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Jump Diffusion Option Valuation in Discrete Time

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ABSTRACT

We develop a simple, discrete time model to value options when the underlying process follows a jump diffusion process. Multivariate jumps are superimposed on the binomial model of Cox, Ross, and Rubinstein (1979) to obtain a model with a limiting jump diffusion process. This model incorporates the early exercise feature of American options as well as arbitrary jump distributions. It yields an efficient computational procedure that can be implemented in practice. As an application of the model, we illustrate some characteristics of the early exercise boundary of American options with certain types of jump distributions.

JUMP DIFFUSION PROCESSES ARE popular in option valuation models. These processes were incorporated by Merton (1976) into the theory of option valuation and constitute an important alternative to the lognormal diffusion process assumed in the Black and Scholes (1973) model. In this paper, we develop a tractable discrete time model for valuing options with these processes.

Many authors have suggested that incorporating jumps in option valuation models may explain some of the large empirical biases exhibited by the Black-Scholes model.[1] However, in many cases, closed form solutions for valuing options are not available. For example, this is true when there is a positive probability of early exercise, when the jump distribution is neither lognormal nor discrete, or when the option is based on the term structure of interest rates (see Ahn and Thompson (1988) and Jarrow and Madan (1991)). In these important cases, the model developed in this paper can be used to value options.

Another aim of this paper is to provide a model which relies on only simple mathematics to price options with jump diffusion processes. It is hoped that the key economic relationships underlying jump diffusion models are transparent in this model. We provide enough details to make the procedure readily accessible.

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[1] For example, see Ball and Torous (1985), Jarrow and Rosenfeld (1984), and Jorion (1988).

To develop our model, we use the binomial model in Cox, Ross, and Rubinstein (1979), hereafter the CRR model, as the starting point. As in the CRR model, we assume that the stock price can either increase by an "uptick" or decrease by a "downtick" in each discrete period, where we define a tick to be the minimum possible change in the stock price. We term these stock price changes "local" price changes. In the continuous time case, a jump can instantaneously cause the stock price to undergo a large change of random size. Therefore, in the discrete time model, we also permit the stock price to change by multiple ticks in a single period. We term these price changes as discrete time jumps. In the discrete time context, all price changes (including the local changes) are jumps since the stock price can take on only discrete values. It should be noted that our definition of jumps includes only nonlocal stock price changes.

As in the CRR model, the risk of local price changes can be hedged in our model. However, the risk induced by the jumps cannot be hedged and correspondingly, the economic model is not complete.[2] Therefore, we rely on equilibrium arguments, where we assume that the jump risk is diversifiable and is not priced in the market. We emphasize that this assumption is needed only to simplify the exposition. In a later section, we demonstrate how systematic jumps can be accommodated.

Our discrete time stock price process converges weakly to the desired continuous time jump diffusion process under some technical regularity conditions. Based on the theory of weak convergence and results from Kushner and DiMasi (1978), we can therefore guarantee that option prices computed from our discrete time model converge to their corresponding continuous time model values. This implies that option values from the discrete time model, with a sufficiently small time step, can be used to approximate the continuous time values.

As a final contribution, we provide characterizations of the early exercise boundaries for American call and put options in the presence of jumps. We show that even small probabilities of jumps can significantly alter the optimal exercise decision relative to the early exercise decision based on the Black-Scholes model. This can lead to some surprising results. For example, the possibility of bankruptcy inducing jumps can significantly postpone the early exercise decision for call options relative to a model in which these jumps are not present.

The rest of the paper is divided into six sections. In the first, we develop the theoretical model. In Section II, we specialize this model by choosing an appropriate state space and time intervals so that the discrete time process converges weakly to the continuous time jump diffusion process of interest. The model is extended to incorporate systematic jump risk in Section III and some computational results with two different types of jump distributions are provided in Section IV. Section V characterizes the properties of the early exercise boundaries for call and put options with two different jump distribu-

[2] See Harrison and Pliska (1981) for a definition of completeness.

tions. Finally, Section VI concludes the paper and provides some directions for future research.

I. The Theoretical Model

We first develop the basic theoretical setup to value options with jump diffusion processes in discrete time. Within this setup, we bring out the underlying economic intuition and develop the valuation relationships necessary to determine option prices. We use the CRR binomial model as the starting point and superimpose jumps on the binomial stock price movements inherent in this model.

In our discrete time securities market, trades occur only on discrete dates indexed by $0, 1, 2, 3, \ldots, T$. We assume that two assets are traded in this economy. The first is a bond which has a riskless rate of return (possibly stochastic over time) of $\hat{r}$ in every period. Therefore, B dollars invested at time i yield $\hat{r}B$ dollars at time $i + 1$. The second asset is a risky stock. We assume that the stock price at date i can take on values only in a discrete set specified exogenously by $S_j(i): j \in \{-\infty, \ldots, -2, -1, 0, 1, 2, \ldots, \infty\}$. The variables i and j index time and state respectively. For illustration, the state space for a two-period model is represented in Figure 1. On this state space, let the transition probabilities of our stock price process be represented by a probability measure Q.

We also assume that this stock pays a dividend in each period. For ease of notation, we represent this dividend as a dividend yield and assume that this dividend yield is determined at least one period in advance. Therefore, if $\hat{D}$ is the dividend yield, the dollar dividend at date $i + 1$ is equal to $\hat{D}S_j(i)$, if the stock was in state j at date i. This dividend yield may depend on both the state and time. However, for notational simplicity, we will suppress this dependence. We can alternately assume that the dividend yield paid at date $i + 1$ is determined only at date $i + 1$. Although this does not change the economic arguments to follow, it makes the resulting algebra more tedious. For most types of dividends, this timing becomes irrelevant in the continuous time limit; therefore, we choose the assumption which yields a simpler analysis.

We now describe the stock price dynamics. In each period, the stock price undergoes either of two different types of mutually exclusive price changes. In most periods, the stock price undergoes only "local" changes. Analogous to the CRR model, the stock price moves up or down one "tick," i.e., the stock price in state j at date i moves to either state $j + 1$ or state $j - 1$ at date $i + 1$. This price change is the discrete counterpart of the stock price changes due to the diffusion component in the continuous time case.

The stock price can also change due to the occurrence of a "rare event." As the name suggests, this rare event has a low probability of occurring in any given period. It corresponds to the arrival of lumpy information which causes a large change in the stock price. When the rare event occurs, the stock price "jumps" to potentially any state (this does not include adjacent states) on the

Date 0	Date 1	Date 2
	$\cdots$	$\cdots$
	$S_5(1)$	$S_5(2)$
	$S_4(1)$	$S_4(2)$
	$S_3(1)$	$S_3(2)$
	$S_2(1)$	$S_2(2)$
	$S_1(1)$	$S_1(2)$
$S(0)$	$S_0(1)$	$S_0(2)$
	$S_{-1}(1)$	$S_{-1}(2)$
	$S_{-2}(1)$	$S_{-2}(2)$
	$S_{-3}(1)$	$S_{-3}(2)$
	$S_{-4}(1)$	$S_{-4}(2)$
	$S_{-5}(1)$	$S_{-5}(2)$
	$\cdots$	$\cdots$

Figure 1. State space for risky stock price dynamics over two periods. This figure defines the state space for the stock price distribution over the first two trading dates. At each date "i," the stock price can take on values in an exogenously specified discrete set indexed by "j." The price $S_j(i)$ denotes the stock price in state "j" at date "i."

state space grid at the next date. This multiple tick jump is the discrete time counterpart of the continuous time jump component. Hereafter, we refer to these multiple tick jumps simply as jumps, with jumps and rare events used interchangeably.

The above definitions imply that if an investor observes a stock price change of either one tick or minus one tick, then a local price change has occurred. If the price change corresponds to any other number of ticks, then a rare event has occurred. We assume that the two price changes are mutually exclusive so that ex post, an investor can distinguish between them from observing only the stock price.[3]

Now suppose that we are given an option with a maturity date T. Let the price of this option at time i and state j be denoted by $C_j(i)$. Further, let its payoff if exercised in this state be $F_j(i)$. We wish to determine $C_j(i)$ as a

[3] The two types of price changes have been assumed to be mutually exclusive only for expositional convenience. We can alternatively permit both types of changes to occur simultaneously. In the continuous time limit, it is irrelevant whether they are mutually exclusive or not.

function of the option values in the states at date $i + 1$. Using backward recursion, we can then compute the option price in each state at every date.

Fix a time point i and state j. Without loss of generality, let $j = 0$. Further, to reduce cumbersome notation, we suppress the subscript and/or the dependence on time when these are unambiguous. As in the CRR model, consider a hedge portfolio with one option, N shares of stock and B dollars in riskless bonds. N is called the hedge ratio since it equals the number of shares required to hedge one option. If we assume that the initial investment in this hedge portfolio is zero, then its value, $V(i)$, is given by

$$V(i) = NS(i) + B + C(i) = 0. \tag{1}$$

Suppose the investor wishes to eliminate the risk of this portfolio due to the local changes in the stock price. By a riskless portfolio, we mean a portfolio whose value in the next period is known in the current period. Therefore, for the portfolio to be riskless with respect to local changes in price, its value must be equal (not necessarily zero) in both the adjoining states ± 1 at the next date. This implies

$$\begin{aligned} V_{\pm 1}(i+1) &= N\left[S_{+1}(i+1) + \hat{D}S(i)\right] + \hat{r}B + C_{+1}(i+1) \\ &= N\left[S_{-1}(i+1) + \hat{D}S(i)\right] + \hat{r}B + C_{-1}(i+1). \end{aligned} \tag{2}$$

Solving for the hedge ratio in the above equation yields

$$N = -\frac{C_{+1}(i+1) - C_{-1}(i+1)}{S_{+1}(i+1) - S_{-1}(i+1)}. \tag{3}$$

The hedge ratio determined above has the same functional form as in the CRR model. Conditional on only local price changes, our stock price dynamics are identical to those in the CRR model. Therefore, if an investor attempts to hedge against only the local price changes, the hedge ratio seems to be the same. However, there is an important difference. The option values in our model are not in general the same as those in CRR. Therefore, even though the functional form of the hedge ratio is identical, its numerical value will be different across the two models. Further, in the CRR model since only local price changes are possible, the hedge portfolio is always riskless. But in our model, the hedge portfolio is not riskless with respect to stock price transitions to nonadjoining (nonlocal) states.

