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Author(s): Steven L. Heston
Source: *The Journal of Finance*, Vol. 48, No. 3, Papers and Proceedings of the Fifty-Third Annual Meeting of the American Finance Association: Anaheim, California January 5-7, 1993 (Jul., 1993), pp. 933-947
Published by: Blackwell Publishing for the American Finance Association
Stable URL: http://www.jstor.org/stable/2329021
Accessed: 25/11/2009 11:33

THE JOURNAL OF FINANCE • VOL. XLVIII, NO. 3 • JULY 1993

Invisible Parameters in Option Prices

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ABSTRACT

This paper characterizes contingent claim formulas that are independent of parameters governing the probability distribution of asset returns. While these parameters may affect stock, bond, and option values, they are "invisible" because they do not appear in the option formulas. For example, the Black-Scholes (1973) formula is independent of the mean of the stock return. This paper presents a new formula based on the log-negative-binomial distribution. In analogy with Cox, Ross, and Rubinstein's (1979) log-binomial formula, the log-negative-binomial option price does not depend on the jump probability. This paper also presents a new formula based on the log-gamma distribution. In this formula, the option price does not depend on the scale of the stock return, but does depend on the mean of the stock return. This paper extends the log-gamma formula to continuous time by defining a gamma process. The gamma process is a jump process with independent increments that generalizes the Wiener process. Unlike the Poisson process, the gamma process can instantaneously jump to a continuum of values. Hence, it is fundamentally "unhedgeable." If the gamma process jumps upward, then stock returns are positively skewed, and if the gamma process jumps downward, then stock returns are negatively skewed. The gamma process has one more parameter than a Wiener process; this parameter controls the jump intensity and skewness of the process. The skewness of the log-gamma process generates strike biases in options. In contrast to the results of diffusion models, these biases increase for short maturity options. Thus, the log-gamma model produces a parsimonious option-pricing formula that is consistent with empirical biases in the Black-Scholes formula.

ALTHOUGH THE BLACK-SCHOLES (1973) formula can be derived in a utility framework (Rubinstein (1976)), it does not depend on impatience or risk aversion parameters. This is not surprising, because the formula uses a stock price and a bond price in lieu of direct information concerning preferences or state prices. But due to a surprising cancellation, the formula is also independent of the parameter governing the average stock return. This "invisible parameter" phenomenon occurs in all dynamic hedging models of option prices. In diffusion models (Black and Scholes (1973)), option prices are independent of the stock drift; in Poisson models (Cox and Ross (1976)), option prices are independent of the Poisson intensity, and in binomial models (Cox, Ross, and Rubinstein (1979)), option prices are independent of the jump probabilities. In order to uniquely replicate options prices with stock and bond prices, these hedging models make strong assumptions about market completeness. In particular, the continuous time hedging models require the implementation of continuous trading in the underlying stock and

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bond. This paper replaces these assumptions with direct assumptions about preferences. Thus, this model is consistent with nontrading and incomplete markets.

This paper derives the conditions under which probability parameters do not appear in contingent claim formulas, and provides some new examples. By further restricting attention to infinitely divisible probability distributions, one can extend the pricing formulas to continuous time. This consideration leads to the gamma process, which is a jump process that generalizes the Wiener process. Unlike the Poisson process, the gamma process is "unhedgeable" because it can jump to a continuum of values. It generates a continuous time option formula that generalizes the Black-Scholes (1973) formula, but can capture strike price biases and time maturity biases with only a single extra parameter. Although this formula contains no preference parameters, it does depend on a parameter that governs the average stock return. This feature may be useful in explaining why implied volatilities appear to be imperfect predictors of future asset volatility.

Section I characterizes option formulas which have missing parameters. Section II uses these results to generalize the Cox-Ross-Rubinstein (1979) and Black-Scholes (1973) formulas. Section III extends the results to continuous time and analyzes the results. Section IV concludes.

I. Missing Parameters

To price arbitrary securities in a state space, one must specify all the state prices, or Arrow-Debreu prices, in the space. One economical way to do this is to severely restrict the dimensionality of the space. For example, Black and Scholes (1973), Cox and Ross (1975), and Cox, Ross, and Rubinstein (1979) used Wiener processes, Poisson processes, and binomial processes respectively, which effectively limits the space to two dimensions. In this manner, one can use two securities, such as a stock and a bond, to price any other claim, such as an option. Alternatively, one can use a much larger state space, and then place additional restrictions on the pricing operator so that it can be characterized by only two securities. Brennan (1979) and Vankudre (1986) called this a risk-neutral pricing relationship. Using a two-dimensional state space is a special case of this, since it is equivalent to using an infinite state space and restricting the state prices to equal zero in all but two of the states. Given the pricing relationship in a state space, one can price contingent claims by arbitrage, although the arbitrage may involve an infinite number of securities.

