THE JOURNAL OF FINANCE • VOL. XLVIII, NO. 2 • JUNE 1993

Spanning with Short-Selling Restrictions

MARTIN RAAB and ROBERT SCHWAGER*

ABSTRACT

In principle, the set of attainable payoff vectors is reduced if assets cannot be sold short. However, we show that the original space of payoff vectors is spanned despite short sale restrictions if there is one additional asset whose payoff is a positively weighted sum of the payoffs of the original assets. For example, this condition is automatically fulfilled if the original assets are stocks and the additional asset is an index future consisting of these stocks.

IN REAL WORLD FINANCIAL markets short sales are restricted. In models of asset markets,[1] however, it is assumed that unlimited short sales of assets are possible. This assumption is crucial since short-selling restrictions affect the completeness of a market system or the degree of its incompleteness, expressed by the difference between the number of states of nature and the number of linearly independent assets. Litzenberger and Ramaswamy (1980) and Dybvig and Ross (1986) are examples of models with short-selling restrictions. However the implication of short-selling restrictions for market completeness is not discussed in these papers nor in any other to our knowledge.

In this note, we start from an original set of assets and impose a short-selling restriction on all these assets. We then introduce one additional asset which is a positively weighted sum of the original assets. We show that this is sufficient to span the same space of payoffs as was spanned by the original assets with the possibility of unlimited short sales. This implies that the assumption of no short-selling restriction in asset market models can be relaxed in most applications with only one additional asset.

I. The Model

Consider a market for at least $F + 1$ assets that pay off in S different states of nature where $F \leq S$. Let y_{sf} denote the (monetary) payoff of asset f in state s. Let $\mathbf{Y}$ represent the $S \times F$ payoff matrix of the first F assets and $\mathbf{y}_{F+1}$ the $S \times 1$ payoff vector of the last asset. We make the following assumptions:

ASSUMPTION 1: *The payoff matrix* $\mathbf{Y}$ *is of full rank.*

* Universität Mannheim, Fakultät für Volkswirtschaftslehre. We would like to thank Matthias Blonski and an anonymous referee of this journal for helpful discussions.

[1] See for example the survey by Geanakoplos (1990) and Ingersoll (1987).

ASSUMPTION 2: *There is a vector* $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_F)'$ *with* $\lambda_f > 0$ *for* $f = 1, \ldots, F$ *satisfying* $\mathbf{Y} \cdot \boldsymbol{\lambda} = \mathbf{y}_{F+1}$.

Assumption 1 states that the assets $f = 1, \ldots, F$ have linearly independent payoffs and hence are nonredundant. Assumption 2 requires that the payoff vector of the last asset can be achieved by a portfolio consisting of a positive amount of *each* of the other assets. Note that we allow for $F < S$ so that the result applies both for complete and incomplete markets. We consider short-selling restrictions for the assets $f = 1, \ldots, F$ but no short-selling restriction for the asset $F + 1$.

The theorem states that any payoff vector which can be achieved by an *arbitrary* portfolio of the assets $f = 1, \ldots, F$ can be achieved by a portfolio of the assets $f = 1, \ldots, F, F + 1$ *which satisfies the short-selling restrictions*.[2] The result is illustrated for two states of nature and two original assets in Figure 1.

THEOREM: *Assume Assumption 1 and Assumption 2 and consider any payoff vector* $\mathbf{x} \in \mathbb{R}^S$ *such that there is some* $\mathbf{m} \in \mathbb{R}^F$ *satisfying* $\mathbf{x} = \mathbf{Y} \cdot \mathbf{m}$. *Then there is a portfolio* $\mathbf{n} = (n_1, \ldots, n_F, n_{F+1})'$ *with* $n_f \geq 0$ *for* $f = 1, \ldots, F$ *such that* $\mathbf{x} = (\mathbf{Y}, \mathbf{y}_{F+1}) \cdot \mathbf{n}$.

Proof: Consider a payoff vector $\mathbf{x} = \mathbf{Y} \cdot \mathbf{m} \in \mathbb{R}^S$. We construct a portfolio $\mathbf{n} = (n_1, \ldots, n_F, n_{F+1})' \in \mathbb{R}^{F+1}$ which yields this return vector and satisfies the short-selling restrictions. It is given by

$$n_{F+1} = \min\{m_f/\lambda_f \mid f = 1, \ldots, F\}$$

and

$$n_f = m_f - \lambda_f n_{F+1} \quad \text{for all } f = 1, \ldots, F.$$

Assumption 2 implies that n_{F+1} is well defined and that $\lambda_f n_{F+1} \leq m_f$ holds for all $f = 1, \ldots, F$. Hence the portfolio $\mathbf{n}$ satisfies the short-selling restrictions. It yields the vector of payoffs

$$(\mathbf{Y}, \mathbf{y}_{F+1}) \cdot \mathbf{n} = \left[\sum_{f=1}^{F} y_{sf}(m_f - \lambda_f n_{F+1}) + y_{s,F+1} n_{F+1} \right]_{s=1,\ldots,S}$$