Now define the single period capital gains return at date $i + 1$ by

$$\Delta_k = \frac{S_{+k}(i+1)}{S(i)} \qquad \text{for } k = \ldots, -1, 0, 1, \ldots.$$

Δ_k therefore corresponds to k upticks if k is positive and $-k$ downticks if it is negative. Note that this return in general depends on the state at date i which we have suppressed. With this definition, we can write down a simple expression for the portfolio payoff at date $i + 1$ in the absence of a rare event by eliminating N and B from equations (1) and (2):

$$V_{\pm 1}(i + 1) = \left[\frac{C_{-1}(i + 1)\Delta_{+1} - C_{+1}(i + 1)\Delta_{-1}}{\Delta_{+1} - \Delta_{-1}}\right] + (\hat{r} - \hat{D})\left[\frac{C_{+1}(i + 1) - C_{-1}(i + 1)}{\Delta_{+1} - \Delta_{-1}}\right] - \hat{r}C(i).$$

Collecting terms containing C_{+1} and C_{-1},

$$V_{\pm 1}(i + 1) = \left[\frac{(\hat{r} - \hat{D}) - \Delta_{-1}}{\Delta_{+1} - \Delta_{-1}}\right]C_{+1} + \left[\frac{\Delta_{+1} - (\hat{r} - \hat{D})}{\Delta_{+1} - \Delta_{-1}}\right]C_{-1} - \hat{r}C(i). \quad (4)$$

By defining the coefficient of C_{+1} to be p, i.e.,

$$p = \frac{(\hat{r} - \hat{D}) - \Delta_{-1}}{\Delta_{+1} - \Delta_{-1}}, \quad (5)$$

the above expression reduces to

$$V_{\pm 1}(i + 1) = pC_{+1}(i + 1) + (1 - p)C_{-1}(i + 1) - \hat{r}C(i). \quad (6)$$

Therefore, if we assume that the state space is defined so that $p \in (0, 1)$, the portfolio value in the absence of a jump (rare event) can be written as an excess expected payoff on the option where the expectation is taken with respect to a probability p of an uptick and a probability $(1 - p)$ of a downtick. Notice that p is exactly the same as the "risk-neutral" probability of an uptick in the CRR binomial model. This is not surprising, since our analysis is so far identical to that in Cox, Ross, and Rubinstein (1979).

However, there are some fundamental differences between our model and the CRR model. In the CRR model, since the initial investment in the hedge portfolio is zero, and this portfolio value is completely riskless at date $i + 1$, absence of arbitrage requires that this value be zero. But in our case we cannot guarantee that the portfolio is riskless when a rare event occurs. Therefore, absence of arbitrage is not sufficient to guarantee that expression (6) equals zero. To hedge all the possible states, we need as many independent securities as the number of possible states at the next date. Since additional securities are not available to hedge the states corresponding to a rare event, an investor cannot hedge against the occurrence of a rare event and will therefore incur an unexpected capital gain or loss when it occurs. Further, we can only ensure equality of the portfolio values in both the adjoining states; we do not yet know their actual values in these states.

Now consider the portfolio value when a rare event occurs. Let Y be the capital gains return on the stock when the rare event occurs and y be the state induced by this rare event at the next date. This implies that if a rare event occurs at date i, the stock price at date $i + 1$ will be equal to $S(i)Y$. Recall that our analysis corresponds to state zero at date i and the definition of a rare event implies that $y \neq \pm 1$. With the above definitions,

$$\text{Portfolio Value} \mid \text{rare event} = V_y(i+1) = NS(i)[Y + \hat{D}] + C_y(i+1) + B\hat{r}. \tag{7}$$

We can eliminate N and B from this equation by substituting (1) and (3) to obtain

$$\begin{aligned} V_y(i+1) = &-\left[\frac{C_{+1}(i+1) - C_{-1}(i+1)}{\Delta_{+1} - \Delta_{-1}}\right][Y - (\hat{r} - \hat{D})] \\ &+ C_y(i+1) - \hat{r}C(i). \end{aligned} \tag{8}$$

As discussed earlier, we cannot simply use the absence of arbitrage opportunities to determine a pricing relationship for option valuation. Therefore, we need to impose a utility-based assumption. Analogous to Merton (1976), we assume that the jump risk is diversifiable. We impose this assumption since a model with diversifiable jumps is perhaps the simplest type of model which illustrates the intuition underlying discrete time option valuation with jumps in asset prices. However, any alternative assumption, which ensures that the option value at every discrete date is the expectation of some function of the option value at the next date, is also acceptable. We will discuss this issue in more detail in Section III.

If we assume that the risk of a rare event is diversifiable across stocks, then in an efficient capital market this risk should not be priced in equilibrium. Our hedge portfolio then contains only diversifiable risk. Therefore, the expectation of the portfolio value in the next period with respect to the distribution of the rare event must be zero. This yields the valuation rule for determining option prices. It is important to keep in mind that the expectation is not taken with respect to the overall distribution of the stock price at the next date, i.e., Q, but is taken with respect to the distribution of the rare event only. Since the investor can hedge against the occurrence of either the uptick or downtick, the relative probabilities of each of these states is irrelevant. In this respect, our model is similar to the CRR model.

Let the probability of a rare event (under Q) in the given time period and state of interest equal $\hat{\lambda}$, and E_Y be the expectation operator with respect to the distribution of Y. Then, taking the expectation of the portfolio value at date $i + 1$ with respect to the occurrence of a jump (rare event) and equating it to zero yields

$$0 = \hat{\lambda} E_Y\left[V_y(i+1)\right] + [1 - \hat{\lambda}]V_{\pm 1}(i+1). \tag{9}$$

Substituting the portfolio values from (6) and (8) into this expression yields

$$\hat{r}C(i) = \hat{\lambda}\left[E_Y\left[C_y(i+1)\right] - \left[\frac{C_{+1}(i+1) - C_{-1}(i+1)}{\Delta_{+1} - \Delta_{-1}}\right]\left[E_Y[Y] - (\hat{r} - \hat{D})\right]\right]$$
$$+(1-\hat{\lambda})[pC_{+1}(i+1) + (1-p)C_{-1}(i+1)]. \quad (10)$$

The above equation represents the option value at date i in terms of its values at date $i + 1$. Therefore, it can be used as the dynamic programming equation for option valuation. Starting from the maturity date of the option, we can use this equation recursively to compute option values at any prior date.

Even though (10) is sufficient to compute option values, it is cumbersome. In the subsequent discussion, we simplify this equation so that the option value can be written as its discounted (by $\hat{r}$) expected value in the next period by defining the expectation with respect to a new measure called the risk-neutral measure (this measure is also called the equivalent martingale measure). Under this measure, the expected rate of return for every security equals the riskless rate.[4] Therefore, we can act as if all investors are risk neutral and value every security by just computing its expected discounted (at the riskless rate) cash flows. This simplifies the valuation task considerably. Further, the existence of such a measure is both necessary and sufficient for our economic model to be arbitrage free (for details, see Harrison and Kreps (1979)). Since prices are meaningless in any model in which there exist arbitrage opportunities, this is a consistency condition which we must ensure is satisfied.

Since we want to write the option price at date i in terms of an expectation with respect to the option price in the possible states at date $i + 1$, we collect terms corresponding to the same state at date $i + 1$ in (10). Further, substi-

[4] The processes defining the option price and the stock price, plus the value of all dividends reinvested into the stock, and discounted by the riskless rate of return are martingales under this measure. Further, this new measure, denoted $\tilde{Q}$, is equivalent to the probability measure Q which governs the evolution of the stock price, i.e., if the probability of a state or event is zero under $\tilde{Q}$, its probability is also zero under Q and vice versa.

If such a measure exists, there can be no arbitrage in the model. An intuitive explanation of this result is as follows. If the prices of all securities (with all dividends reinvested into the security) discounted by the risk-free rate of return are martingales under $\tilde{Q}$ then the expected return under this measure for all securities equals the risk-free rate. This implies that the expected rate of return on every portfolio also equals the riskless rate. Therefore, any portfolio with a zero investment will on average be worthless in the future.

Since an arbitrage profit requires an investor to make a riskless profit without any investment, this implies that no investor can make arbitrage profits in a world governed by $\tilde{Q}$. However, Q and $\tilde{Q}$ agree on zero probability events and therefore on sure events. Since an arbitrage profit is a sure event, this implies that there can be no arbitrage under the true measure, Q, as well.

tuting the value of p from (5),

$$\hat{r}C(i) = \hat{\lambda}E_Y\left[C_y(i+1)\right] + (1-\hat{\lambda})\left[C_{+1}(i+1)\left[\frac{\dfrac{(\hat{r}-\hat{D}) - \hat{\lambda}E_Y[Y]}{1-\hat{\lambda}} - \Delta_{-1}}{\Delta_{+1} - \Delta_{-1}}\right]\right]$$

$$+(1-\hat{\lambda})\left[C_{-1}(i+1)\left[\frac{\Delta_{+1} - \dfrac{(\hat{r}-\hat{D}) - \hat{\lambda}E_Y[Y]}{1-\hat{\lambda}}}{\Delta_{+1} - \Delta_{-1}}\right]\right] \tag{11}$$

We have expressed the above equation to clearly separate the value obtained from the jump states from that obtained from the nonjump states at the next date.

Defining $\tilde{q}$ by

$$\tilde{q} = \frac{\dfrac{(\hat{r}-\hat{D}) - \hat{\lambda}E_Y[Y]}{1-\hat{\lambda}} - \Delta_{-1}}{\Delta_{+1} - \Delta_{-1}}, \tag{12}$$

and noting that the coefficient of C_{-1} without the $(1-\hat{\lambda})$ term equals $(1-\tilde{q})$, (11) becomes

$$\hat{r}C(i) = \hat{\lambda}E_Y\left[C_y(i+1)\right] + (1-\hat{\lambda})[\tilde{q}C_{+1}(i+1) + (1-\tilde{q})C_{-1}(i+1)]. \tag{13}$$

The right-hand side of this equation clearly resembles an expectation over the option price at date $i+1$. The probability measure implicit in this expectation, which we denote by $\tilde{Q}$, is defined as follows. Let the probability of an uptick or downtick conditional on no rare events be given by $\tilde{q}$ and $(1-\tilde{q})$ respectively. Further, let the probability of a rare event be $\hat{\lambda}$ and the distribution of the stock return induced by the rare event be the same as that under the original probability measure Q. These probabilities completely define the probability measure[5] $\tilde{Q}$.