The analysis of this paper is concerned with pricing securities with payoffs that are contingent on the future value of a spot asset, V_1. The state space of the analysis is given by the set of possible future values of the spot asset. In the absence of arbitrage, there must exist a positive pricing operator $M(V_1)$ such that for any contingent payoff $P(V_1)$, the current value, P_0, is given by

$$P_0 = E[P(V_1)M(V_1)], \tag{1}$$

(see Ingersoll (1989), chapter 2). We assume that the probability density of V_1 belongs to a family of densities that depend on an additional parameter θ, $p(V_1; \theta)$. And we assume that the pricing operator has two parameters, β and γ,

$$M(V_1) = \beta m(V_1, \gamma). \tag{2}$$

The parameter β is a discount factor; changing β affects the interest rate. The function $m(V, \gamma)$ is a differentiable function of the asset price and a "risk aversion" parameter γ.[1] The important economic content of equation (2) is that M does not depend on the probability density parameter θ. In words, M depends on the ex post realization of V_1 but not on the ex ante expectation.[2]

The pricing relationship in equations (1) and (2) expresses asset prices as functions of β, γ, and any parameters of the probability distribution. Given two prices, such as a stock and a bond price, one can generally invert them to recover the preference parameters. In this manner, one can price other securities in a "preference-free" fashion that does not directly depend on β and γ. This does not generally resolve any difficulties. It merely substitutes one pair of inputs for another. However, in some cases this procedure also eliminates a parameter of the probability distribution. For example, Rubinstein (1976) showed that if preferences have constant proportional risk aversion, i.e., $m(V_1, \gamma) = V_1^{-\gamma}$, and if the spot asset price is distributed lognormally, then one obtains the Black-Scholes (1973) option formula. The Black-Scholes-Rubinstein formula depends on a stock and bond price, and on a volatility parameter. But it does not depend on the preference parameters, β and γ, or on the mean parameter of the lognormal distribution.[3] This represents a definite reduction because one has replaced two preference parameters, β and γ, and the mean parameter, by only two security prices. Therefore, we investigate general conditions under which this occurs.

Definition: Contingent claims are independent of the parameter θ if the current price, P_0, of any contingent payoff, $P(V_1)$, can be written as a function of the prices of two fixed contingent claims in a manner that does not depend on β, γ, or θ.

The Appendix proves the following proposition.

PROPOSITION 1: *Given the preference structure in equation* (2), *contingent claims are independent of* θ *if and only if the pricing and probability density functions have the form*

$$m(V_1, \gamma) = A(\gamma)B(V_1)C(V_1)^{D(\gamma)}, \tag{3a}$$

[1] The dependence of $m(V, \gamma)$ on V and γ must not be degenerate, i.e., $m\dfrac{\partial^2 m}{\partial V \partial \gamma} \neq \dfrac{\partial m}{\partial V}\dfrac{\partial m}{\partial \gamma} \neq 0$.

[2] Bookstaber and MacDonald (1991) implicitly violate this assumption when they assume that option prices are independent of mean stock returns for arbitrary distributions.

[3] Brenner and Denny (1991) prove related continuous time results.

and

$$p(V_1, \theta) = F(\theta)G(V_1)C(V_1)^{H(\theta)}. \tag{3b}$$

Intuitively, the parameters γ and θ have a similar effects on the exponents $D(\gamma)$ and $H(\theta)$ in equation (3). Since asset prices depend on the product of $m(V_1, \gamma)$ and $p(V_1, \theta)$, contingent claim formulas that are independent of β and γ will also be independent of θ.

Equation (3b) leaves considerable flexibility in the specification of the probability density. It may be desirable to restrict the probability density so that contingent claim formulas are independent of a particular characteristic of the stock return. Here are two examples.

Definition: The parameter θ is *a shift in the mean stock return* if $p(V_1, \theta) = p(V_1 e^{-\theta}, 0)$.

Definition: The parameter θ is *a shift in the scale of the stock return* if $p(V_1, \theta) = p(V_1^{-\theta}, 1)/\theta$.

One can show that if θ is a shift in the mean stock return, and if $p(V_1, \theta)$ has the functional form in equation (3b), then $p(V_1, \theta)$ must be a lognormal density. Rubinstein (1976) showed that the lognormal density combined with a pricing function of the form $m(V_1, \gamma) = V_1^{-\gamma}$ yields the Black-Scholes (1973) formula. These results imply the corollary.

COROLLARY: *If $m(V_1, \gamma) = V_1^{-\gamma}$, then the Black-Scholes-Rubinstein formula is the only contingent claim formula (as a function of a bond and spot asset) that is independent of a shift in the mean stock return.*

Even within a particular family of probability densities or pricing operators (e.g., normal stock returns or HARA utility functions), Proposition 1 still leaves considerable flexibility for the specification of relative prices.[4] It does not tell us whether contingent claim formulas can be independent of a shift in the scale of the stock return, or whether these formulas can be extended to continuous time. The next sections provide affirmative answers to these questions.