From Assumption 2, this is equal to $\mathbf{Y} \cdot \mathbf{m} = \mathbf{x}$. □

II. An Example

Assume there are F linearly independent stocks and a stock index future which consists of these F stocks. Future contracts can typically be held in short positions whereas negative holdings of stocks are precluded either by

[2] Two extensions of the theorem are straightforward. First, in Assumption 2 we may replace "$\lambda_f > 0$ for $f = 1, \ldots, F$" by "$\lambda_f < 0$ for $f = 1, \ldots, F$." In this case there may be a short-selling restriction on the last asset as well. Second, there may be more than one asset without short-selling restrictions in addition to the first F assets. Then the theorem holds if Assumption 2 is satisfied for some payoff vector which can be obtained from a portfolio of the unrestricted assets.

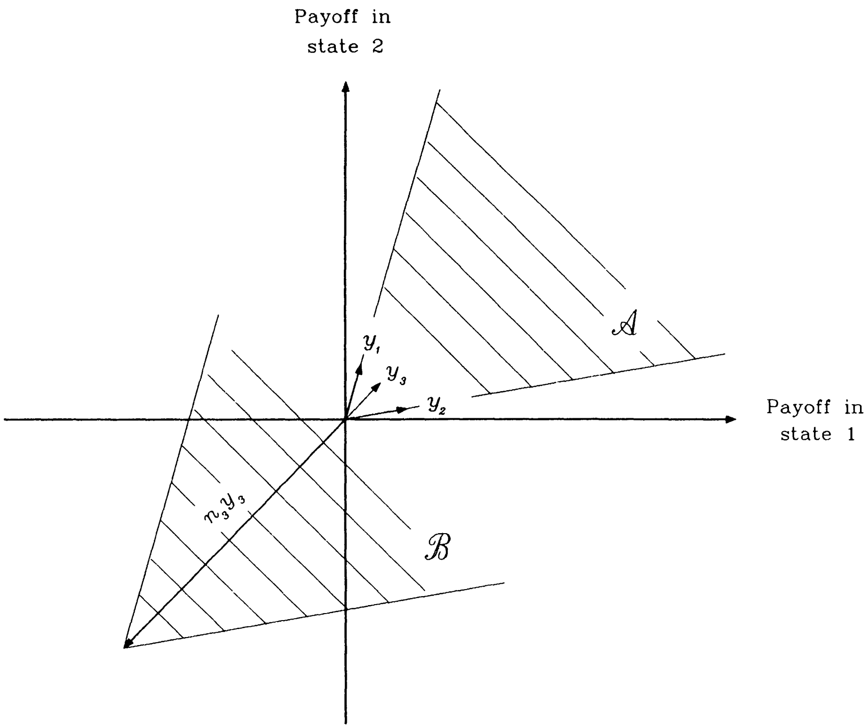

Figure 1. Spanning with short-selling restrictions. If the assets with the payoffs y_1 and y_2 cannot be sold short, they span the convex cone $\mathscr{A}$. If it is possible to sell short an asset with a payoff vector y_3 in the interior of $\mathscr{A}$, the set of attainable payoff vectors is enlarged. For example by selling short $(-n_3)$ units of asset 3 all payoff vectors in the convex cone $\mathscr{B}$ can be obtained. Thus, by selling short sufficiently large amounts of asset 3, the shaded area covers all of $\mathbb{R}^2$. If Assumption 2 would not hold, however, $\mathscr{A}$ and $\mathscr{B}$ would completely lie above or below the straight line given by y_3. Hence without Assumption 2, only the half space above or below this line would be spanned.

legal restrictions or excessive margin requirements (see Sharpe (1991)). By definition, a stock index future is a positively weighted average of the stocks. Hence it is a substitute for short positions in stocks.

REFERENCES

Dybvig, P. H., and S. A. Ross, 1986, Tax clienteles and asset pricing, *Journal of Finance* 41, 751–762.

Geanakoplos, J., 1990, An introduction to general equilibrium with incomplete asset markets, *Journal of Mathematical Economics* 19, 1–38.

Ingersoll, J. E., 1987, *Theory of Financial Decision Making* (Rowman & Littlefield, Totowa, N.J.).

Litzenberger, R. H., and K. Ramaswamy, 1980, Dividends, short-selling restrictions, tax-induced investor clienteles, and market equilibrium, *Journal of Finance* 35, 469–482.

Sharpe, W. F., 1991, Capital asset prices with and without negative holdings, *Journal of Finance* 46, 489–509.