Applying $\tilde{Q}$ to the cum-dividend value of the stock at date $i+1$ yields

$$\begin{aligned} &E_{\tilde{Q}(i)}[S(i+1) + \hat{D}S(i)] \\ &\quad = E_{\tilde{Q}(i)}\left[S(i)\left[[\tilde{q}\Delta_+ + (1-\tilde{q})\Delta_-](1-\hat{\lambda}) + \hat{\lambda}E_Y(Y) + \hat{D}\right]\right], \end{aligned} \tag{14}$$

where $\tilde{Q}(i)$ denotes expectation with respect to $\tilde{Q}$ conditional on the information available at date i. On simplification, the right-hand side equals $\hat{r}S(i)$. Therefore, the expected return on the stock equals the riskless rate under $\tilde{Q}$.

[5] We implicitly assume that the state space is defined so that $0 < \tilde{q} < 1$. This guarantees that $\tilde{Q}$ is a probability measure. Further, Q and $\tilde{Q}$ are equivalent since given a state j at date i, the probability of every state in the state space at date $i+1$ is positive under both measures.

Notice that $\tilde{q}$ in (12) can be obtained from p defined for the CRR model in (5) by replacing $(\hat{r} - \hat{D})$ by a different quantity. The intuition for this difference is as follows. Conditional on the absence of a jump, the expected capital gains return on the stock is given by $[\tilde{q}\Delta_+ + (1 - \tilde{q})\Delta_-]$. By substituting the definition of $\tilde{q}$, and simplifying, this quantity equals

$$\frac{(\hat{r} - \hat{D}) - \hat{\lambda}[E_Y[Y]]}{1 - \hat{\lambda}}.$$

This quantity is equal to the numerator in the definition of $\tilde{q}$ without the Δ_{-1} term (if we had used p as in the CRR model, this would equal $(\hat{r} - \hat{D})$). However, the expected capital gains return under the overall measure $\tilde{Q}$ is equal to $(\hat{r} - \hat{D})$. Since the expected capital gains return conditional on the occurrence of a rare event is equal to $E_Y Y$, which in general does not equal $(\hat{r} - \hat{D})$, this return must be different from $(\hat{r} - \hat{D})$ over the local (uptick and downtick) states. Condition (14) ensures that, on average, the composite return equals the riskless rate, as is required under $\tilde{Q}$. Therefore, $\tilde{q}$ and p must be different.

Once we have defined $\tilde{Q}$, our valuation task is simplified considerably since we can value any stream of payoffs in our model by simply taking the expected value of these payoffs under $\tilde{Q}$ and discounting at the riskless rate. In particular, for the option presently of interest, its value at date i in terms of its value at date $i + 1$ can be written as

$$C(i) = E_{\tilde{Q}(i)}\left[\frac{C(i+1)}{\hat{r}}\right]. \tag{15}$$

This equation is valid for an option that will be held by the investor over the next period. However, if the investor finds that the payoff on the option is worth more in the current period than its value if held into the next period, the investor will choose to exercise the option. Therefore, the option value can be represented by

$$C(i) = \text{Max}\left[F(i), \frac{E_{\tilde{Q}(i)}[C(i+1)]}{\hat{r}}\right], \tag{16}$$

where $F(i)$ is the payoff if the option is exercised in the current state and time. At the terminal date, the option value is a known function of the state space. Therefore, (16) can now be recursively used to obtain option values at each date and in every state by working backwards from the terminal date.

This completes our discussion of the theoretical model.

II. Approximation of Jump Diffusion Processes

In this section, we specialize the discrete time model developed in the preceding section to guarantee that the sequence of stock price processes from the model converges weakly to the continuous time jump diffusion process suggested by Merton (1976). We first define the continuous time process and then choose the state space and transition probabilities for our discrete model to ensure (weak) convergence to the continuous time process.

As in the CRR model, the state space for the discrete model is chosen so that the diffusion component of the process can be easily approximated by constructing a binomial process on this state space. Using this same state space, we approximate the jump component of the jump diffusion process by defining a discrete time "jump process." This jump process is incremented in each period, with a probability equal to the intensity of the continuous time jump process times the width of each time interval. Further, the distribution of the increment to this process is constructed by discretizing the continuous time jump distribution on the state space for the binomial process. Then, the composite discrete time process is constructed by combining the binomial process and the jump process.

A. Continuous Time Model

We now define the continuous time model. To simplify the exposition, we make the following assumption.

ASSUMPTION 1: *The intensity of the jump process (rare event probability in the discrete case) and the distribution of the relative price change induced by the occurrence of a jump are independent of the current state. We also assume that the dividend yield is constant. Further, the variance of the diffusion term is a constant proportion of the contemporaneous stock price. This guarantees that in the absence of jumps, the stock price follows a lognormal process with a constant proportional variance.*

The above assumption does not impose any restriction on the distribution of the jump itself. Further, it is easy to include both a time-varying variance and a time-varying dividend yield by using variable step sizes as suggested by Amin (1991). However, these quantities must be deterministic functions of time. Additional relaxation of these assumptions leads to a significant increase in the complexity of the convergence proofs and the choice of the possible state space of the discrete model. Nevertheless, the underlying intuition is straightforward and the analysis can be extended to include more general processes.

Let the continuous trading interval be $[0, \tau]$ and the probability measure governing the evolution of the stock price be Q^c. The superscript c is used to indicate that the measure corresponds to a continuous time process. The stochastic integral equation representing the standard jump diffusion model of the stock price (see Merton (1976)) is the following:

$$\frac{S^c(t)}{S^c(0)} = \exp\left[\int_0^t \left[\mu - D - \frac{1}{2}\sigma^2 - \lambda K\right] du + \int_0^t \sigma\, dW(u) + \sum_{j=1}^{NJ(t)} \ln(Y(j))\right], \tag{17}$$

where $S^c(t)$ is the stock price at time t, μ is the expected rate of return on the stock, D is the continuous dividend yield, $W(t)$ is a standard Brownian motion, σ is the volatility of the continuous part of the stock price return, $NJ(t)$ is the total number of jumps (Poisson arrivals) up to time t, $Y(j)$'s (generically denoted as Y) are independent and identically distributed random variables corresponding to the Poisson jump magnitudes, λ is the intensity of the jump process, and $K = E_Y(Y-1)$. E_Y is the expectation operator with respect to the distribution function of Y under Q^c. Most of the jump diffusion models in finance can be obtained by choosing particular values of σ, λ, and specific functional forms for the jump distribution.

As in the previous section, we assume that the jump risk is diversifiable. By using the results in Merton (1976) and the risk neutrality argument of Cox and Ross (1976), we can value options by replacing the above equation with

$$\frac{S^c(t)}{S^c(0)} = \exp\left[\int_0^t\left[r - D - \frac{1}{2}\sigma^2 - \lambda K\right]du + \int_0^t \sigma\, d\tilde{W}(u) + \sum_{j=1}^{NJ} \ln(Y(j))\right], \tag{18}$$

where r is the constant spot rate of interest based on continuous compounding and $\tilde{W}(u)$ is a standard Brownian motion under a new probability measure, $\tilde{Q}^c$, which is equivalent[6] to Q^c. The representation of the stock price distribution under $\tilde{Q}^c$ is easier to work with than that under Q^c since the value of μ does not have to be specified under $\tilde{Q}^c$. After specifying a discrete model in which the stock price process converges weakly to the process under $\tilde{Q}^c$ (given by (18)), it is possible to redefine the probabilities to make the same discrete process converge to the process under Q^c (given by (17)).

For convenience, we now switch to a logarithmic scale, with the values of all the state variables specified in terms of their logarithm. Define the drift of the logarithm of the stock price in (18) by

$$\alpha = r - D - {}^1\!/_2\, \sigma^2 - \lambda K. \tag{19}$$

Further, define the process $X^c(t)$ as

$$X^c(t) = \ln\left[\frac{S^c(t)}{S^c(0)}\right] = \int_0^t \alpha(u)\, du + \int_0^t \sigma\, d\tilde{W}(u) + \sum_{j=1}^{NJ(t)} \ln(Y(j)). \tag{20}$$

Our objective in the remainder of this section is to construct a discrete time process which can be used as a suitable approximation to $X^c(t)$.

B. Discrete State Space of the Approximating Process

For every fixed positive integer n, partition the trading interval $[0, \tau]$ into n subintervals of length $h_n = \tau/n$. The subscript denotes dependence on the

[6] Recall from footnote 4 that two probability measures are equivalent if they agree on zero probability events.

particular value of n chosen. Consider the discrete time stochastic process, X_n, defined on the state space $\mathscr{S}_n$ depicted in Figure 2. The probability measure, Q_n, on this state space is currently unspecified. $\mathscr{S}_n$, restricted to a fixed date ih_n, is a grid such that consecutive points on this grid are spaced $\sigma\sqrt{h_n}$ apart. This grid at date ih_n is shifted upwards by αh_n relative to the grid at the date $(i-1)h_n$. At each date, this state space represents the capital gains return on the stock price relative to date "0." Therefore, the stock price in state j and date ih_n is given by $S_j(ih_n) = S(0)\exp[\alpha i h_n + j\sigma\sqrt{h_n}]$.