II. Option Examples

This section presents option formulas that illustrate the theoretical results of Section I. It generalizes the Black-Scholes-Rubinstein lognormal option formula and the Cox-Ross-Rubinstein (1979) log-binomial formula using the log-gamma distribution and log-negative-binomial distribution. In order to generalize these results, we assume the same pricing operator used explicitly by Rubinstein (1976) and implicitly by Black and Scholes (1973), Cox and

[4]Relative pricing formulas for the lognormal, log-binomial, and log-Poisson distributions have been derived by Black and Scholes (1973), Cox and Ross (1975), and Cox, Ross, and Rubinstein (1979), respectively.

Ross (1975), and Cox, Ross, and Rubinstein (1979)

$$m(V_1, \gamma) = V_1^{-\gamma}. \tag{4}$$

The first formula is based on the gamma distribution. The gamma density with degrees of freedom δ is

$$g(z;\delta) = \frac{z^{\delta-1}e^{-z}}{\Gamma(\delta)}, \text{ for } 0 \le z < \infty. \tag{5}$$

Figure 1 shows that for small degrees of freedom, the gamma density is highly skewed. For small degrees of freedom, the density is concentrated near zero, but with a very fat tail. The mean and variance of a gamma variate are both equal to the degrees of freedom, and the sum of two independent gamma variates with degrees of freedom δ_1 and δ_2, respectively, is also a gamma variate, with degrees of freedom $\delta_1 + \delta_2$. By the central limit theorem, the gamma distribution must converge to the normal distribution as the degrees of freedom get large. Figure 2 confirms this by showing a sequence of gamma densities normalized to have zero mean and unit variance. Causal comparison of Figure 2 with stock return histograms suggests that the degrees of freedom parameter, δ, should be fairly large, e.g., at least 6 for monthly returns. The gamma distribution becomes insensitive to the degrees of freedom as δ grows large. Hence, χ^2 probability tables with integer degrees of freedom should suffice for the option formulas below.[5]

To generalize the Black-Scholes-Rubinstein formula, we shall assume that the logarithm of the asset price has a rescaled gamma distribution

$$\ln(V_1) = \mu + \sigma z, \tag{6}$$

where z has the gamma density in equation (5). Note that for negative σ, the asset return is negatively skewed. With power state prices of the form in equation (4), the initial asset price, V_0, must equal

$$V_0 = E[V_1 \times \beta V_1^{-\gamma}] = \beta e^{(1-\gamma)\mu}(1-(1-\gamma)\sigma)^{-\delta}. \tag{7}$$

A unit discount bond price, P, must equal

$$P = E[1 \times \beta V_1^{-\gamma}] = \beta e^{-\gamma\mu}(1+\gamma\sigma)^{-\delta}. \tag{8}$$

If σ is positive, then a call option price, C, with strike price K must equal

$$\begin{aligned} C &= E[\mathrm{Max}(0, V_1 - K) \times \beta V_1^{-\gamma}], \\ &= \beta e^{(1-\gamma)\mu}(1-(1-\gamma)\sigma)^{-\delta}(1-G(h_1;\delta)) \\ &\quad - \beta e^{-\gamma\mu}(1+\gamma\sigma)^{-\delta}(1-G(h_2;\delta)), \end{aligned} \tag{9}$$

[5]A χ^2 variable with ν degrees of freedom is twice a gamma variate with $\nu/2$ degrees of freedom. Johnson and Kotz (1970) and Press *et al.* (1986) provide numerical approximations to compute the cumulative gamma distribution.

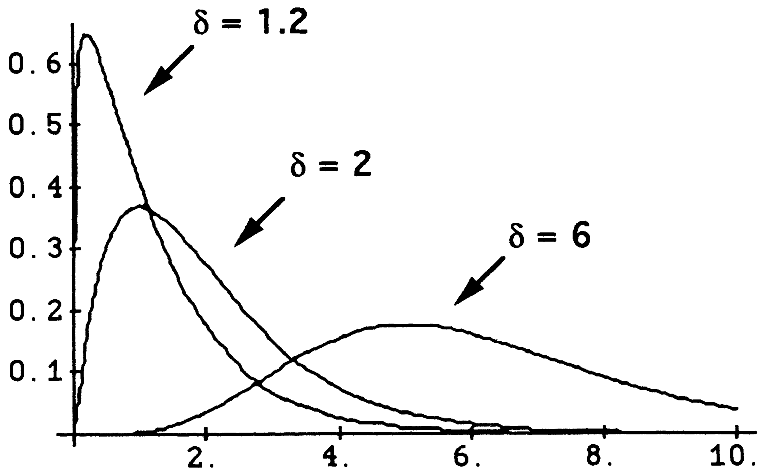

Figure 1. Gamma densities.

