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ABSTRACT

We use an extension of the equilibrium framework of Rubinstein (1976) and Brennan (1979) to derive an option valuation formula when the stock return volatility is both stochastic and systematic. Our formula incorporates a stochastic volatility process as well as a stochastic interest rate process in the valuation of options. If the "mean," volatility, and "covariance" processes for the stock return and the consumption growth are predictable, our option valuation formula can be written in "preference-free" form. Further, many popular option valuation formulae in the literature can be written as special cases of our general formula.

IN THIS PAPER, WE investigate the valuation of individual stock options when the volatility of the underlying stock return is stochastic and has a systematic component which is related to the stochastic volatility of the consumption growth or the market return. Such an investigation is timely and of significant interest because there is now a considerable amount of evidence that the volatility of stock prices is not only stochastic, but that it is also highly correlated with the volatility of the market as a whole. (For example, see Black (1975), Conrad, Kaul, and Gultekin (1991), Jarrow and Rosenfeld (1984), Jorion (1988), and Ng, Engle, and Rothschild (1992). Furthermore, empirical work by Black and Scholes (1972), Gultekin, Rogalski, and Tinic (1982), and Whaley (1982) has shown that the empirical biases inherent in the Black-Scholes option prices are different for options on high and low risk stocks. Because low risk stocks are mainly large firm stocks with stochastic return volatilities that are highly correlated with the return volatility of the market, and high risk stocks are mainly small firm stocks with a less-important systematic volatility component, incorporating systematic volatility effects in option valuation models has the potential to reduce the empirical biases exhibited by prices computed from the Black-Scholes formula.

We use equilibrium arguments that extend the consumption-based representative agent framework of Rubinstein (1976), Brennan (1979), and Stapleton and Subrahmanyam (1984) to allow individual asset returns and the consumption growth to exhibit stochastic volatility. In particular, we permit the volatility of individual asset returns to consist of a systematic component that is related to the consumption (or market) volatility in addi-

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tion to an idiosyncratic component. We also permit our framework to include a broad class of discrete-time processes that do not necessarily converge to diffusion processes.[1] Within this framework, we derive pricing formulae for calls and puts based on equilibrium arguments. Equilibrium arguments are necessary for option valuation since a perfect hedge for an option cannot be constructed in our general discrete time framework. Correspondingly, arbitrage arguments cannot be used.

A direct consequence of permitting the volatility of consumption to be stochastic is that the spot interest rate, which is determined by the consumption volatility in equilibrium, in general is also stochastic. Thus, our model simultaneously incorporates both a stochastic interest rate and a stochastic volatility process for stock returns in the valuation of options. Existing works extending the Black-Scholes model tend to consider either stochastic volatility or stochastic interest rates but not both.

We distinguish between models with predictable and unpredictable processes for the "mean," volatility, and the "covariance" of the stock return and consumption growth and show that the option-pricing results for the two cases are different.[2] If these processes are predictable, then the option-pricing formula is independent of the preference parameters even if the individual stock return volatility is related to the consumption volatility. In particular, since the definition of Ito integrals requires that integrands be predictable, this result is quite useful when the underlying economic model is based on diffusion processes. However, when these processes are not predictable, the option valuation formula generally depends on preference parameters except when the unpredictable component of the volatility of individual stock returns is not related to the consumption volatility.

As a special case of our formula, we also derive a closed form solution for option prices when the asset return follows a lognormal jump-diffusion process with systematic jumps. This permits us to evaluate the effect of systematic jumps in asset prices within the jump-diffusion context. Ball and Torous (1985) find that the existence of idiosyncratic jumps in individual stock returns leads to relatively small deviations between the Black-Scholes and Merton (1976) option valuation models. Therefore, it seems likely that if jumps in asset returns are important for option valuation at all, it is the systematic part rather than the idiosyncratic part that is important. Naik and Lee (1991) use a general equilibrium framework to extend Merton's (1976) model to include options on the market portfolio. By construction, the jumps in their framework must be systematic. However, their model does not permit valuation of options on individual stocks.

[1]Examples of discrete time processes that do not converge to a diffusion, as we shrink the observation interval, are: the finite mixture of normals model of Kon (1984), the t-distribution, the variance gamma process of Madan and Seneta (1987), and many nonlinear autoregressive processes.

[2]A discrete time process is predictable if its value is known one period in advance. For example, the volatility process is predictable if the volatility in period "t" is completely determined from the information set available at date "$t - 1$."

Many popular option valuation formulae in the literature can be written as special cases of our formula. Examples include the Black-Scholes formula, the Hull-White (1987) and Bailey-Stulz (1989) stochastic volatility option-pricing formulae, the Merton (1976) and Naik-Lee (1990) jump-diffusion option-pricing formulae as well as the Merton (1973), Amin-Jarrow (1992), and Turnbull-Milne (1991) stochastic interest rate option-pricing formulae for equity options.

We also present some simulation results with our model. When the "mean" and volatility processes are predictable, we demonstrate that the direction of the bias inherent in prices obtained from the Black-Scholes model is different between stocks with a strong systematic volatility component and those with a strong idiosyncratic volatility component. This results from the fact that the systematic volatility component is related to the stochastic interest rate while the idiosyncratic volatility component is not. Since the effects of stochastic volatility and stochastic interest rates act in opposite directions, the actual bias depends on which effect dominates. Similarly, the effect of mean reversion in volatility on option prices also depends on whether it is from the systematic volatility component or the idiosyncratic component.

Simulation results with our jump-diffusion option-pricing formula indicate that our formula yields option values significantly different from Merton's (1976) jump-diffusion option valuation formula only when the representative investor is significantly more risk averse than one with logarithmic utility. Further, our formula yields higher option values for in-the-money and at-the-money call options and lower values for out-of-the-money call options.

This paper is organized as follows. In Section I, we introduce the basic framework and the stochastic processes used in the paper. We also derive the Euler conditions for utility maximization for the individual agent. In Section II, we derive the general stochastic volatility option-pricing formula. This formula is specialized to predictable volatility processes in Section III and to a jump-diffusion process in Section IV. Some comparative statics results and estimates of the deviations of option prices from those predicted by the Black-Scholes model are also provided in these sections. Finally, Section V concludes.

I. The Framework of Analysis

Our basic economic model consists of a generalization of the multiperiod representative agent economy of Brennan (1979). In this economy, the representative agent chooses consumption and investment plans to maximize her expected discounted utility of consumption subject to her intertemporal budget constraint. The agent maximizes

$$E_0\left[\sum_{t=0}^{\infty} \rho^t C_t^{1-b}/(1-b)\right] \tag{1}$$

where C_t is her consumption at date t, ρ is her time discount factor, and b is her coefficient of relative risk aversion. With $b = 0$, one obtains a risk-neutral agent and with $b \to 1$, one obtains logarithmic preferences.

In this paper, we consider the valuation of simple European call and put options written on a stock with price S_t at date t $(t = 0, 1, 2, \ldots)$.[3] The options mature at date T and have an exercise price K. For simplicity, we also assume that the stock does not pay any dividends before the maturity date T. However, if the stock pays a proportional dividend in each period or a fixed set of dividends on fixed dates, this can easily be accommodated with rather simple modifications.

For added generality, and to include empirical processes that do not have diffusion or jump-diffusion limits, we will set up our economic model in discrete time. Continuous time processes can be obtained by taking appropriate limits.

The relevant uncertainty in our economic model is driven by the realization of a set of $N + M + 2$ random variables at each discrete date. Among them are a random shock to consumption, $\varepsilon_{c,t}$; a random shock to the stock price, $\varepsilon_{s,t}$; a set of N systematic state variables, $[y_{1t}, \ldots, y_{Nt}]' \equiv \underline{Y}_t$, that determine the time-varying "mean," "variance," and "covariance" of the consumption and stock return; and finally a set of M firm-specific state variables, $[u_{1t}, \ldots, u_{Mt}]' \equiv \underline{U}_t$, that determine the idiosyncratic part of the stock return "variance."[4] The investors' information set at time t is represented by the σ-algebra $F_t \equiv \sigma((\varepsilon_{c\tau}, \varepsilon_{s\tau}, \underline{Y}_\tau, \underline{U}_\tau);\ \tau = 0, 1, \ldots, t)$.

We make the following assumptions on the joint evolution of the stock price and aggregate consumption:

ASSUMPTION 1:

$$\ln(S_t) = \ln(S_{t-1}) + \mu_{s,t} - \tfrac{1}{2} \cdot h_{s,t} + \varepsilon_{s,t}, \tag{2a}$$

$$\ln(C_t) = \ln(C_{t-1}) + \mu_{c,t} - \tfrac{1}{2} \cdot h_{c,t} + \varepsilon_{c,t} \tag{2b}$$

where

$$\mu_{c,t} = \mu_c(\underline{Y}_\tau ; \tau \le t), \tag{2c}$$

$$\mu_{s,t} = \mu_s(\underline{Y}_\tau, \underline{U}_\tau ; \tau \le t), \tag{2d}$$

$$\begin{bmatrix} \varepsilon_{s,t} \\ \varepsilon_{c,t} \end{bmatrix} \Bigg| F_{t-1} \vee \sigma(\underline{Y}_t, \underline{U}_t) \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}; \begin{bmatrix} h_{s,t}(\underline{Y}_\tau, \underline{U}_\tau ; \tau \le t) & \sigma_{cs,t}(\underline{Y}_\tau, \underline{U}_\tau ; \tau \le t) \\ \sigma_{cs,t}(\underline{Y}_\tau, \underline{U}_\tau ; \tau \le t) & h_{c,t}(\underline{Y}_\tau ; \tau \le t) \end{bmatrix} \right) \tag{2e}$$

and $N[\cdot]$ is the bivariate normal distribution function.

[3] It is understood that there are also other stocks in the market.

[4] As there are other stocks in the economy, there are also other idiosyncratic state variables related to those stocks. But these additional state variables are irrelevant for the valuation of options on the stock under consideration.

