A General Equilibrium Model of International Portfolio Choice
Author(s): Raman Uppal
Source: *The Journal of Finance*, Vol. 48, No. 2 (Jun., 1993), pp. 529-553
Published by: Blackwell Publishing for the American Finance Association
Stable URL: http://www.jstor.org/stable/2328911
Accessed: 25/11/2009 11:30

THE JOURNAL OF FINANCE • VOL. XLVIII, NO. 2 • JUNE 1993

A General Equilibrium Model of International Portfolio Choice

RAMAN UPPAL*

ABSTRACT

We investigate, in a two-country general equilibrium model, whether a bias in consumption towards domestic goods will necessarily lead to a preference for domestic securities. We develop a model where investors are constrained to consume only from their domestic capital stock and where it is costly to transfer capital across countries. In this model, investors less risk averse than an investor with log utility bias their portfolios towards domestic assets. Investors more risk averse than log, however, prefer foreign assets. Thus, this model suggests that it is unlikely that the portfolios observed empirically can be explained by the high proportion of domestic goods in total consumption.

WE INVESTIGATE, IN A two-country, general equilbrium model, whether a bias in consumption toward domestic goods will necessarily lead to a preference for domestic assets in an investor's optimal portfolio. We constrain investors to consume only from their domestic capital stock and introduce a proportional cost for transferring goods from one country. This cost gives rise to endogenous deviations from the law of one price (LOP). Thus, the home and foreign investors do not hold identical portfolios. We show that an investor's optimal portfolio is biased towards domestic equity only if she is *less* risk averse than an investor with log utility. Investors with relative risk aversion greater than one exhibit a preference for the foreign asset. This is because the exchange rate, derived endogenously, is negatively correlated with the return on the foreign asset, and therefore, the translated return on the foreign stock is less risky than that on the domestic stock. Thus, the results of this model suggest that it is unlikely that the preference for domestic assets that is observed empirically can be explained by the high proportion of domestic goods in total consumption.

There exists a considerable body of literature documenting the gains from international diversification. Grubel (1968) was the first to propose that international diversification allows investors to attain lower return variances than those achievable by diversifying domestically. This proposition relies on the correlation between stock indices across countries being significantly less

* Faculty of Commerce, University of British Columbia. I would like to thank Bernard Dumas for his advice. I would also like to acknowledge comments from Franklin Allen, David Backus, Fischer Black, Paul Kleindorfer, Richard Marston, Neil Pearson, George Pennacchi, Piet Sercu, Rob Heinkel, Maurice Levi, Josef Zechner, and especially Vasant Naik. I am also grateful to two anonymous referees and the editor, René Stulz, for their suggestions. I am responsible for all remaining errors.

than one. The results of Levy and Sarnat (1970), Lessard (1973), Solnik (1974a), and Errunza (1983) support these conclusions. Some of these results are challenged by Jorion (1985) on the grounds that they do not consider estimation risk, especially the difference between ex post and ex ante mean returns. However, Eun and Resnick (1988) find that, even after considering estimation and exchange risk,[1] international portfolio strategies significantly outperform a domestically (U.S.) diversified portfolio in out-of-sample periods.

In spite of this overwhelming evidence on the gains from diversifying internationally, Cooper and Kaplanis (1991) and French and Porterba (1991) document that investors' portfolios have a disproportionately high share invested in domestic assets. Black (1974) and Stulz (1981a) show that the presence of taxes on foreign asset holdings may bias the optimal portfolio toward domestic assets. Solnik (1974b), Sercu (1980), Krugman (1981), Stulz (1981b, 1983), Adler and Dumas (1983), and Branson and Henderson (1985) suggest that since consumption is tilted towards domestic goods, the desire to hedge against home inflation may increase the demand for domestic assets relative to foreign assets.[2]

In the articles mentioned above, except for Stulz (1983), the processes for prices, the exchange rate, and the risk-free interest rate are specified exogenously. We extend these models to a general equilibrium setting. As suggested in Stulz (1981b), we model differences in consumption opportunity sets *explicitly* by introducing a proportional cost for transferring goods from one country to another. This "shipping cost" leads to a difference in the price of the consumption good at home and abroad that exists even in equilibrium. We then show that deviations from LOP cause a preference for home equity only if the investor is *less* risk averse than a log investor. Investors more risk averse than log, bias their porfolios toward the foreign asset. This is for two reasons: (a) the exchange rate change is negatively correlated with the return on the foreign stock, and this correlation increases with risk aversion, and (b) the frequency with which capital is transferred between countries also increases with risk aversion. Thus, the translated return on the foreign stock is less risky in real terms than the return on the domestic stock. Therefore, the foreign asset is preferred by more risk-averse investors.

The above result is in contrast to that of Krugman (1981), Stulz (1983), Branson and Henderson (1985), and Eldor, Pines, and Schwartz (1988). In these models, the bias in an investor's portfolio, towards the domestic risk-free asset, increases as the investor's risk aversion increases. In Stockman and Dellas (1989) the existence of the nontraded good[3] biases the home investor's

[1] Glen (1989) finds that, when sufficiently long periods of time are considered, the performance of portfolios hedged against exchange risk may be no better than those unhedged, on a risk-adjusted basis.

[2] For a recent review of the literature on international portfolio choice, see Stulz (1992).

[3] Stockman and Dellas (1989) assume that utility from the nontraded good is separable from the utility from traded goods. Thus, the domestic investor holds all the shares in the firm producing the nontraded good, which biases the portfolio towards domestic assets.

portfolio towards domestic assets. In our model, by explicitly introducing a cost for transferring goods from one location to another, we identify precisely why a good is not traded in certain periods.

The rest of this article is organized as follows. In Section I, we present the details of the model and describe the solution to the central planner's problem. In Section II, we show how the solution to the central planner's problem may be used, in conjunction with the martingale pricing technique, to solve for the optimal wealth process. Our results on the optimal portfolio choice, in the presence of deviations from LOP, are presented in Section III. We conclude in Section IV. All proofs are in the Appendix.

I. The Central Planner's Problem

Given that our primary interest is in the portfolio choice of the home and foreign representative investors, we could formulate and solve the problem using the standard stochastic dynamic programming framework, as used in Merton (1971). Instead, we use the martingale pricing approach suggested by Cox and Huang (1989). Cox and Huang describe the advantages of this approach over the stochastic dynamic programming approach. There is, however, one important difference between the problem addressed by Cox and Huang and the model that we solve here. The optimal portfolio rules obtained by Cox and Huang are for a partial equilibrium model. Since our model is a general equilibrium one, we must supplement the martingale pricing approach with the results on prices and interest rates obtained by solving the central planner's problem. Thus, we start by outlining the solution to the central planner's problem.

The model that we construct is similar to the one in Dumas (1992) and to the certainty model in Black (1973). We consider a world economy with two countries, each having a single, representative investor. For convenience, we label the countries as the home and foreign country, and use an asterisk (*) to denote all variables for the foreign country. Each representative investor is constrained to consume only the physical good available in her country. Let c_t denote the consumption of the home investor and c_t^* consumption abroad. Since our objective is to examine the effect of the constraint that investors can consume only from the capital stock at home, the home and foreign representative investors are assumed to be identical in all other respects—they start out with equal endowments, they have the same rate of impatience, ρ, and their constant relative risk-averse utility functions have the same degree of risk aversion, $1 - \gamma$.

There are two goods, one in each country, and these goods are perfect substitutes in consumption and production. There is a single risky production process available in each country with instantaneous expected rate of return, μ, and instantaneous standard deviation, σ. The output shocks in the two countries are assumed to be uncorrelated. In both locations the good may either be consumed, transferred abroad, or reinvested in a stochastic constant returns to scale production technology. Let K_t denote the stock of capital at home and K_t^* the stock of capital located in the foreign country.

There is a proportional cost for transferring physical capital from one country to another. However, there is no hindrance to trading financial claims on these physical assets. That is, there are no transactions costs and financial markets are frictionless. Let $1 - s$, $0 < s < 1$, denote the proprotional cost of transferring goods from one location to another.[4] This transfer cost may be considered a proxy for transportation costs, contracting costs, or any other hindrance to international trade.[5] One may also think of the shipping cost as a parsimonious way of modeling factor immobility.