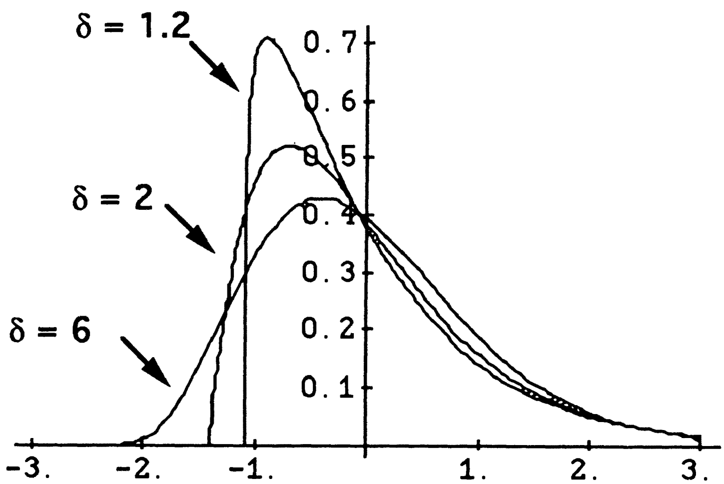

Figure 2. Compensated gamma densities.

where

$$G(x;\delta) = \int_0^x g(z;\delta)\,dz,$$

and

$$h_1 = (\ln(K) - \mu)(\sigma^{-1} + \gamma - 1),$$
$$h_2 = h_1 + \ln(K) - \mu.$$

By solving for the preference parameters, β and γ, in terms of the stock and bond price, one can reexpress the call price in a "preference-free" form.

PROPOSITION 2: *Given the preference structure in equations* (2) *and* (4), *and the distributional assumption in equations* (5) *and* (6), *if* σ *is positive, the value of a call option with strike price K is*

$$C = V_0(1 - G(h_1; \delta)) - KP(1 - G(h_2; \delta)), \tag{10a}$$

where

$$h_1 = (\ln(K) - \mu)\left(\frac{(e^{-\mu}V_0/P)^{-1/\delta}}{1 - (e^{-\mu}V_0/P)^{-1/\delta}}\right),$$

$$h_2 = h_1 + \ln(K) - \mu,$$

and if σ *is negative, the value of the call option is*

$$C = V_0 G(h_1; \delta) - KP - G(h_2; \delta). \tag{10b}$$

Interestingly, the option formula depends on the location parameter μ but is independent of the volatility parameter, σ. Since the volatility parameter, σ, changes both the mean and the variance of the stock return, the option price is not independent of the variance of the stock return. But when μ is zero, the volatility parameter, σ, is a shift in the scale of the stock return. One can show that if σ is a shift in the mean stock return, and if $p(V_1, \sigma)$ has the functional form in equation (3b), then $p(V_1, \sigma)$ must be a log-gamma density with $\mu = 0$. This yields the corollary.

COROLLARY: *If* $m(V_1, \gamma) = V_1^{-\gamma}$, *then formulas* (10a) *and* (10b) *with* $\mu = 0$ *are the only contingent claim formulas (as a function of a bond and spot asset) that are independent of a shift in the scale of the stock return.*

One can also develop parameter-independent formulas using discrete distributions. Suppose that the logarithm of the stock price has a rescaled distribution as in equation (6), where z has a negative binomial density with parameters n and q

$$p(z; n, q) = \left(\frac{\Gamma(n + s)}{\Gamma(n)\Gamma(s + 1)}\right) q^n (1 - q)^z, \quad \text{for } z = 0, 1, 2, \ldots. \tag{11}$$

Then the option formula is independent of the true probability parameter q in analogy with Cox, Ross, and Rubinstein's (1979) binomial formula.

PROPOSITION 3: *Given the preference structure in equations* (2) *and* (4), *and the distributional assumption in equations* (6) *and* (11), *if* σ *is positive, the value of a call option with strike price K is*

$$C = V_0\left[1 - P\left(\frac{\ln(K) - \mu}{\sigma}; n, q_1\right)\right] - KP\left[1 - P\left(\frac{\ln(K) - \mu}{\sigma}; n, q_2\right)\right], \tag{12}$$

where $P(x; n, q)$ is the cumulative negative binomial distribution

$$P(x; \lambda) = \sum_{i<x} p(i; n, q),$$

and

$$q_1 = \frac{(1 - e^{-\sigma})(\ln(e^{-\mu}V_0/P))^{-1/n}}{(\ln(e^{-\mu}V_0/P))^{-1/n} - e^{-\sigma}},$$

$$q_2 = e^{-\sigma}q_1 + 1 - e^{-\sigma}.$$

This section has derived option formulas in a single period setting, so it is not obvious that the formulas can be consistently applied to options of different maturities. In order to do this, the stock price must have a log-gamma distribution or log-negative-binomial distribution at multiple dates. This is indeed possible, because the gamma and negative binomial distributions are additive. By extending these option formulas to continuous time, one can price arbitrary contingent claims at all dates. The resulting state prices define a semigroup operator (see Garman (1985)). The next section defines the appropriate stochastic process, and shows how the option parameters depend on maturity.

III. Extension to Continuous Time

The preceding section derived discrete time option formulas with missing parameters. In order to value options of arbitrary maturity, one must explicitly use multiperiod models. This section shows how to do this by developing stochastic processes that follow a log-negative-binomial distribution or log-gamma distribution at all dates. In this manner it extends the static results of the preceding section to intertemporal frameworks with both discrete and continuous time.