The state space defined above is slightly different from that in Cox, Ross, and Rubinstein (1979) and corresponds to one first suggested by Jarrow and Rudd (1983). This state space is perhaps more intuitive; it permits a direct separation of the mean and variance components as in the continuous time

Time = 0	Time = h_n	$\cdots$	Time = ih_n
	$\cdots$	$\cdots$	$\cdots$
	$\ln \frac{S_5(h_n)}{S(0)} = \alpha h_n + 5\sigma\sqrt{h_n}$	$\cdots$	$\alpha i h_n + 5\sigma\sqrt{h_n}$
	$\ln \frac{S_4(h_n)}{S(0)} = \alpha h_n + 4\sigma\sqrt{h_n}$	$\cdots$	$\alpha i h_n + 4\sigma\sqrt{h_n}$
	$\ln \frac{S_3(h_n)}{S(0)} = \alpha h_n + 3\sigma\sqrt{h_n}$	$\cdots$	$\alpha i h_n + 3\sigma\sqrt{h_n}$
	$\ln \frac{S_2(h_n)}{S(0)} = \alpha h_n + 2\sigma\sqrt{h_n}$	$\cdots$	$\alpha i h_n + 2\sigma\sqrt{h_n}$
	$\ln \frac{S_1(h_n)}{S(0)} = \alpha h_n + 1\sigma\sqrt{h_n}$	$\cdots$	$\alpha i h_n + 1\sigma\sqrt{h_n}$
0	$\ln \frac{S_0(h_n)}{S(0)} = \alpha h_n$	$\cdots$	$\alpha i h_n$
	$\ln \frac{S_{-1}(h_n)}{S(0)} = \alpha h_n - 1\sigma\sqrt{h_n}$	$\cdots$	$\alpha i h_n - 1\sigma\sqrt{h_n}$
	$\ln \frac{S_{-2}(h_n)}{S(0)} = \alpha h_n - 2\sigma\sqrt{h_n}$	$\cdots$	$\alpha i h_n - 2\sigma\sqrt{h_n}$
	$\ln \frac{S_{-3}(h_n)}{S(0)} = \alpha h_n - 3\sigma\sqrt{h_n}$	$\cdots$	$\alpha i h_n - 3\sigma\sqrt{h_n}$
	$\ln \frac{S_{-4}(h_n)}{S(0)} = \alpha h_n - 4\sigma\sqrt{h_n}$	$\cdots$	$\alpha i h_n - 4\sigma\sqrt{h_n}$
	$\ln \frac{S_{-5}(h_n)}{S(0)} = \alpha h_n - 5\sigma\sqrt{h_n}$	$\cdots$	$\alpha i h_n - 5\sigma\sqrt{h_n}$
	$\cdots$	$\cdots$	$\cdots$

Figure 2. State space ($\mathscr{S}_n$) of the discrete approximation of the stock price distribution for a fixed discretisation parameter n. This figure illustrates the state space for the discrete model at date h_n and at some arbitrary date ih_n. The stock price at each date is determined as its time zero value times the exponential of the value of the state variable obtained from the grid in the figure.

diffusion limit. However, with minor modifications, we can also work with the CRR model state space.

To develop the correspondence with the notation in Section I, note that for a fixed n, $\ln \Delta_j(i) = \alpha i h_n + j\sigma\sqrt{h_n}$. Further, since we assume that the dividend yield is constant, and equal to D, $\hat{D} = \exp(Dh_n)$.

C. Transition Probabilities

We now define the transition probabilities of X_n so that it serves as an approximation to X^c under the true measure Q. For every fixed n, let these transition probabilities be represented by the probability measure Q_n. The theory developed in the preceding section can be used to construct the corresponding risk-neutral measure $\tilde{Q}_n$.

First, we specify the probabilities of the local states. Let $X_{1,n}(0) = 0$ and $q_n \in (0, 1)$. Conditional on a local price change, let the transition probabilities of X_n be given by

$$\mathrm{Prob}\left[X_n(t + h_n) - X_n(t) = \alpha h_n + \sigma\sqrt{h_n}\right] = q_n,$$

$$\mathrm{Prob}\left[X_n(t + h_n) - X_n(t) = \alpha h_n - \sigma\sqrt{h_n}\right] = 1 - q_n$$

$$\text{for all } t \in [0, h_n, 2h_n, \ldots, \tau - h_n$$

For the moment, let q_n be unspecified.

Now consider the approximation of the jump component. For the Poisson jump component of the continuous time process, the probability of a single jump occurring in an interval of length h_n is equal[7] to $\lambda h_n + o(h_n)$. The probability of multiple jumps in the same interval equals $o(h_n)$. Therefore, we assume that the probability of a jump (rare event) in the discrete model at any time $t \in [0, h_n, \ldots, \tau - h_n]$ is equal to λh_n. We also assume that multiple jumps at any discrete date cannot occur.

At each discrete date, we approximate the jump distribution on the entire real line by breaking it down over nonoverlapping intervals of equal width, such that each interval is centered around one point from the discrete state space $\mathcal{S}_n$ on that date. We assign the entire probability mass over each of these intervals to the point in $\mathcal{S}_n$ contained in each of these intervals. There is only one exception to the above rule; in order to differentiate between jumps and single tick movements, the interval containing the state "0" is chosen to be larger. We will elaborate on this issue later.

Let the cumulative density function of Y (under Q^c) be given by $\mathcal{N}(x)$ for $x \in \Re$. Further, let $Y_n(t)$ for $t \in [0, h_n, \ldots, \tau - h_n]$ represent their discrete time jumps and $\mathcal{N}_n$ represent their probability distribution under Q_n. Noting that $\mathcal{N}_n$ is a discrete distribution, we specify its probability mass function as follows. For every integer "l" not in the set $\{-1, 0, 1\}$, let $d\mathcal{N}_n(l)$ be given by

$$d\mathcal{N}_n(l) = \mathcal{N}\left(\alpha h_n + \left(l + \tfrac{1}{2}\right)\sigma\sqrt{h_n}\right) - \mathcal{N}\left(\alpha h_n + \left(l - \tfrac{1}{2}\right)\sigma\sqrt{h_n}\right). \tag{21}$$

[7] A function $f(h)$ is termed as $o(h)$ if $\lim_{h \to 0} f(h)/h = 0$.

The probabilities represented by $d\mathcal{N}_n$ correspond to a discretization of the continuous time jump distribution over the states in $\mathcal{S}_n$. However, the probability mass of $\mathcal{N}$ over the region $[\alpha h_n - (1 + 1/2)\sigma\sqrt{h_n}, \alpha h_n + (1 + 1/2)\sigma\sqrt{h_n}]$ remains to be assigned to the points $l \in \{-1, 0, 1\}$. We assign this entire probability mass to the point corresponding to $l = 0$ and assume that $d\mathcal{N}_n(l)$ is zero for $l = \pm 1$. Therefore,

$$d\mathcal{N}_n(0) = \mathcal{N}\left(\alpha h_n + \left(1 + 1/2\right)\sigma\sqrt{h_n}\right) - \mathcal{N}\left(\alpha h_n - \left(1 + 1/2\right)\sigma\sqrt{h_n}\right),$$
$$d\mathcal{N}_n(\pm 1) = 0. \tag{22}$$

To ensure consistency with Section I, we define $\mathcal{N}_n$ in this manner to exclude the jump distribution from having positive mass on any state corresponding to a local price change. However, in the limit as $n \to \infty$, it does not matter whether we define the jump distribution in the above way or over every point in the state space in $\mathcal{S}_n$.

The probability distribution of $\ln Y_n$ under Q_n is now completely specified by

$$\mathrm{Prob}\left[\ln Y_n(t) = \alpha h_n + l\sigma\sqrt{h_n}\right] = d\mathcal{N}_n(l) \qquad \text{for every integer } l. \tag{23}$$

Noting that the probability of a rare event in any period equals λh_n, the transition probabilities of $X_n(t)$ can now be obtained by combining the increments due to local price changes and jumps:

$$\mathrm{Prob}\left[X_n(t + h_n) - X_n(t) = \alpha h_n + \sigma\sqrt{h_n}\right] = q_n(1 - \lambda h_n),$$
$$\mathrm{Prob}\left[X_n(t + h_n) - X_n(t) = \alpha h_n - \sigma\sqrt{h_n}\right] = (1 - q_n)(1 - \lambda h_n),$$
$$\mathrm{Prob}\left[X_n(t + h_n) - X_n(t) = \alpha h_n + l\sigma\sqrt{h_n}\,; l \neq \pm 1\right] = \lambda h_n d\mathcal{N}_n(l). \tag{24}$$

To complete our discrete formulation, we now need to specify the value of q_n. The theory developed in the previous section implies that the actual value of q_n is irrelevant for option valuation and can be replaced by $\tilde{q}_n$ such that

$$\tilde{q}_n = \frac{\dfrac{\exp(rh_n) - \exp(Dh_n) - \lambda h_n E_{Y_n}(Y_n)}{1 - \lambda h_n} - \exp\left(\alpha h_n - \sigma\sqrt{h_n}\right)}{\exp\left(\alpha h_n + \sigma\sqrt{h_n}\right) - \exp\left(\alpha h_n - \sigma\sqrt{h_n}\right)}. \tag{25}$$

From a computational perspective, the above value can be simplified considerably. By construction,[8]

$$E_{Y_n}[Y_n(t)] = K + 1 + O(h_n). \tag{26}$$

Using (25) and (26) and some algebra yields

$$\tilde{q}_n = 1/2 + o(h_n). \tag{27}$$

[8] A function $f(x)$ is defined to be $O(x)$ if $f(x)/x$ remains bounded as $x \to 0$.

From a practical perspective, we can assume that $\tilde{q}_n$ equals $1/2$ if n is sufficiently large. With this definition, the discrete formulation is now complete.

The discrete time process $S(0)\exp(X_n)$, interpolated appropriately to define a continuous time process, converges[9] weakly to the continuous time stock price process given by (18) which defines the stock price distribution under $\tilde{Q}^c$. By defining $q_n = \tilde{q}_n + ((\mu - r)/\sigma)\sqrt{h_n}$, we can also ensure convergence to the stock price process under Q^c. The definition of weak convergence guarantees that the prices of European options computed from our discrete model will converge to their corresponding continuous time values under fairly mild regularity conditions (for example, the option payoff must be uniformly integrable in h_n). However, the definition of weak convergence does not guarantee that the prices of American options will converge.