Given that the investors in the two locations are symmetric and that the output shocks are uncorrelated, ideally, the economic agents would like the stock of capital in the two countries to be equal. Despite this, an imbalance can develop and even persist. This is because the diversification benefits obtained by shipping may be less than the cost of shipping. Instead, within a certain range, it may be optimal to rebalance through different consumption rates—consumers in the land of abundance consuming more than those in the land of scarcity. Within this range of no shipping, the price of the consumption good abroad, relative to that at home, will not always equal one.[6]

Given the presence of proportional shipping costs, there will exist a *region* of the state space denoted by $\boldsymbol{R}(\lambda)$, within which it is optimal not to ship. The region of no shipping is a cone, delineated by two rays with slope λ and $1/\lambda$, originating from the origin (see Figure 1). When the lower edge of the cone is reached, capital abroad is abundant relative to that at home, and therefore, it is transferred from the foreign to the home location. When the upper edge of the cone is reached, capital is shipped from the home to the foreign location. The optimal shipping policy is to ship the minimal amount necessary to stay inside the cone, $\boldsymbol{R}(\lambda)$.[7]

The processes X_t and X_t^*, with $X_0 = X_0^* = 0$, represent the cumulative amounts of the goods shipped since time 0, from the home to the foreign location, and from the foreign to the home location, respectively. These processes, adapted to the filtration generated by K_t and K_t^*, regulate the joint process for the domestic and foreign capital stocks. The stochastic processes X_t and X_t^* are nonnegative, right continuous, and nondecreasing, increasing in value only when (K_t, K_t^*) would have exited from the cone of no shipping, $\boldsymbol{R}(\lambda)$. Formally, the optimally controlled joint process (K_t, K_t^*) is a

[4] Thus, for every unit of capital shipped, the addition to capital stock at the other location is s units, $s < 1$.

[5] See Gagnon (1989) for a study on the importance of these costs.

[6] Also, within the range of no shipping, the usual sharing rules will not apply: individual consumption will be a function not only of aggregate consumption but also of the allocation of the physical capital stock between the home and foreign country.

[7] For details of estimating this range of tolerance, the cone of no shipping, see Dumas (1992). For a formal proof that the optimal region of no shipping is a cone, See Theorem 4.2 of Davis and Norman (1990).

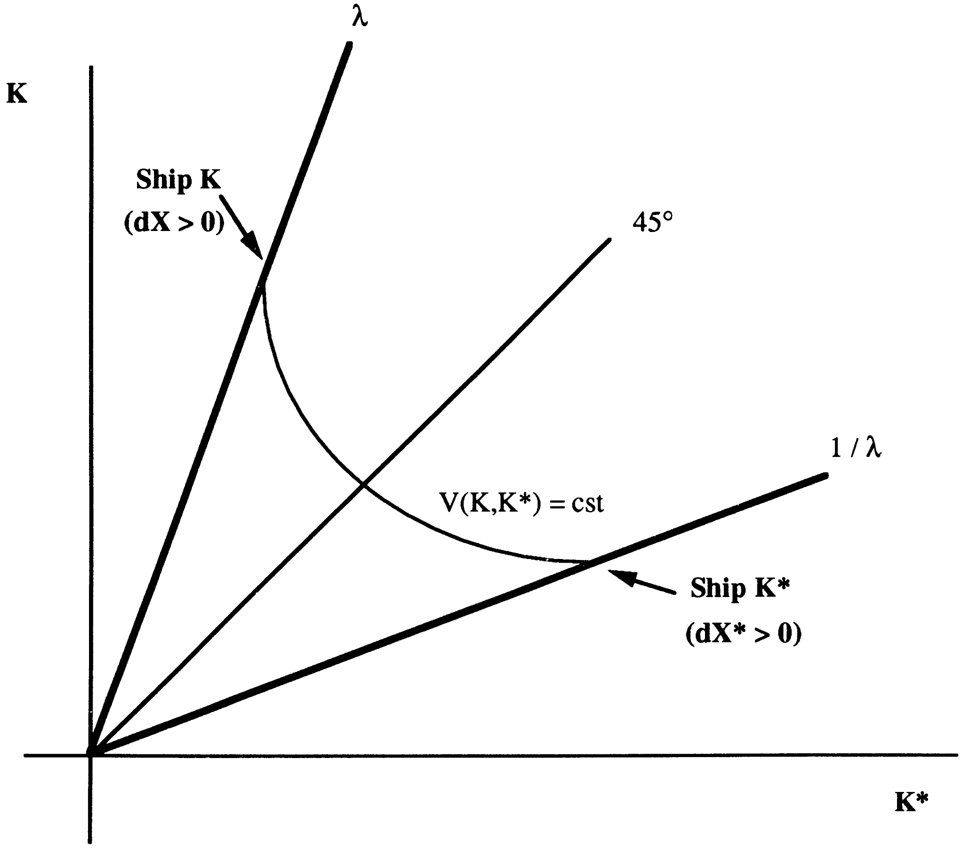

Figure 1. The region of no shipping. The figure shows that within the region $R(\lambda)$, defined by the two rays of slope λ and $1/\lambda$, there is no shipping. When the ratio of the capital stock at home to that abroad, K/K^*, is equal to λ, then capital is shipped from the home to the foreign location ($dX > 0$), and when $K/K^* = 1/\lambda$, capital is shipped from the foreign to the home location ($dX^* > 0$).

reflecting diffusion inside the cone, and the shipping policies, (X_t, X_t^*), are the local times at the boundaries of the cone. Harrison (1985) shows that the processes for K_t and K_t^*, regulated by (X_t, X_t^*), satisfy the Markov property.[8]

We assume that financial markets are complete (see Section II). Then, given that investors in both countries start with equal endowments, the

[8] The mathematical model we construct is based on the theory of regulated Brownian motion developed in Harrison and Taksar (1983) and Harrison (1985). Constantinides (1986), Davis and Norman (1990), and Dumas and Luciano (1991) use this theory to examine optimal portfolios with proportional transaction costs. In contrast to our model, these are partial equilibrium models in which the risk-free rate is given exogenously.

formal problem facing the central planner is:

$$\hat{V}(K_t, K_t^*, t) \equiv \max_{\substack{c,c^* \\ \boldsymbol{R}(\lambda)}} E_{K,K^*} \int_0^{\infty} e^{-\rho t}\left(\frac{c_t^{\gamma}}{\gamma} + \frac{c_t^{*\gamma}}{\gamma}\right) dt, \qquad \gamma < 1, \rho > 0 \quad (1)$$

subject to:

$$dK_t = (\mu K_t - c_t)\,dt + \sigma K_t\,dz_t - dX_t + s\,dX_t^*, \qquad K_0 = \overline{K}_0 \quad (2)$$

$$dK_t^* = (\mu K_t^* - c_t^*)\,dt + \sigma K_t^*\,dz_t^* - dX_t^* + s\,dX_t, \qquad K_0^* = \overline{K}_0^*, \quad (3)$$

or more succinctly,

$$d\mathbf{K}_t = \mu_t\,dt + \Sigma_t d\mathbf{z}_t + \mathbf{s}d\mathbf{X}_t \quad (4)$$

where,

$$d\mathbf{K}_t = \begin{bmatrix} dK_t \\ dK_t^* \end{bmatrix}, \qquad \mu_t = \begin{bmatrix} \mu K_t - c_t \\ \mu K_t^* - c_t^* \end{bmatrix}, \qquad \Sigma_t = \begin{bmatrix} \sigma K_t & 0 \\ 0 & \sigma K_t^* \end{bmatrix},$$

$$d\mathbf{z}_t = \begin{bmatrix} dz_t \\ dz_t^* \end{bmatrix}, \qquad s = \begin{bmatrix} -1 & s \\ s & -1 \end{bmatrix}, \qquad d\mathbf{X}_t = \begin{bmatrix} dX_t \\ dX_t^* \end{bmatrix}.$$