The negative binomial process is a discrete time stochastic process analogous to the binomial random walk. Although one can define the process in continuous time, the discrete time version will suffice here.

Definition: A negative binomial process with parameters δ and q is an adapted stochastic process $z = \{z(i), \Im(i);\ i = 0, 1, 2, \ldots\}$ defined on some probability space $(\Omega, \Im, P)$ with the property that $z(0) = 0$ almost surely, and for $0 \leq i < j$, the increment $z(j) - z(i)$ is independent of $\Im(i)$ and has a negative binomial distribution $p(z; \delta(i - j), q)$.

In contrast to a binomial process that can jump by only one unit at a time, a negative binomial process can jump by any natural number of units. A suitable sequence of binomial processes will converge to a Wiener process, but a sequence of negative binomial process can converge to a gamma process instead. Take a sequence of negative binomial processes $\{z_n\}_1^\infty$ with parameters $\delta_n = 1/n$ and $q_n = 1/n$. Then define the stochastic process $x_n(t) = z_n([nt])/n$, where $[a]$ denotes the greatest integer less than or equal to a.

The sequence x_n converges in distribution to a process with independent increments that follow a gamma distribution. Since the gamma distribution is infinitely divisible, it defines a set of finite dimensional distributions for a stochastic process sampled in continuous time. Instead of defining the gamma process as the limit of negative binomial processes, one can define it directly in terms of its distributions.

Definition: A gamma process is an adapted stochastic process $x = \{x(t), \Im(t); 0 \le t < \infty\}$ defined on some probability space $(\Omega, \Im, P)$ with the property that $x(0) = 0$ almost surely, and for $0 \le s < t$, the increment $x(t) - x(s)$ is independent of $\Im(s)$ and has a gamma distribution with degrees of freedom $t - s$.

Since the definition defines a consistent set of finite dimensional distributions for the gamma process (Daley and Vere-Jones (1988)), Kolmogorov's theorem (Kolmogorov (1950) or Karatzas and Shreve (1988)) shows that the gamma process exists. This justifies applications of the preceding section to options of arbitrary maturity. However, the gamma process fails Kolmogorov's criterion for continuity. The gamma process is strictly increasing and contains no diffusion component; it is a pure jump process. Since the gamma distribution imparts only positive jumps, there can only be a countable number of jumps, and the gamma process must be continuous almost everywhere. As Figure 1 shows, for small degrees of freedom, the gamma distribution is concentrated near zero. Thus, the gamma process is like a Poisson process that takes variable jumps, including many small jumps.[6] This could be a useful description of stock prices, which have many small jumps at the transaction time scale and infrequent crashes on a larger time scale. Note that over long periods, the gamma increments are approximately normal. Hence, this process resembles a Wiener process that is discontinuous under a microscope. The mean and variance of the gamma distribution are both equal to the degrees of freedom. Like the Wiener process and the Poisson process, the gamma process has independent increments with a constant variance per unit of time. We will call a process $x^*(t)$ a compensated gamma process with degrees of freedom δ if $x^*(t) = \delta^{-1/2}(x(\delta t) - \delta t)$, where x is a gamma process. Figure 3 shows simulated sample paths of a compensated gamma process with various degrees of freedom. The paths with small degrees of freedom correspond to short time intervals, and bear an interesting resemblance to intraday stock price transaction data. As Figure 3 illustrates, the compensated gamma process $z(t)$ converges weakly to a Wiener process as the degrees of freedom parameter, δ, gets large. Despite the natural association with the popular Poisson and Wiener processes, the gamma process has been virtually ignored in finance problems.[7]

[6] The gamma process can be represented as the superposition, or sum, of an infinite number of independent Poisson processes with different size jumps.

[7] Ingersoll (1989, p. 266) noted the existence of this process, and Madan and Milne (1991) have used it to scale the variance of a Wiener process.