Fortunately, a similar problem in the context of general optimal control problems has been studied by Kushner and DiMasi (1978). These authors show that if a sequence of processes converges weakly to a limiting jump diffusion process, then the value functions for optimal stopping problems also converge.[10] (For details, see Kushner and DiMasi (1978), pages 797–799.) Therefore, American option prices from our discrete time model converge to their continuous time values. An exposition of the main ideas in the proofs with diffusion processes is provided by Amin and Khanna (1993) in the finance context.

Given our discrete approximation, we can compute option prices by dynamic programming using a backward recursion on the state space that we have developed. This state space can be truncated at very high and very low values of the state variables to ensure that it is finite.

At this point, we should note that there are potentially an infinite number of ways to construct the discrete time process.[11] As long as we ensure weak convergence to the appropriate jump diffusion process, every approximating model will yield the same option values and hedge ratios. The particular approximation provided in this section (and also the model in the preceding section) is the simplest model that we could construct, which serves the dual purpose of: (a) explaining the basic economic properties of jump diffusion option valuation in discrete time and (b) providing a practical computational tool.

From a computational perspective, all we need is the definition of the continuous time stock price process under the risk-neutral measure (Merton's (1976) model provides this in our context). Once a discrete approximation to this process under the risk-neutral measure is available, option values can be

[9] The proof of these convergence results is available from the author.

[10] Kushner and DiMasi (1978) assume that the diffusion coefficients and the option payoff are bounded. However, their proof is valid in our context *under the log transformation* that we have used as long as the option payoff is uniformly integrable in h_n.

[11] This is true for models which approximate simple diffusion processes as well. For example, Amin (1991), Boyle, Evnine, and Gibbs (1989), Jarrow and Rudd (1983), and Page and Sanders (1986) provide several different discrete time counterparts to the Black-Scholes model.

computed by simply evaluating the expected discounted payoff of the option. The economic arguments in the previous section only serve to provide a simple illustration of the underlying intuition. They are unnecessary from a computational perspective. For an elaboration of these arguments in the context of diffusion processes, see Rajasinghim (1990).

III. Models with Systematic Jumps

Jarrow and Rosenfeld (1984), as well as Jorion (1988), find that jumps observed in stock prices are systematic across the market portfolio. Further, events such as the crash of 1987 or the mini-crash in 1989 are clearly systematic. Hence, models incorporating systematic jump risk are of significant interest. Some authors have attempted to model this feature, for example, see Ahn (1991), Amin and Ng (1993), and Naik and Lee (1990). This section discusses applications of our model to value options under this more general scenario.

When the jump risk is systematic, general equilibrium models are the only recourse. However, in every model in which a valuation measure exists, the absence of arbitrage opportunities implies that we can construct a nonnegative process $M(t)$, with $M(0) = 1$, such that the price at time t of any cash flow F payable at time T can be represented as:[12]

$$C(t) = \frac{1}{M(t)} E_{Q(t)}[M(T)F(T)]. \tag{28}$$

For American options, (28) should be replaced by the valuation equation

$$C(t) = \frac{1}{M(t)} \sup_{\theta \in \mathscr{T}[t,T]} E_{Q(t)}[M(\theta)F(\theta)], \tag{29}$$

where $F(\theta)$ is the payoff to the option if exercised at date θ and $\mathscr{T}[t,T]$ is the class of all stopping times (early exercise strategies) in $[t,T]$.

The process $M(t)$ is called a "martingale-pricing process" since $M(t)$ times the price of every security in the economy is a martingale. For example, in a representative agent economy with a single consumption good, and both a time additive (or multiplicative) and state-independent utility function, the process $M(t)$ is the marginal rate of substitution between time t consumption and time 0 consumption (see Huang and Litzenberger (1988), chapter 7).

The martingale-pricing process essentially defines the risk-neutral probability measure that we discussed in Section I. $M(t)$ equals the conditional Radon-Nikodym derivative of the risk-neutral measure, $\tilde{Q}$, with respect to the true measure, Q, discounted by the riskless rate of interest. In the context of Section I, $M(t)$ is defined by the equation

$$\frac{M_j(i+1)}{M_k(i)} = \frac{1}{\hat{r}} \frac{\tilde{q}_{k,j}(i)}{q_{k,j}(i)}, \tag{30}$$

[12] For details, see Harrison and Kreps (1979).

where $M_j(i)$ is the value of the martingale-pricing process in state j and date i, $\tilde{q}_{k,j}(i)$ is the probability that the stock price moves from state k at date i to state j at date $i+1$ under the risk-neutral measure, and $q_{k,j}(i)$ is the corresponding probability under the true measure.

In a model in which every option can be hedged perfectly by trading in the underlying securities, $M(t)$ can be constructed without recourse to equilibrium arguments. This is true, for example, in the CRR model. However, when the option price cannot be hedged perfectly, equilibrium arguments are necessary to construct $M(t)$. In Section I, we assumed that the jump risk was diversifiable to construct the risk-neutral measure. Using (30) above, this was equivalent to constructing the martingale-pricing process. Alternative utility-based assumptions will yield different representations for $M(t)$ and therefore for the risk-neutral measure.

Now consider option valuation in this context. In a discrete time model, the valuation equation given by (29) can be represented in a recursive fashion by

$$C(t) = \mathrm{Max}\left[F(t), \frac{1}{M(t)} E_{Q(t)}[M(t+h_n)C(t+h_n)]\right]. \tag{31}$$

With this representation, we can value American options using our discrete framework developed in Sections I and II, as long as both the payoff function F and the martingale-pricing process M can be represented as functions of a single underlying state variable which follows a jump diffusion process. These conditions will hold if, for example, the model contains a single source of uncertainty generated by a single jump diffusion process. In this framework, we can approximate this state variable by a discrete time process, as in Section II, and compute option prices using (31) above.

As an example, consider the model suggested by Naik and Lee (1990) for valuing options on the market portfolio. In terms of the jump diffusion process analyzed in Section II, their process for the market portfolio is represented by (17) with a normal distribution for $\ln(Y)$. Further, the option valuation relationship in this model is given by

$$C(t) = \sup_{\theta \in \mathscr{T}[t,T]} E_{Q(t)}\left[\exp[-\phi(\theta - t)]\left[\frac{S(\theta)}{S(t)}\right]^{\gamma-1} F(\theta, S(\theta))\right], \tag{32}$$

where $C(t)$ is the option price at time t, $S(t)$ is the price of the market portfolio at time t, γ is a risk-aversion parameter, and ϕ is a constant discounting factor. From (29) and (32), the martingale pricing process, $M(t)$, can be identified as

$$M(t) = [S(t)]^{\gamma-1}\exp(-\phi t).$$

Further, we can discretize the valuation relationship in (32) above by the following equation:

$$C(t) = \sup\left[F(t), E_{Q(t)}\left[\exp[-\phi h_n]\left[\frac{S(t+h_n)}{S(t)}\right]^{\gamma-1} C(t+h_n)\right]\right]. \tag{33}$$

After constructing a discrete time process to approximate the jump diffusion process, as in Section II, we can solve this equation recursively.

Even when the model has multivariate jump diffusion processes driving the state variables, it can sometimes be transformed into a model with a single jump diffusion process. Consider the following situation. First, we simplify the valuation equation represented by (29). In the continuous time case, $\tilde{Q}$ is defined in terms of its conditional Radon-Nikodym derivative with respect to Q as

$$E_{Q(t)}\left[\frac{d\tilde{Q}}{dQ}\right] = \frac{M(t)}{M(0)} \exp\left[\int_0^t r(u)\, du\right]. \tag{34}$$

Applying a change of measure to (29) by using (34) yields

$$C(t) = \sup_{\theta \in \mathscr{T}[t,T]} E_{\tilde{Q}(t)}\left[F(\theta)\exp\left[-\int_t^\theta r(u)\, du\right]\right]. \tag{35}$$

Given $\tilde{Q}$, this equation is defined only in terms of the option payoff at the exercise date. This implies that if we have a single state variable (for example, the price of a risky asset) which follows a jump diffusion process under $\tilde{Q}$, and if $F(\cdot)$ is a function of only this state variable and time, then once again we can apply our discrete model to determine option prices. This transformation can be accomplished in the bivariate jump diffusion models of Ahn (1991) and Amin and Ng (1993).[13] In fact, this is also the reason why these authors are able to derive closed form solutions for European options. Using a change of measure, they transform their respective problems into problems with a single jump diffusion process. Option values are obtained by integrating the terminal option payoff with respect to the distribution of this process.

IV. Numerical Results

In this section, we illustrate our model by providing numerical results with two different types of jump distributions. First, consider Merton's (1976) jump diffusion model in which the proportional jump size has a lognormal distribution. In terms of the model specified by equation (18) in Section II, this corresponds to a normal distribution for $\ln[Y]$. Let the variance of $\ln[Y]$ be δ^2 and its mean be $-\frac{1}{2}\delta^2$ (hence $E_Y[Y] = 1$). For this model, we now illustrate our discrete time counterpart.

In the discrete time model, the option price in any state is obtained by integrating the option price at the next date over the stock price distribution under the risk-neutral measure and discounting at the riskless rate. The

[13] A detailed proof is available from the author on request.

computation of this integral over the two local states requires little effort. But, the corresponding computations over the jump distribution can be substantial if we approximate the jump distribution using every point on the discrete grid at the next date. Given M possible states at each date, these computations at each discrete date are of order M^2, as opposed to order M for the CRR binomial model. However, in practice this computational burden can be reduced to order M by judiciously selecting a small number of points to approximate the jump distribution. For example, the Newton-Cotes formulae (see Abramovitz and Stegum (1965)) available in the numerical analysis literature can be used to accomplish this task.

Our model is implemented as follows. Let N be the number of time steps and M be the number of nodes in the state space at each time step (number of points on the vertical axis in Figure 2). Choose $M = 2N + 1$. This yields N points on each side of the 0 state at each date. For each node, truncate the jump distribution at the next date, on either side, at the closer of the following two points: (a) the point at which the entire state space is truncated at that date (state M or $-M$) or (b) the point N nodes away. Using our notation in Section I, the jump distribution for state j is truncated by the state $\mathrm{Min}(N, j + N)$ at the upper end and the state $\mathrm{Max}(-N, j - N)$ at the lower end.