Since we are solving an infinite horizon problem, and the processes (2) and (3) are time homogeneous and Markovian, the solution to this problem is stationary Markov. Thus, we will work with the undiscounted value function, $V(K_t, K_t^*)$ where, $\hat{V}(K_t, K_t^*, t) \equiv e^{-\rho t}V(K_t, K_t^*)$.[9]

The first order conditions for domestic and foreign consumption are:[10]

$$c(K_t, K_t^*) = V_K(K_t, K_t^*)^{1/(\gamma-1)}, \quad (5a)$$

$$c^*(K_t, K_t^*) = V_{K^*}(K_t, K_t^*)^{1/(\gamma-1)}. \quad (5b)$$

Note, that in the presence of shipping costs, the optimal $c(K_t, K_t^*)$ does not have to be equal to the optimal $c^*(K_t, K_t^*)$. Substituting for the optimal $c(K_t,K_t^*)$ and $c^*(K_t, K_t^*)$, the Hamilton-Bellman-Jacobi partial differential equation that one obtains, for values of K_t and K_t^* in the interior of the cone, is:

$$0 = \left(\frac{1}{\gamma} - 1\right)V_K^{\gamma/(\gamma-1)} + \left(\frac{1}{\gamma} - 1\right)V_{K^*}^{\gamma/(\gamma-1)} - \rho V + V_K\,\mu K_t + V_{K^*}\,\mu K_t^*$$
$$+ 0.5\,V_{KK}\,\sigma^2 K_t^2 + 0.5\,V_{K^*K^*}\,\sigma^2 K_t^{*2}, \qquad \frac{1}{\lambda} < \frac{K_t}{K_t^*} < \lambda. \quad (6)$$

[9] Evidence that this transformation works, can be seen by direct substitution of $e^{-\rho t}V(K_t, K_t^*)$ into the Bellman-Jacobi equation for $\hat{V}(K_t, K_t^*, t)$.

[10] Subscripts denote derivatives.

Also, at the boundaries of the cone, $V(K_t, K_t^*)$ must satisfy the *value-matching* conditions:

$$\text{when } K/K^* = 1/\lambda: \qquad sV_K(K, K^*) - V_{K^*}(K, K^*) = 0, \tag{7a}$$
$$\text{when } K/K^* = \lambda: \qquad -V_K(K, K^*) + sV_{K^*}(K, K^*) = 0. \tag{7b}$$

Since the boundary is determined optimally, $V(K_t, K_t^*)$ must also satisfy the *smooth-pasting* conditions.[11]

$$\text{when } K/K^* = 1/\lambda: \qquad \begin{aligned} sV_{KK}(K, K^*) - V_{KK^*}(K, K^*) &= 0; \\ sV_{K^*K}(K, K^*) - V_{K^*K^*}(K, K^*) &= 0, \end{aligned} \tag{8a}$$
$$\text{when } K/K^* = \lambda: \qquad \begin{aligned} V_{KK}(K, K^*) - sV_{KK^*}(K, K^*) &= 0; \\ V_{K^*K}(K, K^*) - sV_{K^*K^*}(K, K^*) &= 0. \end{aligned} \tag{8b}$$

The solution to the central planner's problem in (1) can be obtained by solving the differential equation in (6), subject to the boundary conditions in (7) and (8). We now state some properties of the solution.

THEOREM 1: *If* $\rho > \gamma[\mu - 1/2(1 - \gamma)\sigma^2]$, *then*

(*a*) *A solution to the problem in* (1) *exists.*
(*b*) *This solution is unique.*
(*c*) *The function* $V(K_t, K_t^*)$ *is homogeneous of degree* γ *in* (K, K^*), *that is,* $V(aK_t, aK_t^*) = a^\gamma V(K_t, K_t^*)$.

Let the home good, K_t, be the numeraire and let $p(t)$ denote the (shadow) price of the foreign good, in terms of the domestic good. Then,

$$p(t) \equiv \frac{V_{K^*}(K_t, K_t^*)}{V_K(K_t, K_t^*)}. \tag{9}$$

This price, $p(t)$, can be interpreted as the real exchange rate. When K_t is equal to K_t^*, $p(t) = 1$. As the imbalance between K_t and K_t^* increases, the deviation from LOP increases. Since the extent of the imbalance between K_t and K_t^* is bounded, the deviation from LOP is also bounded. This bound on $p(t)$ is given by the shipping cost, $1 - s$. When $K_t/K_t^* = 1/\lambda$, and the foreign capital stock is abundant relative to capital at home, then the price of the foreign good, $p(t)$, is equal to s, $0 < s < 1$. Analogously, when $K_t/K_t^* = \lambda$, $p(t) = 1/s$.

The solution to the central planner's problem allows us to determine the optimal consumption and shipping policies, the exchange rate, and the risk-

[11] For a discussion of the value-matching and smooth-pasting conditions, see Dumas (1991) and Dixit (1991).

less interest rate, $r(t)$.[12] With this information, we are ready to solve the individual investor's problem.

II. The Individual Investor's Wealth Process

To compute the optimal portfolios held by the individual representative investors we proceed in two steps. First, using the information that we have from the central planner's problem, we obtain the process for optimally invested wealth. Second, in complete markets there must exist a self-financing portfolio strategy that allows the investor to obtain this wealth. We use this fact to derive, in Section III, the optimal portfolios of the home and foreign investors.

For markets to be dynamically complete in the presence of N sources of uncertainty, Duffie and Huang (1985) show that one requires the existence of at least N risky assets and one risk-free asset.[13] And, if markets are complete, then all nonanticipative consumption plans can be financed. The information structure in the economy being considered is generated by $N(=2)$ Brownian motions. Although there is a technological cost for transferring capital from one location to another, the stochastic processes for K_t and K_t^* in (2) and (3) satisfy all the properties required for the martingale representation theorem to apply. In particular, the regulated Brownian motions in (2) and (3) are right continuous with left limits. Note that the transfer terms, dX and dX^* are adapted to the filtration generated by K_t and K_t^*. Thus, no additional uncertainty is generated because of the shipping cost.

We assume that each investor is allowed to invest in the domestic and the foreign capital stock. Besides the investment in the risky production technologies, each investor may invest in an instant maturity, zero net supply bond that is riskless in terms of the home good. Let $B(t)$ denote the price of the domestic bond. These securities, that is, the two risky assets and the bond, are sufficient to complete the market.[14]

Instead of working with the state variables (K_t, K_t^*), which are quantities in different units of account, we will work in terms of the *values* of K_t and K_t^*. This is the convention in the asset-pricing literature, and it will enable us to relate our analysis to that of option pricing. The value of the domestic equity, S_t, is equal to K_t, since the home good is the numeraire good. The value of the foreign capital stock is S_t^*, where $S^*(K_t, K_t^*) = p(t)K_t^*$. The stochastic processes for S_t and for S_t^*, inside the cone, are derived in

[12] This is the return on an instant-maturity bond that pays one unit of the numeraire good at maturity, and in the home country is given by $r(t) = \rho - (EdV_K)/(V_K\, dt)$.

[13] This result on market completeness relies on the fact that all square-integrable martingales, adapted to σ-fields generated by N Brownian motions, can be represented as a stochastic integral against the N Brownian motions.

[14] We could also have permitted investors to hold a foreign bond, a bond that is riskless in terms of the foreign good. However, given that markets are complete, this foreign bond is redundant. Therefore, it would not be possible to determine what amount of borrowing took place in the domestic bond, and what amount in the foreign bond. Also, see Lemma 2 below.

equations (A3) and (A5) in the Appendix, and can be written as:

$$d\mathbf{S}_t = \zeta_t \, dt + \beta_t \, d\mathbf{z}_t \tag{10}$$

where

$$d\mathbf{S}_t = \begin{bmatrix} dS_t \\ dS_t^* \end{bmatrix}, \quad \zeta_t = \begin{bmatrix} \mu_S S_t - c_t \\ \mu_{S^*} S_t^* - p_t c_t^* \end{bmatrix}, \quad \beta_t = \begin{bmatrix} \sigma S_t & 0 \\ \sigma_{1t} S_t^* & \sigma_{2t} S_t^* \end{bmatrix},$$

$$\text{and } d\mathbf{z}_t = \begin{bmatrix} dz_t \\ dz_t^* \end{bmatrix}.$$

Let $\theta^S(t)$ ($\theta^{S^*}(t)$) denote the fraction of shares of the domestic (foreign) risky asset held by the home investor,[15] and $\theta^B(t)$ the number of bonds that the domestic investor holds. Then, $W(t)$, the wealth of the home investor at time t, is the value of her portfolio:

$$W(t) = \theta^B(t)B(t) + \theta^S(t)S(t) + \theta^{S^*}(t)S^*(t) \tag{11}$$

To derive the optimal portfolio weights using the martingale pricing approach, we would like to interpret $W(t)$ as a contingent claim whose value depends only on S_t and S_t^*. In the following lemma we show that $W(t) = W(S_t, S_t^*)$.