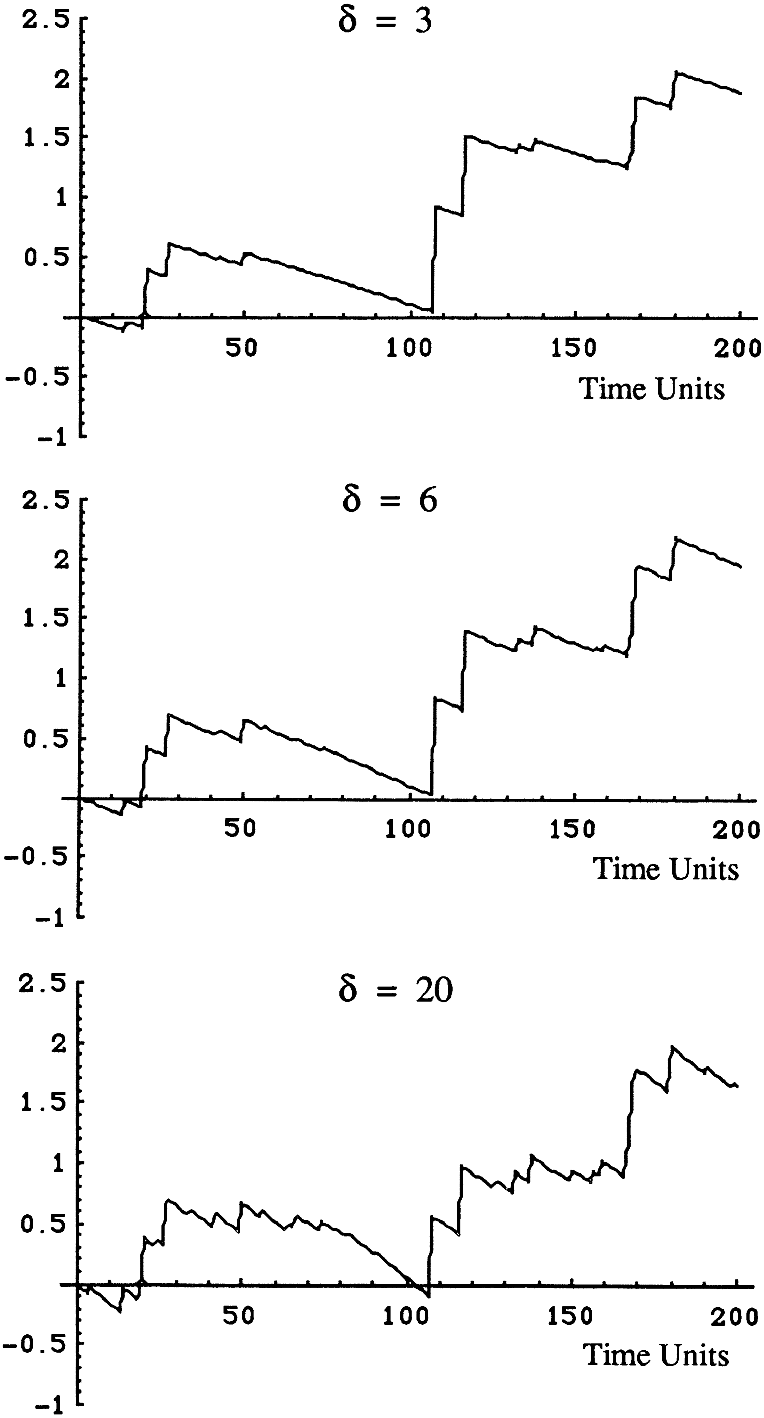

Figure 3. Sample paths of gamma processes simulated with $\Delta t = 1/200$.

Given a gamma process x_t, we can extend the static results of the preceding section by assuming that the asset price, V_t, follows a log-gamma process analogous to equation (6)

$$\ln(V_t) = \ln(V_0) + \nu(t) + \sigma x_t. \tag{13}$$

Given a discount bond price that matures at time t, $P(t)$, we can now apply the discrete log-gamma formula to an option maturing at time t with $\mu = \ln(V_0) + \nu(t)$ and degrees of freedom parameter δt.

In order to compare the log-gamma model to the Black-Scholes (1973) model with a volatility of σ^*, we shall specify $\nu(t) = -\ln(P(t)) - \sigma^*\delta^{1/2}$ if σ is positive, and $\nu(t) = -\ln(P(t)) - \sigma^*\delta^{1/2}$ if σ is negative. In other words, we shall let $\mu = \ln(V_0/P(t)) - \sigma^*\delta^{1/2}$ when using formula (10a), and let $\mu = \ln(V_0/P(t)) + \sigma^*\delta^{1/2}$ when using formula (10b). This choice has two advantages. First, it makes option prices homogeneous of degree one in $V_0/KP(t)$. Note that if we set $\mu = \ln(V_0/P(t)) + \sigma^*\delta^{1/2}$ and also set the true parameter σ equal to $\sigma^*\delta^{-1/2}$, then the expected (continuously compounded) stock return is equal to the expected (continuously compounded) bond return, and has a standard deviation of σ^*. Therefore, this choice of μ approximates the Black-Scholes pricing of at-the-money options.[8] Figure 4 shows the difference between prices from the log-gamma formula (10a) and the Black-Scholes formula when σ^* equals 30 percent for options where the present value of the strike price, KP, is 1. At-the-money options are priced almost identically in both models. But the log-gamma formula (10a) assigns higher prices to out-of-the-money options, and lower prices to in-the-money options. As the degrees of freedom parameter, δ, becomes large, the option value in equation (10a) converges to the Black-Scholes value for all options. For comparison, the Black-Scholes formula gives option prices of 0.035, 0.12, and 0.25, respectively, when the stock price is 0.75, 1, and 1.25. Therefore, the price differences can be economically significant. Intuitively, the upward jumps significantly increase the probability that out-of-the-money options expire in the money. The opposite strike biases appear when using the formula (10b) with downward jumps (Figure 5). These strike biases are similar to the skewness-related biases in diffusion models where stochastic volatility is correlated with stock returns (Heston (1993)). But there is an important distinction. The log-gamma process is a jump process, and the option biases increase for small degrees of freedom, i.e., short times to option maturity. This feature is consistent with the empirical results of Rubinstein (1985) and Knoch (1992). In contrast, stochastic volatility diffusion models do not exhibit significant biases for short times to maturity because the volatility is fairly certain over short intervals. Therefore, the log-gamma model appears to have some empirical advantages over diffusion models of option pricing.