ASSUMPTION 2: *$(\varepsilon_{c,t}, \varepsilon_{s,t})$ conditional on $\sigma(\underline{Y}_\tau, \underline{U}_\tau; \tau \leq t)$ is independent of $[(\underline{Y}_\tau, \underline{U}_\tau), \tau > t]$.*

For subsequent reference, and with a slight abuse of terminology, we will refer to $\mu_{s,t}$ and $h_{s,t}$ as the mean rate (equivalently expected rate) of return and variance (equivalently volatility) of the stock return process, $\mu_{c,t}$ and $h_{c,t}$ as the mean rate of return and variance of the consumption process and $\sigma_{cs,t}$ as their covariance process. These terms are consistent with standard terminology if we condition on $F_{t-1} \vee \sigma(\underline{Y}_t, \underline{U}_t)$. However, for conciseness, we will drop the reference to the conditioning information set.

Assumption 1 describes the dynamics of the stock price and the consumption growth process. The stock price and consumption process are conditionally lognormal in each period. By specifying these variables in terms of their logarithm, we ensure that they are always nonnegative. Further, by choosing the time interval appropriately and by defining the functional forms of the variance and covariance functions as well as the distributions of the state variables (Y_τ, U_τ), we can obtain a very rich class of processes that satisfy Assumption 1. For example, almost all the diffusion models of stock prices in the literature can be accommodated. Other examples will be provided in our subsequent discussion.

Under Assumption 1, the mean consumption growth, $\mu_{c,t}$, and the variance of consumption growth, $h_{c,t}$, are driven only by the systematic state variables $\underline{Y}$ while the mean stock return, $\mu_{s,t}$, the variance of the stock return, $h_{s,t}$, and the covariance between the consumption growth and the stock return, $\sigma_{cs,t}$, can be driven by the systematic and idiosyncratic state variables.

Assumption 2 guarantees that the mean and variance of the stock return and consumption growth at time t are not affected by the values of the future state variables to be realized after time t. Even though it restricts the class of processes that we can consider, this assumption significantly simplifies the analysis and yields tractable formulae. Notice that Assumption 2 is imposed only on the variables that determine the volatility processes and not on the variables $[(\varepsilon_{c,t}, \varepsilon_{s,t}); t = 1, \ldots, T]$. Further, Assumptions 1 and 2 together guarantee that $(\varepsilon_{c,t}, \varepsilon_{s,t})$ for $t = 1, \ldots, T$ are serially uncorrelated in the presence of $[(\underline{Y}_\tau, \underline{U}_\tau); \tau = 1, \ldots, T]$.

The above setup represents an extension of the original setup in Brennan (1979). It is fairly general and can incorporate many important characteristics of stock prices. First, the stock return variance can have a systematic component that changes with the consumption variance. For example, consider the following specification for the stock variance:

$$h_{s,t} = \beta \cdot h_{c,t} + h_{d,t}, \tag{3}$$

where β is a constant and $h_{d,t} = h_d(\underline{U}_\tau; \tau \leq t)$ is some sequence of random variables. In this special case, $\beta \cdot h_{c,t}$ is the systematic component of the stock return variance and $h_{d,t}$ is the idiosyncratic component of the stock return variance which is driven only by the idiosyncratic state variables. The larger the value of β, the higher the sensitivity of the stock return variance to changes in the variance of consumption growth. Since the variance of consumption growth is negatively related to the interest rate in equilibrium (see equation (11) later in the paper), a large value of β implies a strong negative relationship between the individual stock return variance and the interest rate. Given that the variance and the interest rate are two important inputs in the determination of option prices and that they have the opposite effects on call option values,[5] the value of β should therefore be important in determining the net effect of the two inputs.

Second, mean reversion in both the consumption and stock return variance is permitted. For instance, consider the simple case where the function $h_c(\cdot\cdot)$ is a first-order autoregressive model with a single state variable driving the dynamics as in equation (4) below:

$$h_{c,t} = \omega_c + \alpha_c h_{c,t-1} + g_c y_t, \tag{4}$$

where y_t is a random variable independent of $h_{c,t-1}$ and ω_c, α_c, and g_c are constants. Then, $h_{c,t}$ is mean reverting as long as α_c is between 0 and 1. The closer α_c is to 0, the faster the mean reversion. This mean reversion behavior exists as long as $h_{c,t}$ is a stationary autoregressive process. It is not important whether it is first order or not and whether there is more than one state variable driving the uncertainty. However, it is important to note that mean reversion in $h_{c,t}$ is related to mean reversion in the interest rate while the mean reversion in $h_{s,t}$ might or might not be related to the mean reversion in the interest rate. This depends on the importance of the systematic component of the individual stock return variance. Hence, again the valuation of options should depend on the relative importance of the systematic component and the idiosyncratic component.

Third, our framework permits predictable and unpredictable components in both the stock and consumption variance. The component of $h_{c,t}$ and $h_{s,t}$ determined by the realizations of past state variables, $(\underline{Y}_\tau, \underline{U}_\tau; \tau < t)$, constitutes the predictable component while the remaining part determined by the current state variables, $(\underline{Y}_t, \underline{U}_t)$, is the unpredictable component.

If $\mu_{c,t}$, $\mu_{s,t}$, $h_{c,t}$, $h_{s,t}$, and $\sigma_{cs,t}$ are only functions of previous realizations of the state variables and hence are measurable with respect to the information set F_{t-1}, we will say that the economy has predictable mean, variance, and covariance processes. This follows standard terminology in probability. For example, a continuous time representation that is consistent with Assumptions 1 and 2 and the predictability of the mean return, variance, and the covariance is:

$$\begin{aligned} d\ln(C_t) &= \left(\mu_{c,t} - \tfrac{1}{2}\cdot h_{c,t}\right)dt + \sqrt{h_{c,t}}\,dW_{ct}, \\ d\ln(S_t) &= \left(\mu_{s,t} - \tfrac{1}{2}\cdot h_{s,t}\right)dt + \sqrt{h_{s,t}}\,dW_{st}, \end{aligned} \tag{5}$$

where (W_c, W_s) is a two-dimensional Brownian motion with correlation θ, $(\underline{Y}, \underline{U})$ is a $(N + M)$ dimensional Brownian motion, and $(\mu_{c,t}, \mu_{s,t},$

[5]See Section III for additional details.

$h_{c,t}, h_{s,t}, \sigma_{cs,t}$) is continuous and adapted with respect to the filtration generated by $(\underline{Y}_t, \underline{U}_t)$. This representation is of particular interest since the diffusion limits of the popular GARCH(1, 1) and EGARCH(1, 1) processes are special cases of this continuous time specification (see Nelson (1990)). Our basic economic model is also valid with this specification.[6]

If $\mu_{c,t}, \mu_{s,t}, h_{c,t}, h_{s,t}$, and $\sigma_{cs,t}$ are also functions of current realizations of the state variables $(\underline{Y}_t, \underline{U}_t)$ and hence are not measurable with respect to the information set F_{t-1}, we will say that the economy has unpredictable mean return, variance, and covariance processes. This is consistent with pure jump and jump-diffusion models with unpredictable jumps in return and/or return variance.

Within the unpredictable class, our setup can accommodate many discrete time stochastic processes for stock returns proposed in the empirical literature if we impose appropriate assumptions on the distributions of the state variables. For example, if $h_{s,t}$ is i.i.d. multinomial, then we obtain the finite mixture of normals model of Kon (1984). As another example, if $h_{s,t}$ is i.i.d. inverted gamma then the stock return follows the Student t-model. If $h_{s,t}$ is i.i.d. gamma, then we obtain the variance gamma process of Madan and Seneta (1987). Many other distributions can be obtained by making appropriate assumptions on the state variables that govern the mean and variance dynamics.

A. The Valuation of Stocks and Bonds

The optimality conditions of the agent's intertemporal optimization problem require that the date t price $\pi_t(\{\underline{B}\})$ of an asset that pays a stream of future cashflows $\{\underline{B}\} = \{B_{t+1}, B_{t+2}, \ldots, B_\infty\}$ satisfy:

$$\pi_t(\underline{B}) = E_t\left\{ \sum_{s=t+1}^{\infty} \rho^{s-t} \cdot (C_s/C_t)^{-b} \cdot B_s \right\}. \tag{6}$$

Based on (6) we can write down the Euler equations for the determination of the interest rate and the stock price.

LEMMA 1: *The Euler equations for the determination of the one-period interest rate from time t to time $t+1$, r_t, and the stock price are*

$$\exp(-r_t) = E_t[\exp(-\gamma_t)], \tag{7}$$

$$1 = E_t[\phi_t] \tag{8}$$

[6]Strictly speaking, except in the continuous time limit, GARCH and EGARCH processes are not consistent with Assumption 2. However, by making proper assumptions on the dynamics of the state variables, we can, in most cases, construct a volatility process with the same spectrum as the GARCH and EGARCH models. The new process is empirically indistinguishable from the GARCH and EGARCH models given only information on stock prices and consumption. See Harvey, Ruiz, and Shephard (1992) for the relationship between ARCH and stochastic variance models.

where

$$\gamma_t = -\ln \rho + b \cdot \mu_{c,t+1} - {}^1\!/_2 \cdot b(1+b) \cdot h_{c,t+1}, \tag{9}$$

$$\phi_t = \exp(\mu_{s,t+1} - r_t - b \cdot \sigma_{cs,t+1}). \tag{10}$$

Proof: See Appendix A.

Note that the variable γ_t in (7) and (9) can be interpreted as the one-period interest rate at time t if in addition to the information in F_t, the agent also knows the value of the state variables at time $t+1$: $\{\underline{Y}_{t+1}, \underline{U}_{t+1}\}$. Similarly, the variable ϕ_t in (8) and (10) can be interpreted as the ratio of the perfect foresight stock price (the stock price if the agent also knows $\{\underline{Y}_{t+1}, \underline{U}_{t+1}\}$) to the true stock price.

The above Euler equations can be further simplified when the conditional mean and variance of the consumption growth, the mean of the stock return, and the covariance between the stock return and consumption growth are predictable processes.

COROLLARY 1: *If $\mu_{c,t}$, $\mu_{s,t}$, $h_{c,t}$, and $\sigma_{cs,t}$ are only functions of $\{\underline{Y}_\tau, \underline{U}_\tau; \tau < t\}$ and hence are measurable with respect to F_{t-1}, then the Euler equations for the determination of the interest rate and the stock price are*:

$$r_t = -\ln(\rho) + b \cdot \mu_{c,t+1} - {}^1\!/_2 \cdot b(1+b) \cdot h_{c,t+1}, \tag{11}$$

$$1 = \exp(\mu_{s,t+1} - r_t - b \cdot \sigma_{cs,t+1}). \tag{12}$$

Proof: If $\mu_{c,t+1}$, $\mu_{s,t+1}$, $h_{c,t+1}$, and $\sigma_{cs,t+1}$ are measurable with respect to F_t, (7) and (8) reduce to $r_t = \gamma_t$ and $1 = \phi_t$. With (9) and (10), the result follows immediately. Q.E.D.