We can further truncate the jump distribution outside the region $\ln Y_n \in [-3\delta, 3\delta]$, since the probability mass outside this region is negligible. We enlarge this region slightly so that its end points correspond to points on the state space. Therefore, we truncate the jump distribution outside the region $[\alpha h_n + a\sigma\sqrt{h_n}, \alpha h_n - a\sigma\sqrt{h_n}]$, where "$a$" is the smallest nonnegative integer such that this interval contains $[-3\delta, 3\delta]$. The entire probability mass outside the truncation region is assigned to the truncation points.

The value of the option at the maximum and minimum stock price at each date (states $\pm N$) can be set at either their intrinsic value or the European option price computed using the closed form formula. Empirically, these two boundary conditions yield almost identical values. As a final point, the value of $\tilde{q}$ from (27) can be assumed to be $1/2$ as the $o(h)$ term is negligible. Typically, the exact value of $\tilde{q}$ differs from $1/2$ only in the fourth decimal place, if the number of time steps is at least a hundred.

First, consider European put option values computed by integrating the jump distribution over every point on the state space inside the truncation region. These prices are reported in Table I separately for 100 and 200 time steps in columns 2 and 3 respectively. The American option values with 200 time steps are reported in column 9. For comparison, the corresponding values computed using Merton's (1976) closed form solution are reported in column 4 under the heading $N = \infty$. We observe that the computed European option prices are quite accurate even with only 100 time steps. For most cases, the error is less than \$0.02. With 200 steps, the prices reported are accurate within \$0.005 in the vast majority of cases.

Now consider an alternative implementation in which we use the Newton-Cotes formulae to integrate over the jump distribution. We divide the integra-

Table I

Put Option Prices with Lognormal Jumps

This table illustrates the prices of American and European put options using a discrete time approximation to a stock price process with a distribution composed of independent lognormal diffusion and lognormal jump components. The European price with an infinite number of time steps corresponds to the closed form solution. The Newton-Cotes formulae with 15 and 30 steps are used to integrate the jump distribution at each state.

For comparison, put option prices under a lognormal diffusion with a constant variance equal to the total variance of the stock price return in the jump diffusion case are also reported in the last two columns under the heading "Diffusion." The European put option values are computed using the Black-Scholes formula and the American put option prices are computed using the binomial model of Cox, Ross, and Rubinstein (1979) with 200 steps.

Parameter values: initial stock price = \$40; annual interest rate (r) = 8%; annual variance of diffusion component of stock price return (σ^2) = 0.05; variance of stock price return due to each jump occurrence (δ^2) = 0.05.

Definitions:

EU.PR = European put option price using the discrete time model whereby we permit a transition to every possible node at the next date from every node at the current date.

AM.PR = Corresponding American put option price using the discrete time model.

NC30 and NC15 = Prices are computed with the Newton-Cotes formulae using 30 points and 15 points respectively.

	Jump Diffusion								Diffusion	
	EU.PR			AM.PR	EU.PR NC30		EU.PR NC15		EU.PR	AM.PR
	Number of Time Steps									
Strike Price	100	200	∞	200	100	200	100	200	∞	200
Jump Intensity = 5 per year, Maturity = 0.25 years.										
30	0.668	0.669	0.670	0.674	0.668	0.668	0.662	0.665	0.611	0.616
35	1.673	1.674	1.673	1.688	1.673	1.670	1.667	1.669	1.814	1.836
40	3.595	3.594	3.592	3.630	3.594	3.589	3.592	3.593	3.929	4.003
45	6.656	6.656	6.655	6.734	6.655	6.649	6.643	6.645	6.922	7.077
50	10.544	10.545	10.544	10.696	10.541	10.534	10.517	10.518	10.614	10.899
Jump Intensity = 5 per year, Maturity = 1 year.										
30	2.625	2.623	2.621	2.720	2.643	2.629	2.630	2.604	2.677	2.790
35	4.419	4.415	4.412	4.603	4.445	4.423	4.420	4.384	4.525	4.745
40	6.705	6.701	6.696	7.030	6.739	6.710	6.698	6.654	6.849	7.222
45	9.431	9.426	9.422	9.954	9.472	9.436	9.410	9.360	9.589	10.193
50	12.533	12.528	12.524	13.318	12.579	12.536	12.492	12.436	12.682	13.567

tion region into three equally spaced subintervals.[14] European put prices are computed using five- and ten-point Newton-Cotes formulae to integrate over each of these subintervals. Since the integration region consists of three intervals, these calculations require integration over 15 and 30 points respectively at each node.

European put prices using the 15- and 30-point approximation models are reported in Table I in columns 5–6 and 7–8 respectively. We notice that the 15-point approximation model with 200 time steps yields prices for three-month options which are within one cent of their accurate value in the majority of cases. With the 30-point approximation model, almost all the errors are within one cent. However, the 15-point approximation model performs poorly for one-year options. To obtain an acceptable level of accuracy, the 30-point approximation model is necessary.

The above algorithms require only $15N$ and $30N$ computations at each node. Therefore, relative to the CRR model, the computational effort is approximately 20 and 40 times[15] respectively. On modern computers, these calculations do not require substantial effort. Thus, the model can be implemented for option valuation rather effectively.

For our second numerical example, replace the jump distribution specified in the previous example by a discrete distribution, such that

$$\ln(Y) = +\delta \text{ with probability } b \tag{36}$$

$$= -\delta \text{ with probability } 1 - b, \tag{37}$$

for some constant $b \in (0,1)$. With diversifiable jumps, the value of a European call or put option can be derived[16] as

$$ECC[S(0)] = \sum_{n=0}^{\infty} \exp(-\lambda T) \frac{(\lambda T)^n}{n!} \sum_{i=0}^{n} \frac{n!}{i!(n-i)!} b^i (1-b)^{n-i}$$

$$\cdot BSE[S(0)\exp(-\lambda KT + (2i-1)\delta), \sigma],$$

where $ECC(\cdot)$ is the option price, $K = b\exp(\delta) + (1-b)\exp(-\delta) - 1$ and $BSE[S(0)\exp(-\lambda KT + (2i-1)\delta), \sigma]$ is the Black-Scholes price of the option with initial stock price $S(0)\exp(-\lambda KT + (2i-1)\delta)$ and volatility parameter σ.

[14] Since the Newton-Cotes formulae require us to evaluate the integrand at equally spaced intervals, we ensure that the number of points from the state space contained in each subinterval over which we integrate is equal to an integer multiple of the number of points used in the Newton-Cotes formulae. This is accomplished by enlarging the integration interval if necessary.

[15] The CRR model contains a triangular state space. Further, to compute the option price at each date, it requires computations at two nodes at the next date. Since the state space in our model is rectangular, the multiple for the formulae involving 15 points is $15/2 \times 3 = 22.5$.

[16] With diversifiable jumps, the option price is obtained by using the results in Merton (1976). This leads to the following two steps: (a) replace the expected return on the stock by the riskfree rate, (b) then discount the expected payoff on the option by the riskless rate. This yields our formula. The detailed proof is available from the author.

To compute option values at each node, we need to integrate over the two local states and two points in the support of the jump distribution (as defined in Section II, $d\mathcal{N}_n(l)$ is nonzero for only two values of l). Therefore, the computational effort is quite small. Some put option values computed using this model are reported in Table II. As in the earlier case, the accuracy of the model is extremely good.

As a final point, we note that in certain applications in which the jump intensity, λ, is very high, the approximations illustrated above may be

Table II

Put Option Prices with Bivariate Jumps

This table illustrates the prices of American and European put options using a discrete time approximation to a stock price process with a distribution composed of an independent lognormal diffusion and a bivariate proportional jump process. The European price with an infinite number of time steps corresponds to the closed form solution.

For comparison, put option prices under a lognormal diffusion with a constant variance equal to the total variance of the stock price return in the jump diffusion case are also reported in the last two columns under the heading "Diffusion." The European put option values are computed using the Black-Scholes formula and the American put option prices are computed using the binomial model of Cox, Ross, and Rubinstein (1979) with 200 steps.

Parameter values: initial stock price = \$40; annual interest rate (r) = 8%; annual variance of diffusion component of stock price return $(\sigma^2) = 0.05$; variance of stock price return due to each jump occurrence $(\delta^2) = 0.05$. Probability that the log of the proportional jump magnitude is $+\delta$ is 0.5 and that it is $-\delta$ is 0.5.

Definitions:

EU.PR = European put option price using the discrete time model.
AM.PR = American put option price using the discrete time model.

	Jump Diffusion					Diffusion	
	EU.PR			AM.PR		EU.PR	AM.PR
	Number of Time Steps						
Strike Price	100	200	∞	100	200	∞	200
Annual Jump Intensity = 5.0, Maturity = 0.25 years.							
30	0.592	0.595	0.596	0.599	0.600	0.531	0.537
35	1.637	1.638	1.639	1.660	1.660	1.670	1.695
40	3.594	3.594	3.593	3.650	3.649	3.751	3.821
45	6.500	6.497	6.495	6.639	6.636	6.744	6.901
50	10.319	10.318	10.318	10.576	10.574	10.463	10.763
Annual Jump Intensity = 5.0, Maturity = 1.0 years.							
30	2.402	2.406	2.410	2.510	2.513	2.439	2.543
35	4.141	4.140	4.142	4.354	4.355	4.220	4.433
40	6.375	6.371	6.370	6.752	6.752	6.510	6.884
45	9.057	9.048	9.045	9.665	9.662	9.230	9.833
50	12.120	12.110	12.105	13.035	13.033	12.309	13.216

inaccurate. Empirically, levels of λ and h_n which ensure that $\lambda h_n \leq 0.1$ yield an acceptable level of accuracy. Hence, as λ increases, a larger number of steps is required.

V. Early Exercise Behavior with Jump Diffusion Processes

In this section, we compare optimal early exercise strategies under jump diffusion models with the corresponding strategies in the Black-Scholes model.[17] As we will show, jumps in stock prices can induce interesting exercise behavior. Further, information regarding optimal exercise strategies is needed by investors to decide when to exercise.