Although the log-gamma formula contains the Black-Scholes (1973) formula as a limiting case, it is fundamentally different from diffusion or

[8]Call option values are increasing in μ.

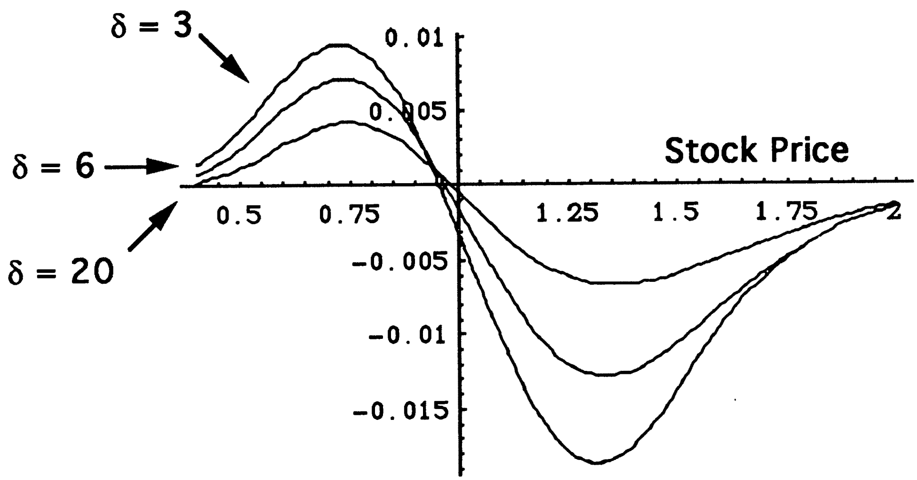

Figure 4. Gamma option price—Black-Scholes price (upward jumps).

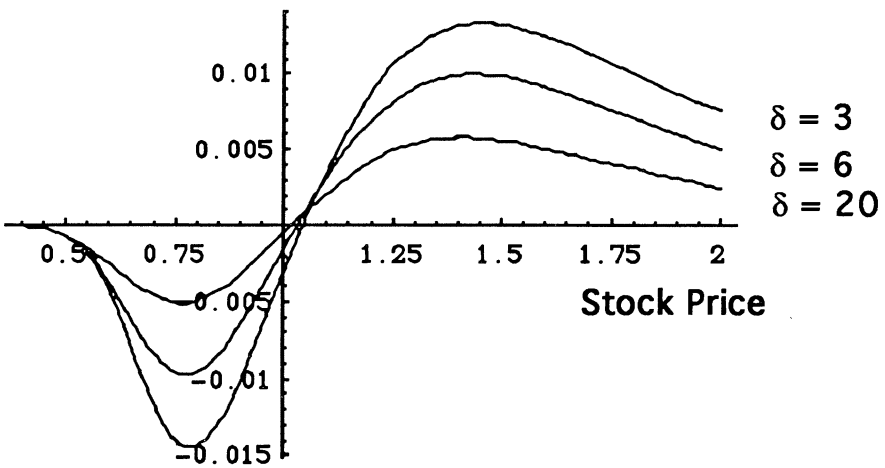

Figure 5. Gamma option price—Black-Scholes price (downward jumps).

Poisson models. Wiener processes and Poisson processes are locally binary. In contrast, the infinitesimal increments of a gamma process can attain an infinite number of values. Since this process can instantaneously jump to a continuum of values, it is fundamentally unhedgeable with a finite number of securities. The log-gamma formula is also surprisingly different from the Black-Scholes (1973) formula because it depends on the mean parameter μ, but not on the scale parameter σ. Therefore, option prices may change in response to a shift in the mean stock return, but be insensitive to changes in volatility. This may explain the empirical puzzle that implied variances from option prices are not optimal predictors of actual stock variance (Lamoureux and Lastrapes (1993)). These issues await empirical examination. One could estimate the parameters of the log-gamma model by matching the mean, variance, and skewness of stock returns or by maximum likelihood (Johnson and Kotz (1970)).

IV. Conclusions

This paper started by asking a very basic question: When are contingent claim formulas independent of parameters in the probability density of the stock return? It answered this question by characterizing these formulas and providing examples. This paper has used a preference-based approach that "is the major competing paradigm to the continuous time [hedging] framework." (Brennan (1979)). In contrast to previous results of Rubinstein (1976) and Brennan (1979), this paper provides continuous time generalizations of the Black-Scholes (1973) formula that cannot be derived with hedging arguments. One of these formulas is based on the gamma process.