Notice that Corollary 1, and in particular equation (12), holds even if $h_{s,t}$ is not measurable with respect to F_{t-1}. In other words, (11) and (12) are consistent with an unpredictable idiosyncratic component in $h_{s,t}$. In Section III, this will permit us to derive a preference-free valuation formula even if $h_{s,t}$ is not predictable.

This completes our description of the basic economic model. We now turn to the problem of option valuation.

II. Pricing European Options on Individual Stocks

Based on equation (6), the time 0 value, Π_0, of a European call option on the stock, with exercise price K and expiration date T, must satisfy:

$$\Pi_0 = E_0\left\{ \rho^T (C_T/C_0)^{-b} \cdot \mathrm{Max}[0, S_T - K] \right\}. \tag{13}$$

Let $G_T \equiv \sigma(\underline{Y}_\tau, \underline{U}_\tau; \tau \le T)$ be the σ-algebra generated by $(\underline{Y}_\tau, \underline{U}_\tau; \tau \le T)$. That is, G_T contains all the information about the state variables up to the expiration date of the option, T. In particular, G_T contains the entire path of the variance processes. Applying the law of iterated expectations by first conditioning on $F_0 \vee G_T$, (13) can be rewritten as:

$$\Pi_0 = S_0 \cdot E_0\Big\{E_0\Big[\rho^T (C_T/C_0)^{-b} \cdot \text{Max}[0, S_T/S_0 - K/S_0]|G_T\Big\}. \tag{14}$$

This expression can be simplified to obtain a useful option-pricing formula.

PROPOSITION 1: *The price of a call option on the stock with a strike price K and maturity date T is*

$$\Pi_0 = E_0\{S_0 Q_1(G_T) \cdot N[d_1(G_T)] - K \cdot Q_2(G_T) \cdot N[d_2(G_T)]\} \tag{15}$$

where,

$$d_1(G_T) = \frac{\ln[S_0 Q_1(G_T)/(K \cdot Q_2(G_T))] + \tfrac{1}{2}\sum_{t=1}^{T} h_{s,t}}{\left[\sum_{t=1}^{T} h_{s,t}\right]^{1/2}},$$

$$d_2(G_T) = d_1(G_T) - \sum_{t=1}^{T} h_{s,t},$$

$$Q_1(G_T) = \exp\left\{\sum_{t=1}^{T} \mu_{s,t} - \gamma_t - b\sigma_{cs,t}\right\},$$

$$Q_2(G_T) = \exp\left\{-\sum_{t=1}^{T} \gamma_t\right\}.$$

Proof: See Appendix A.

Proposition 1 yields the option-pricing formula with time-varying processes for the stock variance and the interest rate where their evolution over time is driven by the evolution of the state variables. Notice that the formula is not preference free in general as $Q_1(G_T)$ and $Q_2(G_T)$ depend on the preference parameter b and parameters determining the consumption process.

As a special case, we can also obtain a valuation formula for options on aggregate consumption. Many authors have interpreted options on aggregate consumption to correspond to options on the market portfolio. As a first step, we need to determine the value of the market portfolio. In many cases, for example with logarithmic utility (Rubinstein (1976)), it can be shown that the value of the market portfolio determined from the Euler equation is proportional to the current value of aggregate consumption. Therefore, the market portfolio is perfectly correlated with aggregate consumption and we can use the above option valuation formula by substituting $h_{c,t}$ for both $h_{s,t}$ and $\sigma_{cs,t}$. The model in Bailey and Stulz (1989) is a special case of our model under these conditions.

The formula in Proposition 1 is the fundamental result on which this paper is based. In subsequent sections, we will specialize this formula to specific situations of interest.

III. Option Pricing with Predictable Mean, Variance, and Covariance

When the mean and variance of consumption growth, the mean of the stock return, and the covariance between the stock return and consumption growth are predictable, our general pricing formula reduces to a preference-free formula. This formula can be written as an expected Black-Scholes formula where the expectation is taken with respect to the state variables that govern the evolution of the stock return variance process and the interest rate. Notice that we do not require the predictability condition on the stock return variance.

PROPOSITION 2: *If $\mu_{c,t}$, $\mu_{s,t}$, $h_{c,t}$, and $\sigma_{cs,t}$ are only functions of $\{\underline{Y}_\tau, \underline{U}_\tau; \tau < t\}$ and hence are measurable with respect to F_{t-1}, then the time 0 value of a European call option, with strike price K and time to maturity T, on the stock is*

$$\Pi_0 = E_0\{S_0 \cdot N[d_1(G_T)] - K \cdot \exp(-R(G_T)T) \cdot N[d_2(G_T)]\} \tag{16}$$

where

$$d_1(G_T) = \frac{\ln[S_0/(K \cdot \exp(-R(G_T)T)] + \frac{1}{2} \cdot \overline{h}_s(G_T)T}{\left[\overline{h}_s(G_T) \cdot T\right]^{1/2}},$$

$$d_2(G_T) = d_1(G_T) - \overline{h}_s(G_T)T,$$

$$R(G_T) = \sum_{t=0}^{T-1} r_t/T,$$

$$\overline{h}_s(G_T) = \sum_{t=1}^{T} h_{s,t}/T.$$

Proof: By observation, the formula (15) becomes a preference-free expected Black-Scholes formula if $Q_1(G_T) = 1$. A necessary and sufficient condition for this condition to hold is that $\mu_{s,t} - \gamma_t - b\sigma_{cs,t} = 0$. When $\mu_{c,t}$, $\mu_{s,t}$, $h_{c,t}$, and $\sigma_{cs,t}$ are predictable, this condition follows from Corollary 1. Q.E.D.

Since the uncertainty only enters the formula through the average interest rate and the average variance, we can think of the expectation as being taken with respect to the joint distribution of the average interest rate and the average variance. Hence, once the stochastic processes for the interest rate and the stock variance are specified, we can value options as simple expected discounted cash flows without explicit information about the consumption process or the utility-based parameters. All the dynamics of the consumption process relevant to option valuation are embodied in the interest rate process. The pricing formula is preference free as it does not explicitly involve the preference parameters other than through the interest rate. Option values using the formula can be computed using Monte Carlo simulation (Boyle (1977)) or other numerical methods.

In the above formula, it is not possible to eliminate the dependence on the stochastic nature of the interest rate and collapse it into the price of a discount bond maturing at date T. In many stochastic interest rate models (e.g., Merton (1973), Amin and Jarrow (1992), and Turnbull and Milne (1991)), this is possible. In general, the stochastic nature of the interest rate creates dynamic hedging demands that must be reflected in option values. In fact, we cannot eliminate this dependence even when the interest rate process is uncorrelated with the stochastic volatility process of the stock. However, if we restrict the stock volatility to be constant and if the distribution of $\exp[-\Sigma_{t=1}^{T} r_{t-1}]$ is lognormal as assumed by these authors, then their result can be obtained. We now demonstrate this result.

From Proposition 2, we can value options using risk-neutral or preference-free valuation. Therefore,

$$\pi(\text{Call}) = E_{\tilde{Q}}\left[\exp\left[-\sum_{t=1}^{T} r_{t-1}\right]\left(S_0\exp\left[\sum_{t=1}^{T}\left(r_{t-1} - {}^1\!/_2 h_{s,t} + \tilde{\varepsilon}_{s,t}\right)\right] - K\right]^{+}\right]$$

$$= E_{\tilde{Q}}\left[S_0\left(S_0\exp\left[\sum_{t=1}^{T}\left(-\frac{1}{2}h_{s,t} + \tilde{\varepsilon}_{s,t}\right)\right] - K\exp\left[-\sum_{t=1}^{T} r_{t-1}\right]\right]^{+}\right] \quad (17)$$

where

$$\tilde{\varepsilon}_{s,t} = \varepsilon_{s,t} + \mu_{s,t} - r_{t-1}$$

and $\tilde{Q}$ is the risk-neutral measure (as defined by Brennan (1979)). Given our assumptions, the distribution of r_t under $\tilde{Q}$ is the same as that under the original probability measure. Therefore, $\exp[\Sigma_{t=1}^{T} r_{t-1}]$ is unconditionally lognormal and further $\tilde{E}[\Sigma_{t=1}^{T} - r_{t-1}]$ is the time "0" price of a discount bond paying \$1 at date "T." Further, under $\tilde{Q}$ and conditional on F_{t-1}, $\tilde{\varepsilon}_{s,t}$ is $N(0, h_{s,t})$. Therefore, using Lemma 2 in Appendix B, the above expectation reduces to:

$$\pi(\text{Call}) = S_0 N(d_1) - KB_0(T)N(d_2), \quad (18)$$

where

$$d_1 = \frac{\ln[S_0/(KB_0(T))] + {}^1\!/_2\,\zeta^2}{\zeta},$$

$$d_2 = d_1 - \zeta,$$

$$\zeta^2 = \text{Variance}\left[\sum_{t=1}^{T}(\varepsilon_{s,t} + r_{t-1})\right].$$

The formula represented by (18) is identical to the stochastic interest rate formula in Merton (1973), Amin and Jarrow (1992), and Turnbull and Milne (1991).

Several other aspects of Proposition 2 are noteworthy. First, as is apparent from the statement of Proposition 2, the above formula is valid even if $h_{s,t}$ is not predictable. However, given that the covariance process is required to be predictable, the unpredictable portion of the stock variance must be restricted to be idiosyncratic.