LEMMA 1: $W(t) = W(S_t, S_t^*)$, *that is,* $W(t)$ *depends solely on* $S(t)$ *and* $S^*(t)$ *and is independent of* t *and of* $\{S(\tau), S^*(\tau)\}$ *for* $0 \leq \tau < t$.

Using Ito's Lemma and the individual investor's budget constraint, we can now derive the partial differential equation for $W(S_t, S_t^*)$.

THEOREM 2: *The partial differential equation, for* $W(S_t, S_t^*)$, *applicable inside the cone of no shipping, is the following*:

$$W_S(rS - c) + W_{S^*}(rS^* - pc^*) + 0.5W_{SS}S^2\sigma^2 + 0.5W_{S^*S^*}S^{*2}(\sigma_{1t}^2 + \sigma_{2t}^2)$$
$$+ W_{SS^*}SS^*(\sigma\sigma_{1t}) - rW + c = 0, \qquad \text{for } \frac{1}{\lambda s} < \frac{S}{S^*} < \lambda s \tag{12}$$

where the arguments of $r(S_t, S_t^*)$, $p(S_t, S_t^*)$, $W(S_t, S_t^*)$, $c(S_t, S_t^*)$ *and* $c^*(S_t, S_t^*)$ *have been suppressed.*[16]

Within the cone, this equation is equivalent to the "Black-Scholes" valuation equation for a security that pays out a dividend of c_t. The payoff of this security depends on the price of two securities, S_t and S_t^*, that themselves pay a dividend of c_t and $p_t c_t^*$, respectively. For a similar equation, see Merton (1973) or Ingersoll (1987).[17]

[15] The total number of shares outstanding, for each firm, is standardized to be one.

[16] The quantities $r(S_t, S_t^*)$, $p(S_t, S_t^*)$, $c(S_t, S_t^*)$ and $c^*(S_t, S_t^*)$ are obtained from the solution to the central planner's problem.

[17] Note that the one term that is missing from the above differential equation is the time derivative of wealth, and the reason for this is the time homogeneity of the infinite horizon problem that we are solving.

To obtain the boundary conditions for this differential equation, note that the policies, X_t and X_t^*, are continuous. Therefore, the path of $W(S_t, S_t^*)$ must be continuous with probability one, which implies that the value of wealth before the transfer must "match" the value of wealth after the transfer. Also, because the shipping costs are proportional, it is optimal to ship only infinitesimal amounts. This allows us to determine the boundary conditions that the differential equation in (12) must satisfy at the points where, $S_t/S_t^* = 1/\lambda s$ and $S_t/S_t^* = \lambda s$:

$$W_S(S_t, S_t^*) - W_{S^*}(S_t, S_t^*) = 0 \tag{13}$$

This equation corresponds to the boundary conditions in (7) that were stated in terms of utility. Equation (13) may be interpreted as a marginal condition that states that transferring capital is optimal only when the return from shipping fully offsets the cost. That is, the effect on wealth of transferring capital is zero. The boundary condition above is similar to the one in Constantinides (1986). Dixit (1991) and Dumas (1991) provide a detailed discussion of these boundary conditions.

To solve (12), one can take advantage of the fact that wealth is homogeneous of degree one in S_t and S_t^* to reduce the partial differential equation to an ordinary differential equation. The details of this transformation, and how the ordinary differential equation is solved, are given in the Appendix. In the next section we use the solution to derive the optimal portfolios of the domestic and foreign investor.

III. Optimal Portfolio Holdings

In this section, we first show how, having obtained the process for optimally invested wealth, one can find the optimal portfolio weights of the investors. We then discuss the conditions that determine the magnitude and direction of the bias in an investor's portfolio and provide the economic intuition for our results. We conclude this section by reporting the results of comparative static exercises with respect to the parameters of the model.

The analogy with option pricing that we alluded to in Section II enables us to see how the optimal portfolio weights are determined. When replicating the value of an option whose payoff depends on the stock price, one needs to figure out how many shares to hold in order to match the cash flow of the option. The replicating strategy then uses bonds to ensure that trading in the stock is self-financing.

In our context, we have shown in Lemma 1 that wealth can be interpreted as a contingent claim whose "payoff" depends only on S_t and S_t^*. We have also derived the process for optimally invested wealth in Section II. We now need to find the trading strategy in the domestic and foreign equity that replicates the optimal wealth process. Not surprisingly, we find below that the optimal number of shares the home investor should hold in the domestic equity, θ_t^S, is given by $W_S(S_t, S_t^*)$. This is similar to the "delta" obtained when finding the number of shares required to replicate the value of a call option. Since wealth depends on S_t and S_t^*, to hedge the second source of uncertainty the investor will need to buy shares in the foreign equity as well. This quantity, $\theta_t^{S^*}$, is equal to $W_{S^*}(S_t, S_t^*)$. The amount invested in the bond, $\theta_t^B B_t$, is given by the difference between the amount of wealth that the investor has, and the amount invested in the risky assets. The optimal portfolio weights, given below, are obtained by dividing the value of the shares by the investor's wealth.

To provide an economically meaningful decomposition of the optimal portfolio policy, we need to define the joint convariation process of two Ito processes.

Definition 1: Let $x_1(t)$ and $x_2(t)$ be two M_1 and M_2 dimensional Ito processes:

$$dx_i = \mu_{x_i}(t)\, dt + \sigma_{x_i}(t)\, dz_{x_i}, \qquad x_i(0) = x_{i0}, \qquad i = 1,2$$

Then, the joint covariation process of $x_1(t)$ and $x_2(t)$ is defined by, $d[x_1, x_2]/dt = \sigma_{x_1}(t)\sigma_{x_2}(t)^T$ and this can be interpreted as the local variance-covariance matrix of $x_1(t)$ and $x_2(t)$.

We now determine the home investor's optimal portfolio rules and provide the economic intuition for this policy.

THEOREM 3: (Theorem 5.1, Huang (1987)) *The home investor's optimal trading strategy, $\{\theta_t^S, \theta_t^{S^*}\}$, and portfolio weights, $\{\pi_t^S, \pi_t^{S^*}\}$, in the domestic and foreign equity are:*

$$\begin{aligned}\{\theta_t^S, \theta_t^{S^*}\} &= (d[W,\mathbf{K}]/dt)(d[\mathbf{K},\mathbf{K}]/dt)^{-1}\Sigma_t\beta_t^{-1} \\ &= \{W_S(S_t, S_t^*), W_{S^*}(S_t, S_t^*)\},\end{aligned} \tag{14}$$

$$\{\pi_t^S, \pi_t^{S^*}\} = \left\{\frac{SW_S(S_t, S_t^*)}{W(S_t, S_t^*)}, \frac{S^* W_{S^*}(S_t, S_t^*)}{W(S_t, S_t^*)}\right\}. \tag{15}$$

The amount invested in the riskless bonds is given by: $\theta_t^B B_t = W_t - [\theta_t^S S_t + \theta_t^{S^*} S_t^*]$.