The gamma process used in this paper appears to be a useful generalization of the Wiener process because it incorporates jumps in the stock price and skewness in the return distribution. It offers an interesting contrast with the Poisson process, because it is a jump process that can jump to a continuum of values. By adjusting the degrees of freedom parameter, one can control the jump intensity of the process to achieve the desired compromise between jump behavior and continuous diffusion behavior. The degrees of freedom parameter may be estimated from the time series properties of spot returns or from the skewness of the return distribution. The results can then be applied to cross-sections of option prices. The jump and skewness properties may be particularly relevant to ascertaining the effects of large information releases or sudden policy changes on asset prices. Further theoretical applications of the gamma process also appear promising. The gamma process may provide theoretical insights about how discontinuities affect early exercise strategies of American options. Additional theoretical work remains to characterize American options and early exercise strategies (as done by Carr, Jarrow, and Myneni (1990)), to approximate short time solutions (as done by Van Moerbeke (1976)), and to develop numerical methods for gamma valuation problems. It remains to derive gamma process models in an equilibrium framework. Gamma process models could also be applied to other areas such as exchange rates or the term structure of interest rates.

In conclusion, the parsimony offered by missing-parameter formulas is convenient because it minimizes the informational requirements for asset pricing. This reduces the estimation burden of the empirical researcher while still allowing a broad family of probability distributions. The log-gamma option formula is testable, and may be useful in future empirical research because it captures strike price biases and short maturity biases with only one extra parameter relative to the Black-Scholes formula.

Appendix

Proof of Proposition 1: It is sufficient to show that we can price Arrow-Debreu securities independently of θ, since Arrow-Debreu securities span all other contingent claims. An Arrow-Debreu price, $p^*(K; \beta, \gamma, \theta)$, is simply the state price times the probability of an asset payoff in that state

$$p^*(K; \beta, \gamma, \theta) = p(K; \theta)\beta m(K, \gamma). \tag{A1}$$

We will suppress the notational dependence of the Arrow-Debreu prices on the parameters. Given two Arrow-Debreu prices $p^*(K_1)$ and $p^*(K_2)$, we can recover the preference parameter γ from the quantity $x = \ln(p^*(K_1)/p^*(K_2))$. Define this function, $G(x; \theta)$, by the implicit relationship

$$G\left(\ln\left(\frac{p(K_1; \theta)m(K_1, \gamma)}{p(K_2; \theta)m(K_2, \gamma)}\right), \theta\right) = \gamma. \tag{A2}$$

The value of a third Arrow-Debreu price, $p^*(K_3)$, in terms of $p^*(K_1)$ and $p^*(K_2)$ is

$$p^*(K_3) = p(K_3; \theta) \times \frac{p^*(K_1)m(K_3, G(\ln(p^*(K_1)/p^*(K_2)), \theta))}{p(K_1; \theta)m(K_1, G(\ln(p^*(K_1)/p^*(K_2)), \theta))}. \tag{A3}$$

This price will be independent of θ if and only if the derivative of equation (A3) with respect to θ equal zero. This implies

$$\frac{\partial}{\partial\theta}(\ln(p(K_3, \theta)) - \ln(p(K_1, \theta))) + \frac{\partial}{\partial\gamma}(\ln(m(K_3, \gamma)) - \ln(m(K_1, \gamma)))\frac{\partial}{\partial\theta}(G(x; \theta)) = 0. \tag{A4}$$

In order to eliminate the function G from equation (A4), we differentiate equation (A2) with respect to θ and γ to obtain

$$\frac{\partial}{\partial x}(G(x; \theta)) \times \frac{\partial}{\partial\theta}(\ln(p(K_1, \theta)) - \ln(p(K_2, \theta))) + \frac{\partial}{\partial\theta}(G(x; \theta)) = 0, \tag{A5a}$$

$$\frac{\partial}{\partial x}(G(x; \theta)) \times \frac{\partial}{\partial\gamma}(\ln(m(K_1, \gamma)) - \ln(m(K_2, \gamma))) = 1. \tag{A5b}$$

Solving for $\partial/\partial\theta(G(x; \theta))$ from equation (A5a, A5b) and substituting into equation (A4) shows

$$\frac{\frac{\partial}{\partial\theta}(\ln(p(K_1, \theta)) - \ln(p(K_3, \theta)))}{\frac{\partial}{\partial\theta}(\ln(p(K_1, \theta)) - \ln(p(K_2, \theta)))} = \frac{\frac{\partial}{\partial\gamma}(\ln(m(K_1, \gamma)) - \ln(m(K_3, \gamma)))}{\frac{\partial}{\partial\gamma}(\ln(m(K_1, \gamma)) - \ln(m(K_2, \gamma)))}. \tag{A6}$$

Note that equation (A6) must hold for arbitrary values of θ and γ. Therefore, the quantities on the left and right sides of the equation must be independent of θ and γ, respectively. This holds only for functions of the form indicated in Proposition 1.

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