Second, even though we have derived the above formula using a discrete time framework, the formula is also valid if we define all our processes to be continuous time diffusion processes as long as two conditions are satisfied. First, the consumption and stock return processes over any finite interval must be lognormal, conditional on the path of some state variables $(\underline{Y}_t, \underline{U}_t)$ governing the mean, variance, and covariance (covariation) processes. Second, the two-dimensional Brownian motion governing the evolution of the stock price and the consumption process at any date t, conditional on $(\underline{Y}_t, \underline{U}_t)$, must be independent of the state variables $(\underline{Y}_\tau, \underline{U}_\tau)$ at any future date τ. These two conditions are the continuous time analogues of Assumptions 1 and 2. We do not need to impose predictability of the mean, variance, and covariance processes since the definition of Ito integrals guarantees that this assumption is automatically satisfied if the stock return and consumption processes are diffusions. In particular, the specification given in equation (5), which includes the continuous time limits of the GARCH(1, 1) and EGARCH(1, 1) processes, satisfies these conditions.

Finally, several other option-pricing formulae in the literature are special cases of equation (16) even though many of them have been derived under different conditions. These include the Black-Scholes formula, the Hull-White (1987) stochastic variance stock option valuation formula, the Bailey-Stulz (1989) stochastic variance index option-pricing formula, and the Merton (1973), Amin and Jarrow (1992), and Turnbull and Milne (1991) stochastic interest rate option valuation formulae.

When the conditional mean and variance of consumption growth are deterministic and time invariant, the interest rate as given by the Euler condition (11) is constant. Equation (16) is then the Hull-White (1987) formula. The call option price is computed as the expected Black-Scholes price with the expectation taken with respect to the stochastic average variance over the life of the option. If the stock variance process is also constant, then we obtain the Black-Scholes formula. Finally, the Bailey-Stulz (1989) stochastic variance formula for index options can be obtained by imposing logarithmic preferences and assuming that $\varepsilon_{c,t}$ and $\varepsilon_{s,t}$ are perfectly correlated. Then, the covariance between the stock return and consumption growth is equal to the variance of the consumption growth. Substituting these conditions yields their formula.

Having derived a general option valuation formula with both stochastic variance and stochastic interest rates, we now demonstrate the effects of incorporating these features on option values relative to those obtained from the Black-Scholes formula.

A. The Effect of Systematic Variance

Since the systematic component of the individual stock return variance is related to the interest rate through the consumption variance, we expect that the direction of the bias inherent in Black-Scholes prices will be different between stocks with a strong systematic variance component and those with a strong idiosyncratic variance component. By similar reasoning, the effect of mean reversion in the variance on option prices should also depend on whether the mean reversion is for the systematic component of the variance or for the idiosyncratic component.

We now illustrate some of these relationships using simple simulation experiments. We emphasize that we wish to only demonstrate that the systematic variance and stochastic interest rate can interact to induce interesting pricing implications. An exhaustive analysis is beyond the scope of this paper. For example, it would also be of interest to document the pricing biases when either/or the interest rate and the stochastic variance are significantly higher or lower than their long-term averages. It is likely that the sign and magnitude of the pricing biases will be different across these different cases.

Table I gives the prices of call options from the Black-Scholes formula and the stochastic variance, stochastic interest rate, option-pricing formula given in Proposition 2. Hereafter, we refer to the latter prices as SVR (for Stochastic Volatility and Interest Rate) prices. To ensure a meaningful comparison, the Black-Scholes prices are based on the average simulated variance and the average simulated interest rate. The exact dynamic model for the variance and interest rate processes used in the experiment is:

$$h_{s,t} = \beta \cdot h_{m,t} + (1-\beta) \cdot h_{d,t} \tag{19}$$

$$\ln(h_{m,t}) = \left[\omega_m - \tfrac{1}{2}\sigma_y^2\right] + \alpha_m \cdot \ln(h_{m,t-1}) + \sigma_y Y_{t-1} \tag{20}$$

$$\ln(h_{d,t}) = \left[\omega_d - \tfrac{1}{2}\sigma_u^2\right] + \alpha_d \cdot \ln(h_{d,t-1}) + \sigma_u U_{t-1} \tag{21}$$

$$r_t = r_{t-1} - \theta(h_{m,t} - h_{m,t-1}) \tag{22}$$

where $h_{m,t} \equiv [E(h_{s,t})/E(h_{c,t})] \cdot h_{ct}$ is a scalar multiple of the consumption variance and is scaled such that it has the same unconditional mean as the stock return variance $h_{s,t}$; $h_{d,t}$ is the idiosyncratic stock return variance and is also taken to have the same unconditional mean as $h_{s,t}$; and Y_t and U_t are independent i.i.d. standard normal random variables.

Equation (19) decomposes the stock variance into a systematic component $(\beta \cdot h_{m,t})$ and an idiosyncratic component $((1-\beta) \cdot h_{d,t})$. A value of β close to one represents a strong systematic component while a value of β close to zero indicates a weak systematic component. We equate the unconditional means of $h_{m,t}$ and $h_{d,t}$ to ensure that the unconditional expectation of the stock return volatility is unchanged as we change the value of β in (19). This is the reason why we specify $h_{m,t}$ instead of $h_{c,t}$ directly.

Our theory implies that the interest rate is proportional to the consumption variance, $h_{c,t}$, (see equation (11)). Therefore, interest rate changes are specified to be proportional to changes in $h_{m,t}$ in (22). Thus, we can either specify

Table I

The Joint Effect of Stochastic Variance and Stochastic Interest Rates on Six-Month Call Option Prices

This table gives the call option prices from the Black-Scholes formula (BS price) and the stochastic variance and stochastic interest rate formula in Proposition 2 (SVR price). The simulations are based on a weekly time interval and the following dynamics:

$$h_{s,t} = \beta \cdot h_{m,t} + (1-\beta) \cdot h_{d,t}$$

$$\ln h_{m,t} = \left[(1-\alpha_m)\ln h_{m,0} - \tfrac{1}{2}\sigma_y^2\right] + \alpha_m \cdot \ln h_{m,t-1} + \sigma_y Y_{t-1}$$

$$\ln h_{d,t} = [(1-\alpha_d)\ln h_{d,0} - \tfrac{1}{2}\sigma_u^2] + \alpha_d \cdot \ln h_{d,t-1} + \sigma_u U_{t-1}$$

where $h_{d,t}$, $h_{m,t}$, and $h_{s,t}$ are the idiosyncratic volatility, the systematic volatility, and the overall stock volatility processes respectively, and (U_t, Y_t) are i.i.d. $N(0,1)$ random variables. Parameter values: $\sigma_y = 1/\sqrt{52}$, $\sigma_u = 2/\sqrt{52}$, $r_0 = 0.1/52$, $h_{m,0} = 0.04/52$, and $h_{d,0} = 0.04/52$.

	Low Systematic Variance ($\beta = 0.1$)			High Systematic Variance ($\beta = 0.9$)		
S_0/X[a, b]	BS price[c]	SVR price[d]	% Dif[e]	BS price	SVR price	% Dif
Panel A: $\alpha_m = 1$, $\alpha_d = 1$						
1.10	7.210	7.232	−0.302	7.214	7.289	−1.02
1.00	4.025	3.946	1.988	4.046	4.082	−0.88
0.90	1.579	1.494	5.73	1.607	1.581	1.645
Panel B: $\alpha_m = 1$, $\alpha_d = 0.8$						
1.10	7.325	7.349	−0.321	7.265	7.332	−0.918
1.00	4.218	4.245	−0.636	4.114	4.145	−0.749
0.90	1.780	1.803	−1.276	1.670	1.644	1.549
Panel C: $\alpha_m = 0.8$, $\alpha_d = 1$						
1.10	7.174	7.174	0.0	7.205	7.211	−0.086
1.00	4.005	3.877	3.293	4.059	4.060	−0.026
0.90	1.576	1.452	8.589	1.632	1.628	0.248
Panel D: $\alpha_m = 0.8$, $\alpha_d = 0.8$						
1.10	7.291	7.292	−0.015	7.243	7.249	−0.076
1.00	4.198	4.188	0.244	4.121	4.123	−0.058
0.90	1.775	1.765	0.579	1.696	1.694	0.144

[a] S_0 = Initial stock price ($50).
[b] X = Exercise price.
[c] BS price = Black-Scholes price.
[d] SVR price = Option price from Proposition 2.
[e] % Dif = {[(BS price)/(SVR price)] − 1} · 100.

the process for the interest rate or $h_{m,t}$. Consistent with the decomposition in (19), we specify the process for $h_{m,t}$. The parameters for $h_{m,t}$ in (20) and "θ" in (22) are chosen to obtain reasonable values of the first and second moments of the interest rate. Importantly, we do not need to specify the risk aversion parameters or the consumption process directly.

The parameters σ_y and σ_u determine the variability of the rate of change of the two variance components $h_{m,t}$ and $h_{d,t}$. The parameters α_m and α_d

measure their degree of persistence. Values close to 1 indicate high persistence and slow mean reversion, while values close to 0 indicate low persistence and fast mean reversion. Finally, we set the drift terms ω_m and ω_d as $(1 - \alpha_c)\ln(h_{m,0})$ and $(1 - \alpha_d) \cdot \ln(h_{d,0})$ respectively to ensure that the unconditional expectations of $\ln(h_{m,t})$ and $\ln(h_{d,t})$ are equal to $\ln(h_{m,0})$ and $\ln(h_{d,0})$, respectively. This permits us to isolate the effect of changing the speed of mean reversion (α_m and α_d) from the effect of changing the unconditional expectations of $\ln(h_{m,t})$ and $\ln(h_{d,t})$.

The length of each time interval is taken to be one week. The value of σ_y and σ_u are fixed at $1/\sqrt{52}$ and $2/\sqrt{52}$, respectively. For the case where $\alpha_m = \alpha_d = 1$, these values represent approximately a 14 percent standard deviation for $(\ln h_{m,t} - \ln(h_{m,t-1}))$ and a 28 percent standard deviation for $(\ln(h_{d,t}) - \ln(h_{d,t-1}))$ on an *annualized* basis. By setting σ_u to be bigger than σ_y, we take the position that the idiosyncratic stock return variance is more variable than the systematic component. We fix θ to be 3.0. When $\alpha_m = \alpha_d = 1$, this yields an annualized standard deviation of the interest rate to be 3.72 percent.

The date 0 values: r_0, $h_{m,0}$, and $h_{d,0}$ are taken to be 0.1/52, 0.04/52, and 0.04/52 respectively. Forcing $h_{m,0}$ and $h_{d,0}$ to be equal ensures that when we change β the unconditional standard deviation of the stock return does not change.