We consider two examples. In the first, the stock becomes worthless on the first jump occurrence. In the second, jumps are lognormal; if $Y - 1$ is the stock price return (in proportional terms) induced by each jump occurrence, $\ln Y$ is normal with variance δ^2 and mean $-\frac{1}{2}\delta^2$. In both examples, we assume that the jump risk is diversifiable. A significant probability of early exercise is induced for call options by assuming a fairly high dividend yield. Further, to ensure a meaningful comparison between the early exercise decisions in the Black-Scholes and jump diffusion models, we equate the stock return variance in the Black-Scholes case to the total variance of the stock price return in the jump diffusion case.

With bankruptcy-inducing jumps, we find that early exercise is "postponed" for both puts and calls relative to the Black-Scholes model for all maturities. By "postponed" (accelerated), we mean that for a given stock price, early exercise is less (more) likely to occur. The result for calls seems quite surprising; the possibility of a ruinous jump actually postpones early exercise!

With lognormal jumps, we find that relative to the early exercise decisions in the Black-Scholes model, the possibility of jumps causes early exercise to be postponed when the time to maturity is small and accelerated at longer maturities. This is true for both put and call options.

We now provide some additional details.

A. Early Exercise with Bankruptcy-inducing Jumps

We first consider jumps which induce the stock to become worthless. The graphs representing the critical stock price at which exercise occurs (exercise boundary) are depicted in Figures 3 and 4 for puts and calls respectively. As the jump always reduces the stock price, our intuition might suggest that relative to the Black-Scholes model, jumps should accelerate early exercise for calls and postpone early exercise for puts. However, the result is the opposite for call options. The early exercise boundary for the jump diffusion model is deeper in the money relative to the Black-Scholes model, i.e., the

[17] For an analysis of early exercise strategies in the Black-Scholes model, see Bodurtha and Courtadon (1991).

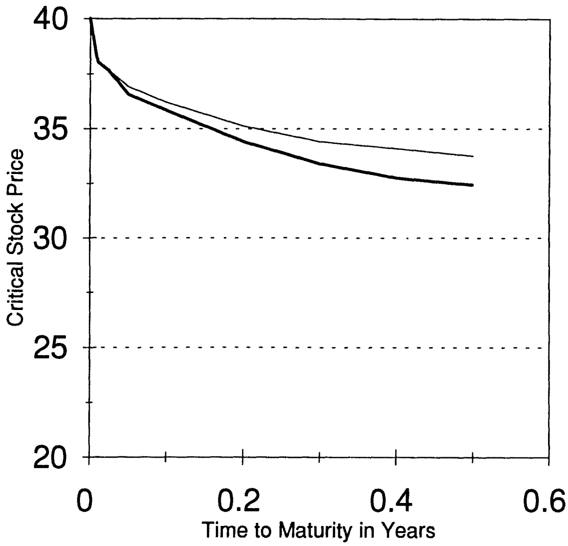

Figure 3. Put exercise boundary with bankruptcy-inducing jumps and corresponding boundary with a lognormal diffusion process. This figure presents the early exercise boundary for a put option when the underlying process follows a combination of a lognormal diffusion process and a Poisson jump process which induces bankruptcy on the first jump occurrence (*thick line*). The corresponding early exercise boundary under a lognormal diffusion process with a constant variance equal to the total variance of the stock price return in the jump diffusion case is also shown for comparison (*thin line*). Parameter values: strike price = \$40; annual interest rate = 0.08; annual dividend yield = 0.0; annual jump intensity = 0.1; annual variance of lognormal diffusion component (σ^2) = 0.05.

stock price has to increase by a larger amount before early exercise occurs in the jump diffusion case. Therefore, jumps reduce the likelihood of early exercise or alternately cause it to be postponed. An explanation of this result is as follows.

Recall from Sections I and II that the option price is obtained by first transforming the stock price distribution into a "risk-neutral distribution" whereby the expected return on the stock is equal to the riskless rate. This yields the drift (α) in the stock price distribution given by equation (19) as

$$\alpha = r - D - \tfrac{1}{2}\sigma^2 - \lambda K.$$

As the jump causes bankruptcy, $Y = 0$. Therefore, $K = E[Y - 1] = -1$. So,

$$\alpha = r - D - \tfrac{1}{2}\sigma^2 + \lambda.$$

In the Black-Scholes model, the drift equals $r - D - \tfrac{1}{2}\sigma^2$. Thus, the drift of the stock price is higher in the jump diffusion case by λ. Therefore, in the absence of a jump, the stock price increases at a faster rate on average in the jump diffusion model.

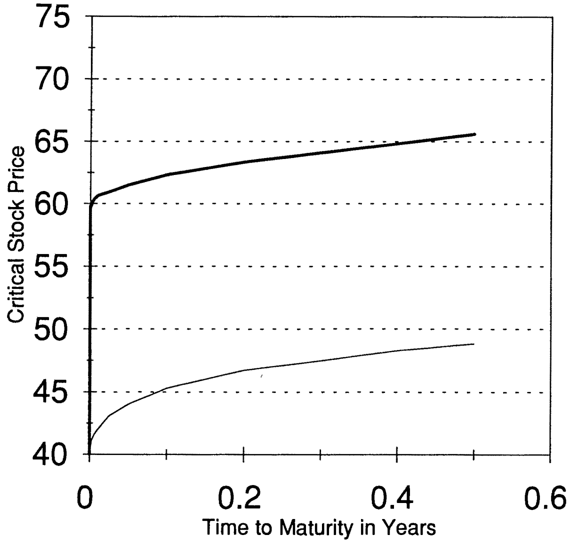

Figure 4. Call exercise boundary with bankruptcy-inducing jumps and corresponding boundary with a lognormal diffusion process. This figure presents the early exercise boundary for a call option when the underlying process follows a combination of a lognormal diffusion process and a Poisson jump process which induces bankruptcy on the first jump occurrence (*thick line*). The corresponding early exercise boundary under a lognormal diffusion process with a constant variance equal to the total variance of the stock price return in the jump diffusion case is also shown for comparison (*thin line*). Parameter values: strike price = \$40; annual interest rate = 0.08; annual dividend yield = 0.12; annual jump intensity = 0.1; annual variance of lognormal diffusion component (σ^2) = 0.05.

Now consider the situation faced by the owner of a deep in-the-money call who is trying to decide whether she should exercise her option. Suppose that the current stock price equals the critical stock price at which early exercise occurs under the Black-Scholes model. If the investor does not exercise the call, she continues to capture almost the full effect of the higher drift; deep in-the-money calls increase in value almost dollar for dollar with the stock price. However, she bears only a part of the downside risk of bankruptcy since the call payoff is truncated at the exercise price. But, the owner of the stock must obtain a higher drift in the stock price return to fully compensate her for bearing the bankruptcy risk. Hence, the call owner captures the full upside potential of the higher drift but avoids a part of the downside risk. Therefore, she finds it attractive to hold the option even though she would have exercised it under the Black-Scholes model, and early exercise is postponed.

The results for put options are as expected. The possibility of bankruptcy reduces the likelihood of early exercise (the early exercise boundary moves further in the money) because a jump occurrence can only cause a favorable effect for put owners. There are no surprises here since owners of in-the-money puts both bear the entire cost of a larger drift and reap the entire benefit of the jump.

Finally, consider the effect of increasing the jump intensity. As expected, the likelihood of early exercise further decreases for both calls and puts.

B. Early Exercise with Lognormal Jumps

The early exercise boundary for American puts (or calls) with lognormal jumps and the corresponding Black-Scholes early exercise boundary are plotted in Figure 5 (or Figure 6) for a representative set of parameter values. For both calls and puts, the results are similar. At short maturities, the possibility of a large jump in a favorable direction makes early exercise less likely under the jump diffusion model than under Black-Scholes. This is apparent since the jump diffusion exercise boundary is deeper in the money relative to the Black-Scholes exercise boundary. But, this difference in the likelihood of early exercise is the opposite for long maturities; we observe that the early exercise boundaries (for the Black-Scholes and jump diffusion models) cross in Figures 5 and 6. The intuition behind these results is given below.

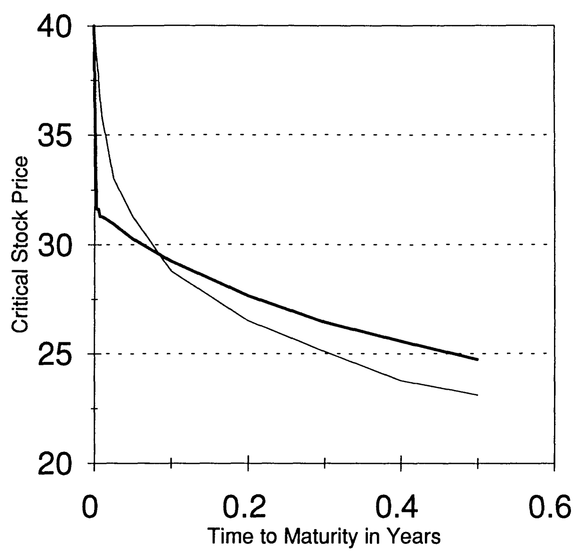

Figure 5. Put exercise boundary for a jump diffusion process with a lognormal jump distribution and corresponding boundary with a lognormal diffusion process. This figure presents the early exercise boundary for a put option when the underlying process follows a combination of a lognormal diffusion process and lognormally distributed Poisson jumps (*thick line*). The corresponding early exercise boundary under a lognormal diffusion process with a constant variance equal to the total variance of the stock price return in the jump diffusion case is also shown for comparison (*thin line*). Parameter values: strike price = \$40; annual interest rate = 0.08; annual dividend yield = 0.0; annual jump intensity = 5; annual variance of lognormal diffusion component (σ^2) = 0.05; variance of lognormal jump (δ^2) = 0.05.

