) becomes

$$J_K(\mu K_{t+1}, \lambda_{t+1}) = \frac{A_K(K_{t+1}, I_{t+1})}{A_I(K_{t+1}, I_{t+1})}, \tag{24}$$

where we have used the assumption that $A(\cdot, \cdot)$ is linear homogeneous and therefore that the partial derivatives depend only on the ratio I_{t+1}/K_{t+1}. Since μ appears only on the LHS of equation (24) it must be that the LHS does not depend on K_{t+1}. Formally, we can therefore write that

$$J_K(\lambda_{t+1}) = \frac{A_K(1, I_{t+1}/K_{t+1})}{A_I(1, I_{t+1}/K_{t+1})} \equiv H(\lambda_{t+1}). \tag{25}$$

Integration yields

$$J(K_{t+1}, \lambda_{t+1}) = H(\lambda_{t+1})K_{t+1} + h(\lambda_{t+1}, t+1). \tag{26}$$

Then, from equation (20)

$$J(K_t, \lambda_t) = E_t^*\{-I_t + F(K_{t+1})\lambda_{t+1} + J(K_{t+1}, \lambda_{t+1})\}. \tag{27}$$

Substitution of equation (26) into equation (27) yields

$$\begin{aligned} &H(\lambda_t)K_t + h(\lambda_t, t) \\ &\quad = E_t^*\{-I_t + F(K_{t+1})\lambda_{t+1} + H(\lambda_{t+1})K_{t+1} + h(\lambda_{t+1}, t+1)\}. \end{aligned} \tag{28}$$

The homogeneous solution of equation (26) satisfies equation (27). Indeed, substituting the homogeneous solution in equation(27) and using the linear homogeneity properties of $F(\cdot)$ and $A(\cdot, \cdot)$ we obtain equation (21). From equation (28), h has to satisfy the condition

$$h(\lambda_t, t) = E_t^*\{h(\lambda_{t+1}, t+1)\}. \tag{29}$$

Notice that as long as we have not assumed a terminal condition for the Bellman equation, equation (29) admits an infinite number of solutions. However, assuming bubbles away, we obtain that $h = 0$ at all times.

By using again the assumption of linear homogeneity of $A(\cdot, \cdot)$, the ex-dividend market value of the firm becomes $V_t = J_t + I_t = A_K(\lambda_t)K_t/A_I(\lambda_t) + I_t = K_{t+1}/A_I(\lambda_t)$. Q.E.D.

Proof of Lemma 3:

$$\begin{aligned}
\sum_{j=t+1}^{+\infty} u_j DA_j &= \sum_{j=t+1}^{+\infty} \sum_{s=1}^{+\infty} u_j G(s, j-s) I_{j-s} \\
&= \sum_{j=t+1}^{+\infty} \sum_{l=-\infty}^{j-1} u_j G(j-l, l) I_l \\
&= \sum_{j=t+1}^{+\infty} \sum_{l=-\infty}^{t-1} u_j G(j-l, l) I_l + \sum_{j=t+1}^{+\infty} \sum_{l=t}^{j-1} u_j G(j-l, l) I_l \\
&= \sum_{j=t+1}^{+\infty} \sum_{l=-\infty}^{t-1} u_j G(j-l, l) I_l + \sum_{l=t}^{+\infty} \sum_{j=l+1}^{+\infty} u_j G(j-l, l) I_l \\
&= \sum_{j=t+1}^{+\infty} \sum_{l=-\infty}^{t-1} u_j G(j-l, l) I_l + \sum_{l=t}^{+\infty} \sum_{j=1}^{+\infty} u_{j+l} G(j, l) I_l \\
&= \alpha_t + \beta_t .
\end{aligned}$$

The first equality comes from substitution of the definition of DA_j. A change of variables with $l = j - s$, going from $-\infty$ to $t - j$ when s goes from 1 to $+\infty$ yields the second equality. The third one is obtained by just splitting $\sum_{l=-\infty}^{+\infty}$ into two well chosen parts. Inverting the second group of summations while being careful about the domain of summation yields the fourth equality. The last equalities follow from straightforward algebraic manipulations. Q.E.D.

Proof of Lemma 4: From Lemma 3, the objective function of the firm can be written as

$$E_t^* \left\{ -(1 - w_t - z_t) I_t + \sum_{j=t+1}^{+\infty} \big[[1 - u_j] F(K_j) \lambda_j - [1 - w_j - z_j] I_j \big] + \sum_{j=t+1}^{+\infty} \sum_{l=-\infty}^{t-1} u_j G(j-l, l) I_l \right\}.$$

Noticing that the rightmost term is independent of the manager's present action, this program is equivalent to the following recursive one

$$J_t = \max_{\{I_t\}} E_t^* \left\{ -(1 - w_t - z_t) I_t + \big[(1 - u_j) F(K_j) \lambda_j - J_{t+1} \big] \right\},$$

$$\text{s.t.} \quad K_{t+1} = A(K_t, I_t),$$

whose first order conditions are those stated in equations (15) and (16). Using algebra similar to the one required to prove Lemma 1, one obtains:

$$J_t = (1 - w_t - \hat{z}_t) \frac{A_K(t)}{A_I(t)} K_t . \tag{30}$$

along an optimal investment path.

The ex-dividend market value of the firm can be written as

$$V_t = J_t + E_t^* \left\{ \sum_{j=t+1}^{+\infty} \sum_{l=-\infty}^{t-1} u_j G(j-l, l) I_l \right\} + (1 - w_t) I_t. \tag{31}$$

Therefore, from the definitions of $\hat{\beta}_t$ and $\hat{z}_t$, equation (30), and the capital accumulation rule we obtain that

$$V_t = J_t + \hat{\beta}_t - \hat{z}_t I_t + (1 - w_t) I_t = (1 - w_t - \hat{z}_t) \frac{K_{t+1}}{A_I(K_t, I_t)} + \hat{\beta}_t.$$

Q.E.D.

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