In the experiment, we vary the values of β, α_m, and α_d. Choosing different values of β permits us to compare changes in the Black-Scholes bias when the systematic component of the stock return variance becomes more or less important. Changing the values of α_m and α_d permits us to compare the effect of mean reversion in the systematic and idiosyncratic components of the variance.

Table I reports the prices of six-month call options on a stock with varying ratios, (1.10, 1.00, and 0.90), of the stock price to the exercise price. Table II reports the results for one-year options. These tables have four panels. Panel A corresponds to the case when there is no mean reversion in either the systematic variance or the idiosyncratic stock return variance. Panel B reports option values when there is mean reversion in the idiosyncratic stock return variance but there is no mean reversion in the systematic variance. Panel C corresponds to the case when there is mean reversion in the systematic variance but no mean reversion in the stock return variance. Finally, Panel D gives the results when there is mean reversion in both the variance components. In each panel, the first three columns correspond to a small systematic variance component ($\beta = 0.1$) and the last three columns correspond to a large systematic component in the stock return variance ($\beta = 0.9$).

Strictly speaking, the Black-Scholes call price is a concave function of the variance for at-the-money options and a convex function of the variance for sufficiently deep in-the-money or deep out-of-the-money call options.[7] However, for almost all exchange-traded options, the exercise price is within 10

[7]See Hull and White (1987) for details.

Table II

The Joint Effect of Stochastic Variance and Stochastic Interest Rates on One-Year Call Option Prices

This table gives the call option prices from the Black-Scholes formula (BS price) and the stochastic variance and stochastic interest rate formula in Proposition 2 (SVR price). The simulations are based on a weekly time interval and the following dynamics:

$$h_{s,t} = \beta \cdot h_{m,t} + (1-\beta) \cdot h_{d,t}$$

$$\ln h_{m,t} = \left[(1-\alpha_m)\ln h_{m,0} - \tfrac{1}{2}\sigma_y^2\right] + \alpha_m \cdot \ln h_{m,t-1} + \sigma_y Y_{t-1}$$

$$\ln h_{d,t} = [(1-\alpha_d)\ln h_{d,0} - \tfrac{1}{2}\sigma_u^2] + \alpha_d \cdot \ln h_{d,t-1} + \sigma_u U_{t-1}$$

where $h_{d,t}$, $h_{m,t}$, $h_{s,t}$ are the idiosyncratic volatility, the systematic volatility, and the overall stock volatility processes respectively, and (U_t, Y_t) are i.i.d. $N(0,1)$ random variables. Parameter values: $\sigma_y = 1/\sqrt{52}$, $\sigma_u = 2/\sqrt{52}$, $r_0 = 0.1/52$, $h_{m,0} = 0.04/52$, and $h_{d,0} = 0.04/52$.

	Low Systematic Variance ($\beta = 0.1$)			High Systematic Variance ($\beta = 0.9$)		
$S_0/X^{a,b}$	BS price[c]	SVR price[d]	% Dif[3]	BS price	SVR price	% Dif
			Panel A: $\alpha_m = 1$, $\alpha_d = 1$			
1.10	9.467	9.600	−1.383	9.476	9.768	−2.997
1.00	6.367	6.409	−0.647	6.400	6.662	−3.936
0.90	3.555	3.470	2.456	3.607	3.722	−3.077
			Panel B: $\alpha_m = 1$, $\alpha_d = 0.8$			
1.10	9.668	9.821	−1.557	9.621	9.886	−2.676
1.00	6.683	6.892	−3.037	6.582	6.809	−3.332
0.90	3.941	4.159	−5.251	3.801	3.890	−2.291
			Panel C: $\alpha_m = 0.8$, $\alpha_d = 1$			
1.10	9.445	9.407	0.408	9.476	9.483	−0.074
1.00	6.377	6.104	4.471	6.425	6.427	−0.042
0.90	3.594	3.159	13.771	3.652	3.649	0.097
			Panel D: $\alpha_m = 0.8$, $\alpha_d = 0.8$			
1.10	9.652	9.653	−0.008	9.579	9.586	−0.069
1.00	6.684	6.678	0.083	6.578	6.584	−0.082
0.90	3.959	3.949	0.240	3.834	3.836	−0.042

[a] S_0 = Initial stock price ($50).
[b] X = Exercise price.
[c] BS price = Black-Scholes price.
[d] SVR price = Option price from Proposition 2.
[e] % DIF = {[(BS price)/(SVR price)] − 1} · 100.

percent of the stock price. Ignoring interest rate effects for the moment, for these options, Jensen's inequality implies that the stochastic nature of the variance reduces the call price relative to the Black-Scholes price evaluated at the average variance level. In other words, the stochastic variance acting by itself causes the Black-Scholes formula to overprice call options (which are not extremely deep in or out of the money). However, the stochastic variance induces more complicated effects if it is systematic and hence is related to the stochastic interest rate.

Assuming that stock and bond prices are positively correlated (which is normally the case), then following Merton (1973), stochastic interest rates increase the option price (relative to a constant interest rate model) above the Black-Scholes value when the variability of the interest rate is high. The effect is reversed if the variability of the interest rate is low. Of course, when the interest rate variability is low, the effect of the stochastic nature of the interest rate is small. Hence, it is very likely to be the case that stochastic interest rates induce higher option prices, i.e., the Black-Scholes formula will underprice call options.

In summary, for the options of most interest, stochastic volatility causes option values to increase and stochastic interest rates cause option values to decrease if each of these effects act by themselves. However, their combined effect should depend on the relative importance (variability) of each of these two processes.

First, we analyze the results in Table I for six-month call options. Comparing the fourth column of Panel A (labeled "% Dif") to the fourth column of Panel B, we observe that mean reversion in only the idiosyncratic part of the stock return variance can cause the Black-Scholes bias to change from positive (the Black-Scholes formula overprices the option) to negative (the Black-Scholes formula underprices the option). This is logical since mean reversion in the idiosyncratic variance effectively reduces the importance of the stochastic variance relative to the stochastic interest rate induced by the systematic volatility component (alternatively, stochastic variance of consumption).

Comparing the fourth column of Panels A and C reveals that mean reversion in the systematic variance alone causes the Black-Scholes overpricing to be even more significant. Again, this is intuitively reasonable since mean reversion in the systematic variance is accompanied by mean reversion in the interest rate. Therefore, the importance of the stochastic interest rate declines relative to the overall stochastic variance. Thus, the difference between Black-Scholes values and SVR values is higher in Panel C than in Panel A.

In Panel D, the prices correspond to significant mean reversion in both the systematic and idiosyncratic variance components. Since mean reversion reduces the effect of the stochastic nature of both these inputs, the Black-Scholes biases are quite small.

When the systematic component of the stock variance is large (last three columns in Table I), the combined effect of the stochastic variance and the stochastic interest rate is more complicated since the two are now highly correlated. In this case, the stochastic variance also works through the stochastic interest rate. Since the direct and indirect effects of the stochastic variance are of opposite direction, the composite effect of the stochastic variance is not very significant. Therefore, the relative difference between the

Black-Scholes price and the SVR price is smaller when the systematic variance component is large (column 7) relative to when the systematic variance component is small (column 4). This fact is apparent by comparing the "% Dif" columns in Panels A and B when the systematic variance component is low or high.

Now consider the results for one-year call option prices reported in Table II. By and large, we notice that the SVR prices are much larger than the Black-Scholes prices relative to those reported for six-month calls in Table I. For the purpose of option valuation, Proposition 2 implies that we can view the stock distribution as risk neutral with an expected rate of return in each period equal to the interest rate. Therefore, the stochastic interest rate appears as a stochastic drift in the stock price distribution. The additional uncertainty due to a stochastic drift induces the total unconditional variance over the life of the option to increase relative to a model in which the interest rate is deterministic. Further, the longer the maturity, the bigger the increase in the total variance due to the stochastic drift. Therefore, SVR call values are higher due to stochastic interest rates and this overpricing increases even further as the maturity increases. Correspondingly, one-year options exhibit much larger overpricing by the SVR formula.

The notable exceptions to the above argument are the option prices in Panel C. These prices correspond to a high rate of mean reversion in the interest rate. With high mean reversion, the stochastic interest rate has only a marginal effect, therefore the stochastic volatility effect dominates. Thus, the Black-Scholes prices are larger.

IV. Option Valuation with Systematic Jumps

Our model development in Section II has been very general. For predictable mean return, variance, and covariance processes, we have a very simple and useful formula. However, when these processes are not predictable, the formula is not so simple. For it to be useful, we need to make it more specific. In general, this will depend on the specific characteristics of the aggregate consumption process and the stock price process. In this section, we consider a specific example that leads to an analysis with jump-diffusion processes. In particular, these processes have received considerable attention in the literature.

We will specify the stock price return as well as the return on aggregate consumption in each period to have a distribution described by a mixture of normal distributions. This distribution has been previously studied by several authors, including Kon (1984), who demonstrates that this fits return data much better than the (log)normal distribution. Further, we will show that by an appropriate parameterization and taking limits, one can obtain a bivariate jump-diffusion process for consumption and asset returns.

Let the time between each successive information release be Δ. Define $\underline{\delta}_t \equiv (\underline{Y}_t, \underline{U}_t)$ to be a composite random variable that corresponds to a sequence of i.i.d. scalar-valued Bernoulli random variables such that

$$\underline{\delta}_t = 1 \text{ with probability } \lambda\Delta,$$
$$= 0 \text{ with probability } 1 - \lambda\Delta,$$

where λ is a constant. As $\underline{\delta}_t$ is scalar valued, we will drop the vector notation for subsequent analysis. We further assume that the distribution of the random shocks $(\varepsilon_{c,t}, \varepsilon_{s,t})$ is given by

$$\begin{bmatrix} \varepsilon_{s,t} \\ \varepsilon_{c,t} \end{bmatrix} \Bigg| \delta_t \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}; \begin{bmatrix} \delta_t \sigma_{js}^2 + \sigma_{ds}^2 \Delta & \delta_t \sigma_{jsc} + \sigma_{dsc}\Delta \\ & \delta_t \sigma_{jc}^2 + \sigma_{dc}^2 \Delta \end{bmatrix} \right) \tag{23}$$

where all the parameters on the right-hand side are assumed to be constant. One can think of the variance component as being composed of two parts. In every period, the stock return has a variance component equal to $\sigma_{ds}^2\Delta$. However, in some periods there is an additional component corresponding to the occurrence of some rare event that causes a large increment to the variance equal to σ_{js}^2. A similar principle applies to the consumption return and the correlation between the two processes. Alternately, when δ_t equals 1, jumps in both the consumption as well as the stock price variance occur, otherwise no jump occurs. Of course, large increases in the variance will in general be accompanied by large jumps in the stock return and the consumption process. In fact, when we take limits to the continuous time case, these two concepts are indistinguishable. Therefore, we will use them interchangeably.