To interpret the trading strategy $\{\theta_t^S, \theta_t^{S^*}\}$, decompose the expression in (14) into two parts: (a) $(d[W,\mathbf{K}]/dt)\ (d[\mathbf{K},\mathbf{K}]/dt)^{-1}$ and (b) $\Sigma_t\beta_t^{-1}$. Let us look at part (b) first. The first row of the matrix process $\Sigma_t\beta_t^{-1}$ is that trading strategy on S_t and S_t^* whose accumulated gains process is perfectly correlated with respect to the first-state variable K_t. The second row gives the trading strategy perfectly correlated with the second-state variable, K_t^*.[18] Thus, each row of this matrix process can be interpreted as a hedging mutual

[18] Huang (1987, p. 135) shows that the jth row of $\Sigma_t\beta_t^{-1}$ is proportional to the solution to the problem where one chooses $\nu \in R^N$ to maximize $\nu^T\beta_t\Sigma_{jt}^T$ subject to $\nu^T\beta_t\beta_t^T = k$, where k is a constant and Σ_{jt} is the jth row of Σ_t.

fund, a fund that provides a perfect hedge against that state variable.[19] Part (a) of the expression can be interpreted as the *betas* of dW_t on dK_t and dK_t^*. Thus, the beta process tells the investor *how* to hold strategies whose accumlated gains are locally perfectly correlated with the state variables K_t and K_t^*.

Before presenting the results on the optimal portfolio holdings, we show that in this general equilibium model the proportion of wealth invested in the instantaneous riskless assets is zero. This is surprising because one would expect the investor in the country with the smaller capital stock to have an incentive to borrow from the investor in the capital rich country. But, it turns out that borrowing or lending is not required to ensure that the portfolio strategies in Theorem 3 are self-financing. This is because wealth is homogeneous of degree one in S_t and S_t^*, which is a consequence of the linear production technology and the homogeneity of the constant relative risk-averse utility function. This result is shown in the following lemma.

LEMMA 2: *The optimal investment in the domestic bond is zero, implying that investors do not lend or borrow among themselves.*

We now analyze the optimal portfolio of the home investor, $\{\pi_t^S, \pi_t^{S^*}\}$, and compare it to that of the central planner, $\{\Pi_t^S, \Pi_t^{S^*}\}$, which is given by the world market portfolio weights, $\left\{\frac{S_t}{S_t + S_t^*}, \frac{S_t^*}{S_t + S_t^*}\right\}$.[20]

Recall that the home and foreign investors are similar in all respects except that each investor is constrained to consume only from the domestic capital stock. Given this constraint, one may expect agents to invest only in the domestic asset. However, the possibility of shipping in the future affects today's portfolio choice. To see that this is the case, note that the differential equation in (12) can be satisfied by holding only the domestic equity within the cone of no shipping. That is, choosing $W(S_t, S_t^*) = S_t$ will solve the differential equation. But, this portfolio policy will not be optimal at the boundaries of the cone, where capital is transferred, because the policy $W(S_t, S_t^*) = S_t$ does not satisfy the boundary condition given in equation (13).

Since an investor's portfolio is related to the likelihood of shipping, we derive the optimal portfolio weights as functions of the imbalance in the physical capital stock.[21] We define this imbalance by the variable $\omega = \ln(K/K^*)$. The advantage of using this particular definition of the imbalance is that it makes it easier to interpret the results. This is because the conditional variance of the process for ω is constant, and therefore, the drift

[19] The hedge is perfect since markets are complete.

[20] The central planner's portfolio is also equal to a weighted average of the portfolios of the home and foreign investors, where the weights are the relative wealth of the investors, measured in terms of the numeraire K.

[21] The homogeneity of wealth, in K and K^*, allows us to do this.

of this process is sufficient to characterize the frequency of shipping. The process for ω, derived using equations (2) and (3) is,

$$d\omega = \left[-\frac{c}{K} + \frac{c^*}{K^*} \right] dt + \sigma\sqrt{2}\, dz', \tag{16}$$

where $dz' = (dz - dz^*)/\sqrt{2}$ is a standardized Brownian motion. When $K = K^*$ or $\omega = 0$, the home and foreign investors have equal consumption, and therefore, the drift of ω is zero. The sign of the drift is positive when $K > K^*$ and negative when $K < K^*$. Thus, the process tends to move away from the mean rather than being mean reverting. Also, the drift of this process is an increasing function of ω, which implies that ω is likely to be close to the boundaries most of the time. The properties of the process for ω are discussed in greater detail in Dumas (1992).

We are now ready to explain why the home investor does not hold the market portfolio, and how the difference between the investor's portfolio and the market portfolio varies with the imbalance in the physical capital stock.

PROPOSITION 1: *Within the cone of no shipping, the home investor's portfolio differs from the market portfolio that the central planner holds. This difference is largest when the capital stock at home is equal to the capital stock abroad, and smallest at the boundaries of the cone.*

Figure 2 shows the proportion of the home and foreign investor's wealth invested in the domestic capital stock and that of the central planner, for the case where $(1 - \gamma) < 1$. The difference between the market portfolio that the central planner holds and the individual investors' portfolios represents the excess demand for the domestic asset. This excess demand is a consequence of the transfer cost that restricts investors to consume only from the capital stock located in their country of residence.

The difference between the home investor's portfolio and the market portfolio depends on the likelihood of transfer of capital from abroad, which is given by the drift of the process for ω. We know that when the capital stock is perfectly balanced across the two nations, the drift of ω is zero, and therefore, a transfer of capital is least likely. Thus, at this point, the incentive to hedge against the likelihood of a transfer of capital is the smallest, and therefore, the bias in the investor's portfolio is the largest. As the imbalance in the capital stock increases, the drift of the process for ω, and the likelihood of shipping increase, and thus, the bias in the investor's portfolio decreases.

We now relate the direction of the bias in the home investor's portfolio to her degree of relative risk aversion. The direction of the bias in the optimal portfolio will depend on two factors: (a) the frequency of shipping and (b) the variance of the return on the foreign equity, translated into domestic terms. Given the general equilibrium nature of the model, these factors are not independent of the degree of risk aversion. The frequency of shipping is directly related to the investors risk aversion. As risk aversion increases, the

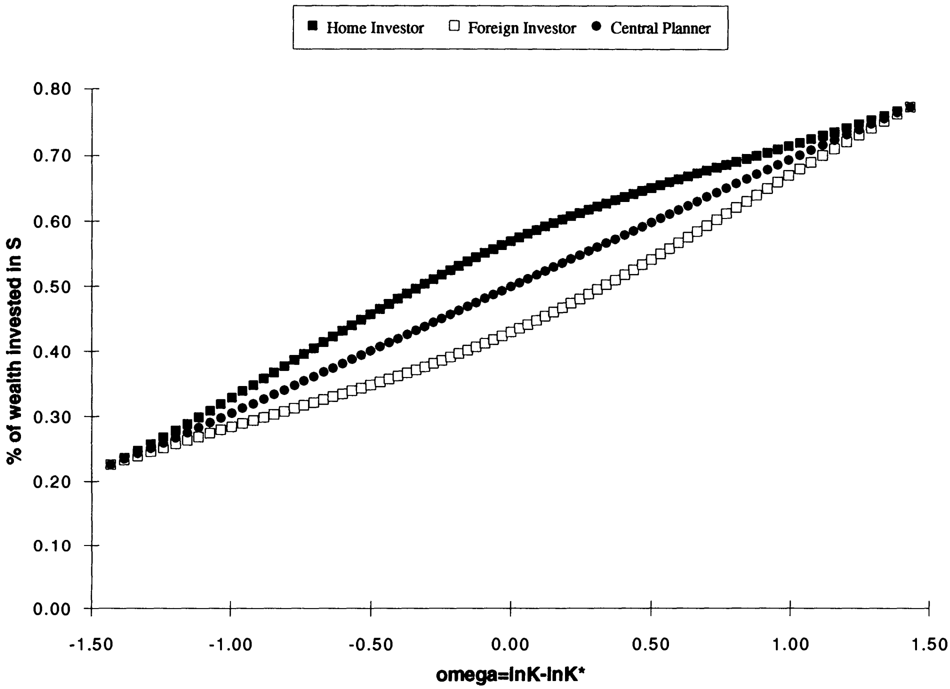

Figure 2. Comparison of portfolios. The figure shows that when the degree of relative risk aversion, $1 - \gamma$, is less than 1, the domestic agent invests a larger proportion of her wealth in the domestic equity, S, relative to the foreign investor, or the central planner. In this figure, the degree of relative risk aversion, $1 - \gamma$, is 0.50, the instantaneous volatility of the home and foreign production processes, σ, is 0.50, the instantaneous mean return of the production processes, μ, is 0.11, the degree of impatience, given by ρ, is 0.10, and the shipping cost, $1 - s$, is 0.18.

diversification benefits from balancing capital stocks increase, and therefore, the cone narrows and the frequency of shipping increases.