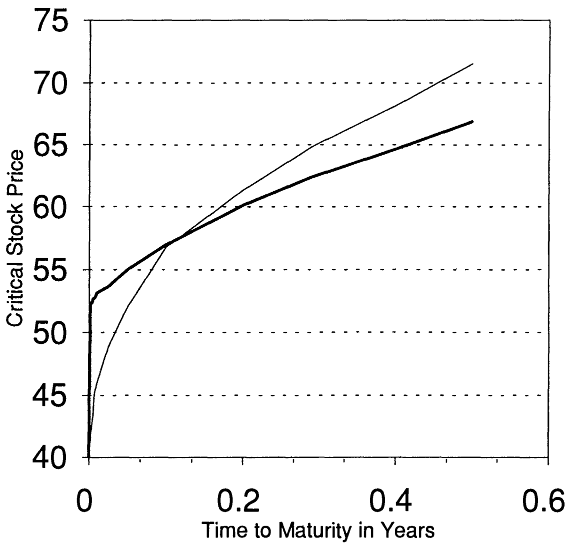

Figure 6. Call exercise boundary for a jump diffusion process with a lognormal jump distribution and corresponding boundary with a lognormal diffusion process. This figure presents the early exercise boundary for a call option when the underlying process follows a combination of a lognormal diffusion process and lognormally distributed Poisson jumps (*thick line*). The corresponding early exercise boundary under a lognormal diffusion process with a constant variance equal to the total variance of the stock price return in the jump diffusion case is also shown for comparison (*thin line*). Parameter values: strike price = \$40; annual interest rate = 0.08; annual dividend yield = 0.12; annual jump intensity = 5; annual variance of lognormal diffusion component (σ^2) = 0.05; variance of lognormal jump (δ^2) = 0.05.

When the maturity is small, the diffusion volatility can cause only relatively small changes in the stock price. However, a single jump can cause a relatively large change in the stock price. Therefore, the jump distribution has a large impact on the early exercise decision. Essentially, close to the maturity date, jumps induce much thicker tails in the terminal stock price distribution for a given total variance. For both calls and puts, the possibility that a jump will cause the stock price to move by a large amount in a favorable direction reduces the likelihood of early exercise (the early exercise boundary moves further in the money). The possibility of a large unfavorable jump is not sufficient to counteract this reduction. Since both puts and calls are convex functions of the stock price, a change in the stock price in a favorable direction results in a much larger gain than is lost due to a similar move in the opposite direction.

On the other hand, far from the maturity date, the jump distribution has a different effect. This is due to two reasons. First, over longer periods, multiple jumps can cancel each other out. Second, the diffusion component as well as the stock price drift can also cause significant changes in the stock price over the relatively longer period before the option matures. In fact, from Figures 5 and 6, we observe that early exercise is more likely in the jump diffusion model at longer maturities (jump diffusion boundary is less in the money relative to the Black-Scholes boundary) for both calls and puts. The reason is apparent if we compare the terminal stock price distribution functions in the Black-Scholes and jump diffusion models for long maturity dates. Plotting the two distributions for different maturities reveals that the jump diffusion process has a smaller probability mass in the tails at longer maturities.[18] This leads to a lower value for longer term options (both calls and puts). Therefore, early exercise is more likely in the jump diffusion case.

We note that the two boundaries once again converge when the maturity becomes very large.[19] Over very long periods, a combination of lognormal jumps and a lognormal diffusion process yields essentially the same terminal stock price distribution as a lognormal diffusion process (assuming that the total mean and variance of the stock return are held constant). Therefore, option values as well as early exercise decisions are very similar under both models.

Next, we consider comparative statics,[20] beginning with the jump intensity, λ. For comparison, we hold the total stock price return variance ($\sigma^2 + \lambda\delta^2$) constant by adjusting σ as we change either λ or δ. Close to the maturity date, the likelihood of extreme changes in the stock price increases as we increase the jump intensity. This magnifies the effect of the jumps; the likelihood of early exercise reduces further under the jump diffusion model. Over longer maturities, the early exercise boundary once again converges with the Black-Scholes boundary, however the two boundaries meet further away from the maturity date. The strong impact of the larger jump likelihood at short maturities takes longer to die out.

The effect of increasing the jump volatility, δ, is the same as that for λ. This follows since the composite effect of the jumps can be measured by the total induced variance in the stock price return, which equals $\lambda \times \delta^2$. Therefore, the same effect is obtained whether we increase the jump intensity or the volatility of the jump.

VI. Summary and Extensions

This study builds a tractable, discrete time model for the valuation of options on securities when the underlying process follows a jump diffusion process. It also yields a framework which uses only simple mathematics and yet permits us to study option valuation with these processes. The model incorporates the early exercise feature of American options, arbitrary jump distributions and, with suitable transformations, systematic jumps. Further, using numerical analysis, an efficient computational tool is provided.

[18] Recall that the stock return variance is equal under the two distributions.

[19] To highlight the differences in the plots at short maturities, the graph is truncated at half a year. The longer maturity graphs are not presented for conciseness. They are available from the author on request.

[20] Graphs are available from the author on request.

We characterize the optimal early exercise policies for American call and put options under two types of jump distributions relative to the corresponding policies under the Black-Scholes model. With lognormal jumps, we find that jumps significantly postpone early exercise at short maturities. However, at longer maturities the likelihood of early exercise is not very different from that in the Black-Scholes model. Bankruptcy-inducing jumps delay the exercise decision for both calls and puts.

The model developed here can also be used in other contexts. For example, it can be used to approximate the jump diffusion term structure model of Ahn and Thompson (1988) or a jump diffusion model corresponding to more general diffusion processes. Beginning with a discrete time model which converges weakly to the diffusion component of the jump diffusion process (such as that suggested by Nelson and Ramaswamy (1990)), we can superimpose jumps (rare events) on the existing local price changes. Indeed, the technique developed in this paper is applicable as long as the process of interest has independent increments (i.e., the distribution over short intervals is independent of the history of the process). Processes which satisfy these criteria include for example, univariate diffusions, the finite mixture of normals process, and the Variance Gamma process.[21]

The model can be augmented to provide characterizations of optimal investment policies in portfolio-consumption problems with jump diffusion processes. It can also be used in more general contexts. Corporate finance models involving bankruptcies and takeover contests which induce jumps in asset prices are candidates. Finally, the discrete model can be used to estimate the parameters of jump diffusion processes which do not have simple expressions for the moments or the likelihood function. The model essentially generates an approximation of the transition probability density of the process and hence can be used to numerically compute the moments or the likelihood function of the distribution. This technique has been used by He (1991) within the context of diffusion processes. His analysis can be extended to more general processes.

[21] See Kon (1984) and Madan and Seneta (1990).

REFERENCES

Abramowitz, M., and I. Stegum, 1965, *Handbook of Mathematical Functions* (Dover Publications, New York).

Ahn, C. M., 1991, Option pricing when jump risk is systematic, Working paper, Michigan State University.

——— and H. E. Thompson, 1988, Jump-diffusion processes and the term-structure of interest rates, *Journal of Finance* 43, 155–174.

Amin, K., 1991, On the computation of continuous time options prices using discrete approximations, *Journal of Financial and Quantitative Analysis* 26, 477–497.

Amin, K., and A. Khanna, 1993, Convergence of American option values from discrete to continuous-time financial models, Mathematical Finance, Forthcoming.

Amin, K., and V. Ng, 1993, Option valuation with systematic stochastic volatility, *Journal of Finance*, 48, 881–910.

Ball, C., and W. Torous, 1985, On jumps in common stock prices and their impact on call option pricing, *Journal of Finance* 40, 155–173.

Black, F., and M. Scholes, 1973, The pricing of options and corporate liabilities, *Journal of Political Economy* 8, 637–659.

Bodurtha, J., and G. Courtadon, 1991, The probability of early exercise, Working paper, University of Michigan.

Boyle, P. P., J. Evnine, and S. Gibbs, 1989, Numerical evaluation of multivariate contingent claims, *Review of Financial Studies* 2, 241–250.

Cox, J. C., and S. A. Ross, 1976, The valuation of options for alternative stochastic processes, *Journal of Financial Economics* 3, 145–166.

——— and M. Rubinstein, 1979, Option pricing: A simplified approach, *Journal of Financial Economics* 7, 229–263.

Harrison, J. M., and D. M. Kreps, 1979, Martingales and arbitrage in multiperiod security markets, *Journal of Economic Theory* 20, 381–408.

Harrison, J. M., and S. Pliska, 1981, Martingales and stochastic integrals in the theory of continuous trading, *Stochastic Processes and their Applications* 11, 215–260.

He, H., 1991, Method of moments estimation using discrete models, Working paper, University of California-Berkeley.

Huang, C.-F., and R. Litzenberger, 1988, *Foundations for Financial Economics* (Prentice Hall, New York).

Jarrow, R., and D. Madan, 1991, Option pricing using the term structure of interest rates to hedge systematic discontinuities in asset returns, Working paper, Cornell University.

Jarrow, R., and A. Rudd, 1983, *Option Pricing* (Richard D. Irwin, Inc., Homewood, Ill.).

Jarrow, R., and E. Rosenfeld, 1984, Jump risks and the intertemporal capital asset pricing model, *Journal of Business* 57, 337–351.

Jorion, P., 1988, On jump processes in the foreign exchange and stock markets, *Review of Financial Studies* 1, 427–445.

Kon, S., 1989, Models of stock returns—a comparison, *Journal of Finance* 39, 147–165.

Kushner, H., and G. DiMasi, 1978, Approximations for functionals and optimal control on jump diffusion processes, *Journal of Mathematical Analysis and Applications* 40, 772–800.

Madan, D., and E. Seneta, 1990, The variance gamma model for share market returns, *Journal of Business* 63, 511–524.

Merton, R. C., 1976, Option pricing when underlying stock returns are discontinuous, *Journal of Financial Economics* 3, 125–144.

Naik, V., and M. Lee, 1990, General equilibrium pricing of options on the market portfolio with discontinuous returns, *Review of Financial Studies* 3, 493–522.

Nelson, D., and K. Ramaswamy, 1990, Simple binomial processes as diffusion approximations in financial models, *Review of Financial Studies* 3, 393–430.

Page, F., and A. Sanders, 1986, A general derivation of the jump process option pricing formula, *Journal of Financial and Quantitative Analysis* 21, 437–446.

Rajasinghim, I., 1990, Convergence of discrete time event trees to continuous time economies and the pricing of contingent claims, Working paper, International Monetary Fund, Washington, D.C.