It is possible to augment δ_t such that separate jumps in the stock price and in aggregate consumption are permitted. Any jump in aggregate consumption that does not affect the stock price will not directly affect the option value. The effect will enter into the option price only through the endogenously determined interest rate. Therefore, there does not seem much point in explicitly considering this. Further, any jump that occurs in the stock price when the consumption process does not jump is idiosyncratic. Ball and Torous (1985) show that the presence of idiosyncratic jumps in stock prices leads to relatively minor deviations in prices from corresponding Black-Scholes values. This implies that only those jumps that simultaneously affect both aggregate consumption and the stock price will be important for determining option values. Therefore, we concentrate on such jumps. However, except for tedious algebra and a much larger number of parameters, the other types of jumps mentioned can easily be incorporated.

To complete the model specification, we need to specify $\mu_c(\cdot)$, $\mu_s(\cdot)$ and ρ. Analogous to the variance, let

$$\mu_c(\delta_t) = \mu_{dc}\Delta + \delta_t \cdot \mu_{jc},$$
$$\mu_s(\delta_t) = \mu_{ds}\Delta + \delta_t \cdot \mu_{js}.$$

We will assume that μ_{js}, σ_{js}^2, μ_{jc}, σ_{jc}^2, and σ_{jsc} are also constants and they do not vary with the length of the trading interval. Further, to facilitate continuous discounting in the subsequent derivation, redefine ρ as

$$\rho = \exp(-\beta\Delta),$$

where β is also a constant. With these definitions, the discounting parameter $\gamma(\delta_t)$ defined in Section I can be decomposed into two components,

$$\gamma(\delta_t) = \gamma\Delta + \delta_t\gamma_j \tag{24}$$

where

$$\gamma \equiv \beta + b \cdot \mu_{dc} - \frac{b(1+b)}{2} \cdot \sigma_{dc}^2,$$

$$\gamma_j \equiv b \cdot \mu_{jc} - \frac{b^2}{2} \cdot \sigma_{jc}^2. \tag{25}$$

We first focus on the stochastic process followed by the aggregate consumption and the stock price as the time between successive information releases is reduced to zero. In this case, it can be shown that the joint distribution of (C_T, S_T) is given by (see Appendix A for additional details)

$$C_T = C_0 \exp\left\{\left[\left[\mu_{dc} - \frac{1}{2}\sigma_{dc}^2\right]T + \sigma_{dc}W_{dc}(T) + \sum_{j=0}^{NJ(T)} Y_c(j)\right]\right\},$$

$$S_T = S_0 \exp\left\{\left[\left[\mu_{ds} - \frac{1}{2}\sigma_{ds}^2\right]T + \sigma_{ds}W_{ds}(T) + \sum_{j=0}^{NJ(T)} Y_s(j)\right]\right\}, \tag{26}$$

where $NJ(T)$ is the number of jumps up to time T corresponding to a compound Poisson process with parameter λ and $\sum_{j=0}^{NJ(T)} Y_c(j)$ and $\sum_{j=0}^{NJ(T)} Y_s(j)$ are the cumulative increments up to time T to the logarithm of the consumption and the stock price corresponding to the Poisson jumps. The number of jumps for both C_T and S_T are identical and $Y_c(j)$ is normally distributed with mean $(\mu_{ji} - 1/2\,\sigma_{ji}^2)$ and variance σ_{ji}^2 for $i = c, s$ respectively and the covariance of $Y_c(j)$ and $Y_s(j)$ is equal to σ_{jcs}. Further, $(W_{ds}(T), W_{dc}(T))$ has the same distribution as a two-dimensional Brownian motion evaluated at date T with a correlation equal to $\sigma_{dsc}/(\sigma_{ds}\sigma_{dc})$.

This joint distribution for the stock return and the consumption growth process corresponds to a bivariate jump-diffusion process. The stock price distribution is the same as that used by Merton (1976) and the representation for the consumption process is the starting point for the analysis in Naik and Lee (1990).

Given these distributions, we can now substitute the definition of the parameters from (23), (24), and (25) into our general option valuation formula (Proposition 1) and take limits to obtain the continuous time version of our option valuation formula. This yields

$$\Pi_0(\text{Call}) = \sum_{n=0}^{\infty} \exp(-\lambda T)\frac{(\lambda T)^n}{n!} \cdot [S_0 Q_1(n) N[d_1(n)] - K Q_2(n) N[d_2(n)]]$$

where

$$Q_1(n) = \exp\left\{[\mu_{ds} - \gamma - b\sigma_{dsc}]T + n\left[\mu_{js} - \gamma_j - b\sigma_{jsc} + \tfrac{1}{2}\cdot\sigma_{js}^2\right]\right\},$$

$$Q_2(n) = \exp[-\gamma T - n\gamma_j],$$

$$d_1(n) = \frac{\ln[S_0 Q_1(n)/(K Q_2(n))] + \tfrac{1}{2}\left[\sigma_{ds}^2 + \sigma_{js}^2 n/T\right]T}{\left[\left(\sigma_{ds}^2 + \sigma_{js}^2 n/T\right)T\right]^{1/2}},$$

$$d_2(n) = d_1(n) - \left[\left(\sigma_{ds}^2 + \sigma_{js}^2 n/T\right)T\right]^{1/2}. \tag{27}$$

This is our jump-diffusion call option valuation formula with systematic jump risk. Similarly, one can write down a formula for put options.

The above formula can also be used to value options on the market portfolio. One just needs to substitute the parameters for the market instead of the parameters for the stock and the market variance parameter for the covariance parameters. This is true since, with our utility function and our current assumptions, it can be shown that the value of the market portfolio is proportional to the value of current aggregate consumption where the proportionality factor is determined to satisfy the Euler equation for the market. The result then follows from the observation that the market portfolio is perfectly correlated with aggregate consumption. This is the special case analyzed by Naik and Lee (1990). Correspondingly, their formula is a special case of the above formula.

The Euler equation for stock valuation (equation (8)) implies that for a small time interval Δ,

$$\exp(-[\mu_{ds} - \gamma - b\sigma_{dsc}]\Delta) = \lambda\Delta \cdot \exp\left[\mu_{js} - \gamma_j - b\sigma_{jsc} + \tfrac{1}{2}\sigma_{js}^2\right] + (1 - \lambda\Delta) + o(\Delta).$$

By taking a Taylor series expansion and taking limits as $\Delta \to 0$, this yields

$$\mu_{ds} - \gamma - b\sigma_{dsc} = \lambda\left[1 - \exp\left(\mu_{js} - \gamma_j - b\sigma_{jsc} + (1/2)\sigma_{js}^2\right)\right]. \tag{28}$$

Similarly, the Euler condition corresponding to discount bond prices (equation (9)) is

$$r = \gamma - \lambda(\exp[-\gamma_j] - 1). \tag{29}$$

These two conditions must be satisfied in equilibrium. They yield two additional restrictions that can be imposed on the model to reduce the number of parameters.

When jumps in stock returns are idiosyncratic, there are no jumps in the consumption return. Therefore, γ_j and σ_{jsc} are both zero. The Euler conditions then imply that $r = \gamma$ and $\mu_{ds} - \gamma - b\sigma_{dsc} = \lambda[1 - \exp(\mu_{js} + \frac{1}{2} \cdot \sigma_{js}^2)]$. Substituting these relationships in the general jump-diffusion formula derived earlier yields Merton's (1976) jump-diffusion formula with idiosyncratic jump risk. This is a preference-free option valuation formula and does not depend on any of the utility and consumption parameters.

To understand the effect of systematic jumps on option prices, we provide a sample of European call option prices for a representative set of parameter values using our formula (expression (27)) as well as Merton's (1976) jump-diffusion formula with idiosyncratic jump risk in Table III. As an illustrative example, we assume that the jumps in the stock price and the consumption process are positively correlated. Further, the parameters used to compute prices with Merton's formula are the same as those used in expression (27) to

Table III

The Effect of Systematic Jumps versus Idiosyncratic Jumps on Call Option Prices

This table gives the call option prices from Merton's (1976) Jump-Diffusion Formula and the Systematic Jump Risk Formula (equation 27). Parameter values with time measured in years:

$$\sigma_{dc}^2 = 0.025, \quad \sigma_{jc}^2 = 0.015, \quad \sigma_{ds}^2 = 0.05, \quad \sigma_{js}^2 = 0.025, \quad \sigma_{dsc} = \tfrac{1}{2}\sigma_{ds}\sigma_{dc},$$

$$\sigma_{jsc} = 0.75\sigma_{js}\sigma_{jc}, \quad \ln\rho = -0.07, \quad \mu_{dc} = 0.1, \quad \mu_{jc} = -\tfrac{1}{2}\sigma_{jc}^2,$$

$$\mu_{js} = -\tfrac{1}{2}\sigma_{js}^2, \quad \lambda = 2.0.$$

μ_{ds} and "r" are determined in equilibrium by the Euler conditions.

	$b = 0.5$			$b = 1.0$			$b = 3.0$		
S_0/X[a, b]	IJR[c]	SJR[d]	% Dif[e]	IJR	SJR	% Dif	IJR	SJR	% Dif
Panel A: Option Maturity $= \frac{1}{12}$ Years									
1.1	5.20	5.21	−0.23	5.25	5.28	−0.48	4.97	5.07	−1.99
1.0	1.84	1.84	−0.16	1.87	1.88	−0.47	1.69	1.74	−2.84
0.9	0.34	0.33	3.43	0.35	0.33	6.41	0.32	0.27	17.26
Panel B: Option Maturity $= \frac{1}{2}$ Years									
1.1	8.31	8.33	−0.24	8.55	8.60	−0.54	7.27	7.49	−2.93
1.0	5.54	5.55	−0.22	5.75	5.78	−0.57	4.69	4.86	−3.57
0.9	3.17	3.16	0.13	3.31	3.31	−0.06	2.58	2.65	−2.65
Panel C: Option Maturity = 1 Year									
1.1	11.14	11.16	−0.19	11.58	11.63	−0.44	9.28	9.56	−2.86
1.0	8.56	8.58	−0.19	8.96	9.01	−0.50	6.90	7.15	−3.46
0.9	6.05	6.06	−0.09	6.39	6.42	−0.42	4.69	4.87	−3.72

[a] S_0 = Initial stock price ($50).
[b] X = Exercise price.
[c] IJR = Option values with idiosyncratic jump risk (Merton's (1976) formula).
[d] SJR = Option values with systematic jump risk.
[e] % Dif = [(IJR/SJR) − 1] · 100.

compute option prices. To ensure a "fair" comparison, the interest rate used in Merton's formula is equated to the interest rate obtained in the model with systematic jump risk.