We characterize two properties of the instantaneous variance of the translated return on foreign equity, $\mathrm{var}(dS^*/S^*)$ where $S^* = pK^*$. First, the variance of the translated foreign return in less than that before translation: $\mathrm{var}(dS^*/S^*) \leq \mathrm{var}(dK^*/K^*) = \mathrm{var}(dS/S)$.[22] This implies that, from the domestic investor's point of view, the foreign security is less risky than the domestic asset. This is a consequence of the covariance between the exchange rate change and the rate of return on the foreign stock being negative. That is, from (3) and (A4), $\mathrm{cov}(dK^*/K^*, dp/p)/dt = (\sigma^2 K^* p_{K^*})/p$. The sign of this expression depends on that of p_{K^*}, which is negative. This is because the exchange rate, p, is defined as the relative price of the foreign good in terms

[22] Note that $\mathrm{var}(dS/S) = \mathrm{var}(dK/K)$, since the domestic good is the numeraire, that is, $S = K$. From (2) and (3), $\mathrm{var}(dK^*/K^*)$ is the same as that on domestic stock, $\mathrm{var}(dK/K)$, and is equal to σ^2.

of the domestic good. Thus, the price of K^* decreases as it becomes relatively more abundant. Second, the variance of the foreign translated return, $\mathrm{var}(dS^*/S^*)$, decreases as the degree of risk aversion increases. These two results are shown in Figure 3.

Now we can explain why, as the degree of risk aversion increases, the bias towards the domestic asset decreases. As is well known, when the home and foreign investors have log utility, they hold identical portfolios. Investors less risk averse than log, bias their assets towards the more risky asset, the domestic stock. In the limit, when investors are risk neutral, the cone of no shipping is infinitely wide. Hence, a transfer of capital never occurs and the domestic investor never consumes from the foreign capital stock. Moreover, under risk neutrality the exchange rate is always equal to one, and therefore, the nominal risk of investing abroad or at home is the same. Therefore, a risk-neutral investor has no incentive to invest in the foreign stock and holds only the domestic asset.

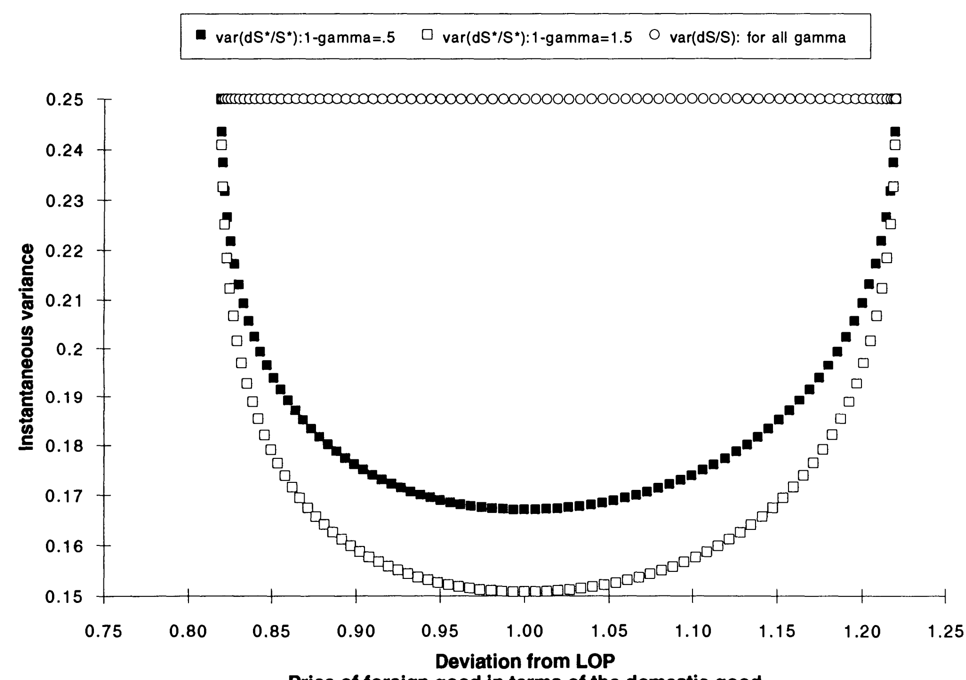

Figure 3. Riskiness of the translated foreign return as a function of risk aversion. The figure shows that, as the degree of risk aversion, $1 - \gamma$, increases, the instantaneous variance of the return on the foreign equity, translated into domestic terms, dS^*/S^*, decreases. Also note that the instantaneous variance of the domestic equity, dS/S, is constant and equal to σ^2. In this figure, the instantaneous volatility of the home and foreign production processes, σ, is 0.50, the instantaneous mean return of the production processes, μ, is 0.11, the degree of impatience, ρ, is 0.10, and the shipping cost, $1 - s$, is 0.18.

Investors more risk averse than log choose to hold more of the relatively less-risky asset.[23] Given the negative covariance between the exchange rate change and the return on foreign equity, this is the foreign stock. Moreover, the frequency of shipping increases with risk aversion. Thus, the home investor consumes more often from the foreign capital stock, which reinforces the incentive to invest in foreign equity since this is even less risky in real terms.

To summarize: As risk aversion increases: (a) The frequency of shipping, and thus, the likelihood of consuming from the foreign capital stock increases, and (b) The riskiness of the return from the foreign asset, translated into domestic terms, declines. This is because the exchange rate and the return on the foreign asset are negatively correlated. Thus, the more risk-averse investor chooses to bias her portfolio towards the asset that is less risky in real terms—the foreign asset.[24] This result is stated formally below and is shown in Figure 4.

PROPOSITION 2: *For the case where the investor is less risk averse than log,* $(1 - \gamma) < 1$, *the home investor exhibits a bias for the local equity. For the case where* $(1 - \gamma) > 1$, *the direction of the bias is reversed. That is, if the investor is more risk averse than log, then she will invest a larger porportion of her wealth in the foreign equity, relative to the portfolio of the central planner.*

The difference between the above results and that of the models of Krugman (1981) and Stulz (1983) can be explained using Jensen's inequality.[25] Comparing our result with that of Krugman's first, note that in Krugman's model the exchange rate is given exogenously. The only assets available to the investor are the domestic and foreign bonds, and therefore, the covariance between the return on these securities and the exchange rate is zero. As is well known, when investors have log utility the foreign and domestic investors choose the same portfolios—in Krugman's model as well as ours. However, in Krugman's model, investors less risk averse than log take on more risk by increasing their investment in the foreign bond—an asset that has a positive expected excess return because of the effect of Jensen's inequality. In our model, investors less risk averse than log take on more risk by investing a larger proportion in domestic equity.

The source of the difference between our results and those of Krugman (1981) is the effect that a change in risk aversion has on the frequency of shipping and the risk of investing abroad. In our model, a change in risk aversion affects the frequency of shipping and also the variance of the exchange rate and its covariance with the return on the foreign stock. As risk aversion decreases, exchange rate volatility declines and the effect of Jensen's inequality is reduced. Also, the decrease in the frequency of shipping implies that it is unlikely that the home investor will ever consume from the foreign

[23] Note, from (A3) and (A5), that the return on S and S^* is positively correlated.

[24] Recall, from Lemma 2, that the equilibrium investment in the bonds is always zero.

[25] I thank the referee for suggesting this explanation.