From Table III, we notice that the systematic jump risk becomes important only when the risk aversion coefficient becomes larger than one (i.e., the representative agent is more risk averse than one with logarithmic utility). Further, the model with systematic jumps yields higher option values relative to Merton's formula for in-the-money and at-the-money ($S_0/X \geq 1$) options and significantly lower values for short-term, out-of-the-money options. These biases result from the interaction of two different effects, which are described below.

First, if jumps in the stock price as well as consumption are positively correlated, the stock return premium is higher under the systematic jump case relative to the idiosyncratic jump case since the systematic jump is not diversifiable. This is reflected in the Euler equation (28) for stock valuation. The value of μ_{ds} is higher when σ_{jsc} is positive than when σ_{jsc} is equal to zero. The consequence is that the stock price drifts upwards at a faster rate under systematic jumps than with idiosyncratic jumps. We term this the "drift effect." Second, the marginal rate of substitution tends to jump with the stock return under systematic jumps but not under idiosyncratic jumps. As a consequence, jumps in the stock price that increase the value of the call also tend to increase, on average, the discount rate applied to discount the cash flows on the option. We term this the "discounting effect."

The drift effect causes the call option to be worth more relative to a model with idiosyncratic jumps, whereas the discounting effect leads to the opposite result. For a short term out-of-the-money call option whose value is mainly derived from the possibility of a jump, the discounting effect dominates and hence the price given by Merton's (1976) formula is higher than the price given by our systematic jump formula. However, for longer term options and for in-the-money and at-the-money options whose values are not derived mainly from the potential of having a jump, the drift effect dominates. Hence, the price given by Merton's formula is lower than the price given by our systematic jump formula.

Even though the prices are not reported, the bias in option values reverses itself when the correlation between the consumption and asset jumps is negative. The discounting effect and the drift effect now influence the call option price in exactly the opposite way. Therefore, the sign of the biases is also reversed as expected.

V. Conclusions

This paper provides a model for the valuation of individual stock options when the variance of the underlying stock return is stochastic and is correlated with the variance of the consumption growth or the market return. The variance of the consumption growth, in general, induces a stochastic interest rate. Therefore, our model simultaneously incorporates a stochastic variance as well as a stochastic interest rate in the valuation of options.

Using a generalization of the consumption-based equilibrium approach of Rubinstein (1976) and Brennan (1979), we have derived a general option-pricing formula that is consistent with stochastic stock return variance, stochastic consumption growth variance, stochastic interest rates, as well as the existence of a systematic component in the stock return variance. Further, our framework can include a broad class of popular empirical processes that do not necessarily have diffusion limits. In the special case in which the mean return of both the stock and consumption, the consumption variance and the covariance between the stock and consumption returns are predictable processes, our option valuation formula is preference free.

We have shown that the direction of the bias inherent in prices obtained from the Black-Scholes model is different between stocks with a strong systematic variance component and those with a strong idiosyncratic variance component. Further, the effect of mean reversion in variance on option prices is dependent on whether the mean reversion is related to the interest rate or whether the mean reversion is for the idiosyncratic component that is not related to the interest rate.

We have also derived a closed form formula for call options on stocks with systematic jumps in the stock return. Relative to Merton's (1976) jump-diffusion formula with idiosyncratic jump risk, the systematic jump formula gives higher prices for long maturity calls and short maturity in-the-money calls if the jumps in the stock return and aggregate consumption are positively correlated. However, for short maturity out-of-the-money calls, Merton's formula yields significantly higher values under the same conditions. Further, the direction of these biases is reversed if the jumps in the stock return and aggregate consumption are negatively correlated.

The theory developed here calls for additional research. We have documented that economy wide volatility and interest rate changes can induce different types of pricing biases for options on different types of stocks. Further, it is also likely that different biases will be induced when the interest rate or/and the volatility are significantly different from their long run values. Mean reversion in these processes from extreme values will likely yield different biases depending on whether these variables are lower or higher than their long run values. Future empirical research should focus on incorporating the systematic volatility and stochastic interest rate factors studied here. This research can potentially explain some of the empirical biases documented by authors such as Whaley (1982) and Rubinstein (1985).

Appendix A

Proof of Lemma 1: Let $c_t \equiv \ln[\rho(C_t/C_{t-1})^{-b}]$ and $s_t \equiv \ln[S_t/S_{t-1}]$. Based on (6), the one-period interest rate at time t, r_t, must satisfy

$$\exp(-r_t) = E_t[\exp(c_{t+1})].$$

In terms of stock valuation, since the stock does not pay any dividends prior to date T, the stock return process in the time interval $[0, T]$ must satisfy

$$1 = E_t[\exp(c_{t+1} + s_{t+1})].$$

Using the law of iterated expectations, the above two equations can be written respectively as

$$\exp(-r_t) = E_t[E_t[\exp(c_{t+1})|\sigma(\underline{Y}_{t+1}, \underline{U}_{t+1})]], \tag{A1}$$

$$1 = E_t[E_t[\exp(c_{t+1} + s_{t+1})|\sigma(\underline{Y}_{t+1}, \underline{U}_{t+1})]]. \tag{A2}$$

Under our setup, for any t,

$$\begin{bmatrix} c_t \\ s_t \end{bmatrix} \Bigg| F_{t-1} \vee \sigma(\underline{Y}_t, \underline{U}_t) \sim N\left(\begin{bmatrix} \ln \rho - b\mu_{c,t} + .5bh_{c,t} \\ \mu_{s,t} - 0.5h_{s,t} \end{bmatrix}; \begin{bmatrix} b^2 h_{c,t} & -b\sigma_{cs,t} \\ -b\sigma_{cs,t} & h_{s,t} \end{bmatrix} \right).$$

Evaluating the inner conditional expectations in (A1) and (A2) yields equations (7) through (10). Q.E.D.

Proof of Proposition 1: Define $x_T \equiv \sum_{t=1}^{T} s_t = \ln(S_T/S_0)$, $z_T \equiv \sum_{t=1}^{T} c_t = \ln[\rho^T (C_T/C_0)^{-b}]$ and $\kappa \equiv \ln(K/S_0)$ where c_t and s_t are defined as in the proof for Lemma 1. Equation (14) can now be rewritten as

$$\Pi_0 = S_0 \cdot E_0\{E_0[\exp(z_T) \cdot \mathrm{Max}[0, \exp(x_T) - \exp(\kappa)]|G_T]\}. \tag{A3}$$

Under Assumptions 1 and 2,

$$\begin{bmatrix} z_T \\ x_T \end{bmatrix} \Bigg| F_0 \vee G_T \sim N\left(\begin{bmatrix} \overline{\mu}_{z,T} T \\ \overline{\mu}_{x,T} T \end{bmatrix}; \begin{bmatrix} \overline{h}_{z,T} T & \overline{\sigma}_{zx,T} T \\ \overline{\sigma}_{zx,T} T & \overline{h}_{x,T} T \end{bmatrix} \right)$$

where,

$$\overline{\mu}_{z,T} \equiv \sum_{t=1}^{T} \left(\ln \rho - b \cdot \mu_{c,t} + \frac{1}{2} \cdot b \cdot h_{c,t} \right) / T, \tag{A4a}$$

$$\overline{\mu}_{x,T} \equiv \sum_{t=1}^{T} \left(\mu_{s,t} - \frac{1}{2} \cdot h_{s,t} \right) / T, \tag{A4b}$$

$$\overline{h}_{z,T} \equiv \sum_{t=1}^{T} b^2 h_{c,t} / T, \tag{A4c}$$

$$\overline{h}_{x,T} \equiv \sum_{t=1}^{T} h_{s,t} / T, \tag{A4d}$$

$$\overline{\sigma}_{zx,T} \equiv \sum_{t=1}^{T} - b \cdot \sigma_{cs,t} / T. \tag{A4e}$$

To evaluate the inner conditional expectation in equation (A3), we can apply a change of measure conditional on $F_0 \vee G_T$, taking $\exp(z_T - \overline{\mu}_{z,T}T - 1/2 \cdot \overline{h}_{z,T}T)$ as the Radon-Nikodym derivative. Using Lemma 1 in Appendix B, the conditional distribution of x_T given $F_0 \vee G_T$ under the new measure is normal with mean $\overline{\sigma}_{zxT}T$ and variance $\overline{h}_{xT}T$. Integrating over this distribution yields

$$E_0[\exp(z_T)\cdot \mathrm{Max}[0, \exp(x_T) - \exp(\kappa)]|G_T] = Q_1(G_T)\cdot N[d_1(G_T)] - \exp(\kappa)\cdot Q_2(G_T)\cdot N[d_2(G_T)] \qquad \text{(A5)}$$

where,

$$Q_1(G_T) \equiv \exp\Big[\Big(\overline{\mu}_{z,T} + \overline{\mu}_{x,T} + 1/2\cdot \overline{h}_{z,T} + \overline{\sigma}_{zx,T} + 1/2\cdot \overline{h}_{x,T}\Big)T\Big],$$

$$Q_2(G_T) \equiv \exp\Big[\Big(\overline{\mu}_{z,T} + 1/2\cdot \overline{h}_{z,T}\Big)T\Big],$$

$$d_1(G_T) \equiv \Big[-\kappa + \Big(\overline{\mu}_{x,T} + \overline{\sigma}_{zx,T} + \overline{h}_{x,T}\Big)T\Big]\Big/\Big(\overline{h}_{x,T}T\Big)^{0.5},$$

$$d_2(G_T) \equiv d_1(G_T) - \Big(\overline{h}_{x,T}T\Big)^{0.5}.$$

But, given the definitions (A4a) to (A4e) and equations (9) and (10),

$$Q_1(G_T) = \exp\left\{\sum_{t=1}^{T} \mu_{s,t} - \gamma_t - b\sigma_{cs,t}\right\},$$

$$Q_2(G_T) = \exp\left\{-\sum_{t=1}^{T}\gamma_t\right\},$$

$$d_1(G_T) = \frac{\ln[S_0 Q_1(G_T)/(KQ_2(G_T))] + 1/2 \sum_{t=1}^{T} h_{s,t}}{\left[\sum_{t=1}^{T} h_{s,t}\right]^{1/2}},$$

$$d_2(G_T) = d_1(G_T) - \sum_{t=1}^{T} h_{s,t}.$$

Multiplying (A5) through by S_0 now yields the required formula. Q.E.D.