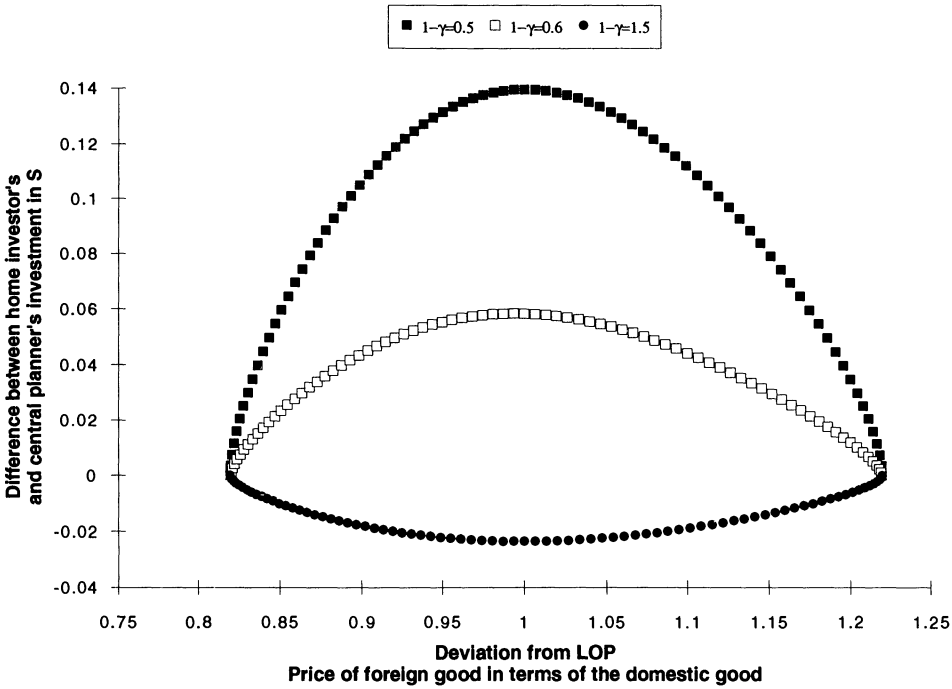

Figure 4. Comparative static results on bias in portfolios with respect to risk aversion. The figure shows that as risk aversion, $1 - \gamma$, increases, the home investor decreases her investment in domestic equity, and therefore, the *difference* between the domestic investor's and central planner's portfolio declines. This figure is drawn for the case where the instantaneous volatility of the home and foreign production processes, σ, is 0.50, the instantaneous mean return of the production processes, μ, is 0.11, the degree of impatience, given by ρ, is 0.10, and the shipping cost, $1 - s$, is 0.18.

capital stock. In the limit, when investors are risk neutral, the effect of Jensen's inequality is eliminated altogether, capital is never transferred, and the home investor does not invest any funds abroad. Thus, in Krugman's model and in our model, as risk aversion decreases, the investor is doing the same thing: holding a more risky portfolio. But, in Krugman's model this is accomplished by investing more in the foreign bond and in our model it is achieved by investing more in the domestic stock.

Stulz (1983) extends the Krugman (1981) model by considering utility functions that do not restrict expenditure shares to be fixed. Stulz then shows that with an increase in risk avesion an investor is less willing to take on price gambles. The agent does this by investing a larger proportion in the domestic bond. The investor in our model behaves the same way: as risk aversion increases she holds a portfolio that is weighted towards the less-risky asset. The only difference is that in our model the less-risky asset is the foreign stock. Note that in our model the expenditure share, that is, the proportion of total consumption that comes from the foreign capital stock, is determined endogenously and depends on the frequency of shipping. When investors are very risk averse, capital is transferred frequently. Thus, the home investor consumes often from the foreign capital stock, which augments the inducement to invest in foreign equity.

We can also compare the results of our model with those of Solnik (1974b), Sercu (1980), and Adler and Dumas (1983). Solnik and Sercu assume the absence of any home inflation and find that the optimal portfolio of an investor can be decomposed into a log investor's portfolio and the domestic risk-free asset. Adler and Dumas show that, in the absence of an asset that is riskless in real terms, an investor's optimal portfolio is a combination of the log portfolio and the minimum variance portfolio, where the minimum variance portfolio will typically consist of the domestic T-bill. Thus, in these models an infinitely risk-averse agent would hold only the domestic risk-free asset, while in our model this investor's portfolio is dominated by the foreign asset for the reasons described above.

Most asset-pricing studies find that relative risk aversion is greater than one.[26] Thus, the results of our model suggest that it is unlikely that the portfolios that one observes empirically can be explained by the bias in consumption towards domestic goods. Even if one were willing to accept that relative risk aversion is less than one, the extent of the portfolio bias explained by the constraint on consumption is quite small. This is especially true given that the process for ω is close to the boundaries for most of the time where the portfolio bias is quite small. One would have to increase the shipping cost to extremely high levels, or reduce the volatility significantly, before one could capture the magnitude of the empirically observed preference for domestic assets. Comparative static results relating the portfolio bias to these parameters are presented below.

PROPOSITION 3: *The bias toward domestic assets increases as*: (*a*) *risk*, σ, *decreases*, *or* (*b*) *the shipping cost*, $1 - s$, *increases*.

We know that in choosing her investment in the foreign equity, the investor hedges against the likelihood of shipping. A decrease in risk, σ, leads to a decrease in the frequency of shipping,[27] and therefore, a fall in the demand for hedging via investment in the foreign equity. Thus, a decrease in σ increases the bias of a portfolio away from the market weights, as shown in Figure 5. Also, reasoning along the same lines, we conclude that as the shipping cost, $1 - s$, increases, and the frequency of shipping declines, investors will hold a smaller proportion of foreign equity. At the other extreme, if the shipping cost were zero, then the foreign and the domestic investors would hold identical portfolios.

[26] See, for example, the survey article by Singleton (1990).

[27] See Dumas (1992, Table 1) for the relationship between the region of no shipping and the parameters of the model.

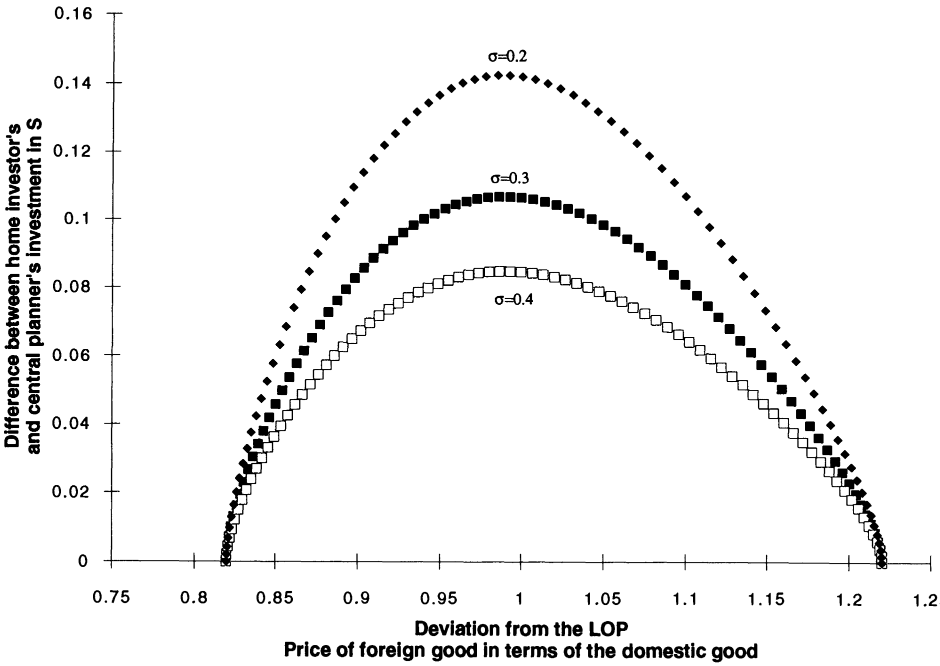

Figure 5. Comparative static results on bias in portfolios with respect to riskiness of production processes (σ). The figure shows that as risk increases, for an investor with $(1 - \gamma) < 1$, the bias towards domestic equity decreases, and therefore, the *difference* between the domestic investor's and central planner's portfolio declines. In this figure, the degree of relative risk aversion, $1 - \gamma$, is 0.50, the instantaneous mean return of the home and foreign production processes, μ, is 0.11, the degree of impatience, ρ, is 0.10, and the shipping cost, $1 - s$, is 0.18.