Convergence Proof for Section IV:[8] With the definitions in Section IV, the consumption process for any fixed date "T" can be represented as

$$C_T = C_0\exp\left\{\sum_{t=1}^{T/\Delta}\left[[\mu_{dc}\Delta + \delta_t\mu_{jc}] - \frac{1}{2}\left[\sigma_{dc}^2\Delta + \delta_t\sigma_{jc}^2\right] + \delta_t\sigma_{jc}u_j(t) + \sigma_{dc}u_d(t)\sqrt{\Delta}\right]\right\},$$

[8] For conciseness, we provide only the general ideas in the proof. More details and formal proofs are available from the authors.

where $(u_d(t), u_j(t))$ for $t = 1, \ldots, T/\Delta$ are i.i.d. bivariate normal (0, 1) random variables such that for a fixed date t, $u_d(t)$ and $u_j(t)$ are independent of each other. Simplifying,

$$C_T = C_0 \exp\left\{ \sum_{t=1}^{T/\Delta} \left[\mu_{dc}\Delta - \frac{1}{2}\sigma_{dc}^2\Delta + \sigma_{dc}u_d(t)\sqrt{\Delta} \right] \right\}$$
$$\cdot \exp\left\{ \sum_{t=1}^{T/\Delta} \delta_t \left[\mu_{jc} - \frac{1}{2}\sigma_{jc}^2 + \sigma_{jc}u_j(t) \right] \right\}.$$

In the limit, as $\Delta \to 0$, the first exponent is equal to $\exp\{[\mu_{dc} - \frac{1}{2}\sigma_{dc}^2]T + \sigma_{dc}W_{dc}(T)\}$ where $W_{dc}(T)$ is a normally distributed random variable with mean zero and variance T which we may as well assume to be a standard Brownian motion evaluated at T. The second exponent has a distribution given by

$$\sum_{n=0}^{T/\Delta} \frac{\left(\dfrac{T}{\Delta}\right)!}{n!\left(\left(\dfrac{T}{\Delta} - n\right)!\right)} (\lambda\Delta)^n (1 - \lambda\Delta)^{(T/\Delta)-n} \, \mathrm{LN}\left[\left(\mu_{jc} - \frac{1}{2}\sigma_{jc}^2\right)n, \sigma_{jc}^2 n\right],$$

where $\mathrm{LN}[x, y]$ is the lognormal distribution with mean x and variance y. In the limit, as $\Delta \to 0$, this expression equals

$$\sum_{n=0}^{\infty} \exp(-\lambda T)\frac{(\lambda T)^n}{n!} \mathrm{LN}\left[\left(\mu_{jc} - \frac{1}{2}\sigma_{jc}^2\right)n, \sigma_{jc}^2 n\right].$$

This is the distribution corresponding to a compound Poisson process evaluated at time T, with a lognormal distribution for the jump magnitude. Therefore,

$$C_T = C_0 \exp\left\{ \left[\left[\mu_{dc} - \frac{1}{2}\sigma_{dc}^2 \right] T + \sigma_{dc}W_{dc}(T) + \sum_{j=0}^{NJ(T)} Y_c(j) \right] \right\}$$

where $\sum_{j=0}^{NJ(T)} Y_c(j)$ is a compound Poisson process evaluated at T whose increments at jump times are Normal$(\mu_{jc} - \frac{1}{2}\sigma_{jc}^2, \sigma_{jc}^2)$ increments at jump times.

Similarly, one can show that the stock price process is given by:

$$S_T = S_0 \exp\left\{ \left[\mu_{ds} - \frac{1}{2}\sigma_{ds}^2 \right] T + \sigma_{ds}W_{ds}(T) + \sum_{j=0}^{NJ(T)} Y_s(j) \right] \right\}$$

where $\sum_{j=0}^{NJ(T)} Y_s(j)$ is a compound Poisson process evaluated at "T" whose increments at jump times are Normal$(\mu_{js} - \frac{1}{2}\sigma_{js}^2, \sigma_{js}^2)$ and the distribution of $W_{ds}(T)$ is that of a Brownian motion evaluated at T. Given our assumptions, the number of jumps as well as jump times for C_T and S_T are identical.

Further, it is apparent that Covariance($\sigma_{ds} W_{ds}(T)$, $\sigma_{dc} W_{dc}(T)$) $= \sigma_{dsc} T$ and Covariance($Y_c(j), Y_s(j)$) $= \sigma_{jcs}$. Q.E.D.

Appendix B

LEMMA 1: *Consider a probability measure Q and two random variables X and Y such that (X, Y) is normally distributed under Q with mean (μ_x, μ_y) and variance-covariance matrix*

$$\begin{bmatrix} \sigma_x^2 & \rho_{xy}\sigma_x\sigma_y \\ \rho_{xy}\sigma_x\sigma_y & \sigma_y^2 \end{bmatrix}.$$

Define $\tilde{Q}$ such that its Radon-Nikodym derivative with respect to Q is given by

$$\frac{d\tilde{Q}}{dQ} = \exp\left[-\mu_y - \frac{1}{2}\sigma_y^2 + Y\right].$$

Then, $X \sim N[\mu_x + \rho_{xy}\sigma_x\sigma_y, \sigma_x^2]$ under $\tilde{Q}$.

Proof: First, we notice that $\tilde{Q}$ is a probability measure since $d\tilde{Q}/dQ \geq 0$ and $E_Q[d\tilde{Q}/dQ] = 1$. To show the appropriate distribution of X under $\tilde{Q}$, consider its density function $\tilde{f}_x(x)$ under $\tilde{Q}$. By the Radon-Nikodym Theorem (Halmos (1974), p. 128),

$$\tilde{f}_x(x) = \int_{-\infty}^{\infty} \exp\left[-\mu_y - \frac{1}{2}\sigma_y^2 + y\right] f_{xy}(x, y)\, dy$$

where $f_{xy}(\cdot, \cdot)$ is the bivariate normal density under Q:

$$f_{xy}(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{(1-\rho_{xy}^2)}} \cdot \exp\left[\frac{-1}{2(1-\rho_{xy}^2)}\left\{\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho_{xy}\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right\}\right]$$

Define

$$\bar{y} \equiv \frac{y-\mu_y}{\sigma_y\sqrt{(1-\rho_{xy}^2)}}, \qquad \bar{x} \equiv \frac{x-\mu_x}{\sigma_x\sqrt{(1-\rho_{xy}^2)}}.$$

We can now rewrite $\tilde{f}_x(x)$ as

$$\tilde{f}_x(x) = g(x)\int_{-\infty}^{\infty} \frac{1}{\sqrt{(2\pi)}\sigma_y\sqrt{(1-\rho_{xy}^2)}} \cdot \exp\left[-\frac{1}{2}\bar{y}^2 + y + \rho_{xy}\bar{x}\bar{y}\right] dy$$

where

$$g(x) = \frac{1}{\sqrt{(2\pi)}\sigma_x} \cdot \exp\left[-\mu_y - \frac{1}{2}\sigma_y^2 - \frac{1}{2}\bar{x}^2\right].$$

Making a change of variable from y to $\bar{y}$,

$$\tilde{f}_x(x) = g(x) \cdot \exp\left[\mu_y + \frac{1}{2}\left[\sigma_y\sqrt{\left(1-\rho_{xy}^2\right)} + \rho_{xy}\bar{x}\right]^2\right]$$
$$\cdot \int_{-\infty}^{\infty} \frac{1}{\sqrt{(2\pi)}} \cdot \exp\left[-\frac{1}{2}\left[\bar{y} - \left(\sigma_y\sqrt{\left(1-\rho_{xy}^2\right)} + \rho_{xy}\bar{x}\right)\right]^2\right] dy.$$

Noting that the integral equals 1 and simplifying the remaining expression yields

$$\tilde{f}_x(x) = \frac{1}{\sigma_x\sqrt{(2\pi)}} \cdot \exp\left[-\frac{1}{2}\left[\bar{x}\sqrt{\left(1-\rho_{xy}^2\right)} - \sigma_y\rho_{xy}\right]^2\right]$$
$$= \frac{1}{\sigma_x\sqrt{(2\pi)}} \cdot \exp\left[-\frac{1}{2}\left\{\frac{x-\mu_x}{\sigma_x} - \sigma_y\rho_{xy}\right\}^2\right]$$
$$= \frac{1}{\sigma_x\sqrt{(2\pi)}} \cdot \exp\left[-\frac{1}{2}\{x - \mu_x - \rho_{xy}\sigma_x\sigma_y\}^2/\sigma_x^2\right].$$

This is the normal density with mean $\mu_x + \rho_{xy}\sigma_x\sigma_y$ and variance σ_x^2. Q.E.D.

LEMMA 2: *Given that X and Y are normal random variables with mean zero and variances σ_x^2 and σ_y^2 respectively,*

$$E\left[K_1 \cdot \exp\left(X - \frac{1}{2}\sigma_x^2\right) - K_2 \cdot \exp\left(Y - \frac{1}{2}\sigma_y^2\right)\right]^+$$
$$= K_1 \cdot N(\eta) - K_2 \cdot N(\eta - \zeta)$$

where

$$\eta = \frac{\ln(K_1/K_2) + \tfrac{1}{2}\zeta^2}{\zeta^2}$$

and ζ^2 is the variance of $(X - Y)$.

Proof: See Lemma 5, Appendix A in Amin and Jarrow (1992).

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