IV. Conclusion

In this article, we examine the impact of imperfections in the physical goods market on portfolio decisions. We model a two-country world in which there is a proportional cost for transferring capital from one country to another that generates deviations from the law of one price. Despite the constraint that an agent can only consume from the stock of capital at home, we find that an investor's optimal portfolio is biased towards local assets only if she is less risk averse than a log investor. When an investor's relative risk aversion is greater than one, the investor's portfolio is biased towards foreign equity. This is a consequence of the negative covariance between the exchange rate and the return on the foreign equity, which implies that the riskiness of the translated return on the foreign stock is less than that of domestic equity.

Most asset-pricing studies find that the relative risk aversion parameter is larger than one. Thus, our work suggests that it is unlikely that the high proportion of domestic goods in consumption can explain the observed bias in portfolios towards local assets. Even if one is willing to accept a value of less than one for the risk aversion parameter, the portfolio bias that is explained by the high share of domestic goods in consumption is much smaller than that reported in the empirical work of Cooper and Kaplanis (1991) or French and Porterba (1991).

To explain the observed bias in investors' portfolios, one may wish to consider the following extensions to the model analyzed in this article. One could introduce money to analyze the relationship between the optimal portfolio and nominal quantities, especially inflation, in a general equilibrium setting. Another dimension in which the current model could be extended is to introduce a different structure for the transfer costs. Finally, one may wish to examine other factors, such as the additional information costs incurred for analyzing foreign stocks, to explain why investors do not invest a larger proportion of their wealth abroad.

Appendix

Proof of Theorem 1: (a) Essentially, the proof proceeds by showing the existence of an upper bound and a lower bound for $V(K_t, K_t^*)$ when $s = 1$ and $s = 0$, respectively. First consider the case for the upper bound where $s = 1$, that is, when there is no shipping cost. Then the problem is identical to the one considered in Merton (1971) and the exact solution is:

$$\overline{V}(K_t, K_t^*) = \frac{\alpha}{\gamma}(K_t + K_t^*)^\gamma \qquad \text{where } \alpha = \left[\frac{\rho - \gamma[\mu - 0.5(1-\gamma)\sigma^2]}{1-\gamma}\right]^{\gamma-1} \tag{A1}$$

Next, consider the case when shipping costs are infinite. Then no shipping ever takes place, and the autarky level of welfare is given by,

$$\underline{V}(K_t, K_t^*) = \frac{\alpha}{\gamma}\left[(K_t)^\gamma + (K_t^*)^\gamma\right] \tag{A2}$$

Thus, given that $\overline{V}(K_t, K_t^*)$ and $\underline{V}(K_t, K_t^*)$ exist when $0 \leq s \leq 1$, $V(K_t, K_t^*)$ exists.

(b) The uniqueness of the solution to the problem in equation (1) follows from the concavity of $V(K_t, K_t^*)$, which is a result of the concavity of the utility function and the linearity of the constraints in equations (2) and (3).

(c) See Davis and Norman (1990). Basically, given that the constraints are linear and $V(K_t, K_t^*)$ is isoelastic, $V(K_t, K_t^*)$ is homogeneous of degree γ. □

Proof of Lemma 1: The proof that $W(t)$ depends solely on S(t) and S^*(t) but is independent of t and of the paths of $\{S(\tau), S^*(\tau)\}$ for $0 \leq \tau < t$, proceeds in three steps:

Step 1: First establish that $W(t) = W(S_t, S_t^*, t)$, that is, the portfolio choice decisions are indepedent of the history of S_t and S_t^*. This is done by applying

Theorem 5.2 and Corollary 5.2 of Huang (1987), adapted to our context, that is, where the dynamics of the state variables are given by the "regulated" Brownian motions described in (2) and (3). To do this we:

a. Prove that the appropriate state variables, S_t and S_t^*, are Markov processes even in the presence of shipping costs. This follows because the regulated Brownian motions described in (2) and (3) satisfy the Markov property, as shown in Harrison (1985). Therefore, the value function, $V(t)$, depends on only the current value of the state variables, K_t and K_t^*.
b. Show that the optimal consumption process is path independent. This is the case since the optimal consumption process is a solution to the Markov stationary problem in (1). That is, $c(K_t, K_t^*) = V_K(K_t, K_t^*)^{1/(\gamma-1)}$ and therefore, it is independent of the history of K_t and K_t^*.

Step 2: To show that the portfolio choice decisions are independent of time, t, note that since we are solving an infinite horizon problem, where the dynamics of S_t and S_t^* are time homogeneous, the solution is stationary (Markov). This can also be seen by direct substitution of $\hat{V}(K_t, K_t^*, t) \equiv e^{-\rho t}V(K_t, K_t^*)$ into the Bellman-Jacobi equation for $\hat{V}(K_t, K_t^*, t)$.

Step 3: Finally, to show that $B(t)$ and $p(t)$ are independent of t and the history of $\{S_t, S_t^*\}$ (or the history of $\{K_t, K_t^*\}$), note that these prices are functions of $V(K_t, K_t^*)$ and its derivatives. Thus, given that $V(K_t, K_t^*)$ is independent of the history of $\{K_t, K_t^*\}$ and of t, it follows that these price processes are independent of the history of $\{K_t, K_t^*\}$ and of t, as well.

Given that all the variables determining wealth are independent of the past realizations of S_t, S_t^*, and independent of time, we conclude that $W(t) = W(S_t, S_t^*)$. □

Proof of Theorem 2: To derive the partial differential equation for $W(S_t, S_t^*)$, we use the results in Lemma 1 and the fact that the change in the investor's optimally invested wealth, as given by the investor's budget constraint, should be the same as the one obtained from applying Ito's Lemma to $W(S_t, S_t^*)$. To use Ito's Lemma we first derive the processes for S_t and S_t^* within the cone of no shipping.

Since $S_t = K_t$, the process for S_t, inside the cone, is the same as the one for K_t, given in (2):

$$dS_t = (\mu_S S_t - c_t)\, dt + \sigma S_t\, dz_t, \tag{A3}$$

where $\mu_S = \mu$. To derive the process for $S_t^*(K_t, K_t^*) = K_t^* p(K_t, K_t^*)$, we first obtain the process for $p(K_t, K_t^*)$:

$$dp_t = \mu_p\, dt + \sigma K p_K\, dz + \sigma K^* p_{K^*}\, dz^*, \tag{A4}$$

where $\mu_p = [\,p_K(\mu K - c) + 0.5 p_{KK}(\sigma K)^2 + p_{K^*}(\mu K^* - c^*) + 0.5 p_{K^*K^*} (\sigma K^*)^2]$.

Now, using this result, we derive the process for $S_t^*(K_t, K_t^*)$:

$$dS_t^* = (\mu_{S^*} S^* - p_t c_t^*)\, dt + \sigma_{1t} S^*\, dz + \sigma_{2t} S^*\, dz^*, \tag{A5}$$

where, $\mu_S^* = \mu + \mu_p/p_t + \sigma^2 K^* p_{K^*}/pt$, $\sigma_{1t} = \sigma K p_K/p_t$ and $\sigma_{2t} = \sigma(1 + K^* p_{K^*}/p_t)$.

Applying Ito's Lemma to $W(S_t, S_t^*)$, within the region of no shipping, and using the dynamics of S_t and S_t^* given in (A3) and (A5), we get:

$$\begin{aligned} dW(S_t, S_t^*) = \{ & W_S(\mu S_t - c) + W_{S^*}(\mu S_t^* - p_t c^*) + W_{SS^*} S S_t^*(\sigma \sigma_{1t}) \\ & + 0.5 W_{SS}(\sigma S_t)^2 + 0.5 W_{S^*S^*}(\sigma_{1t}^2 + \sigma_{2t}^2) S_t^{*2} \}\, dt + W_S \sigma S_t\, dz \\ & + W_{S^*} S_t^*(\sigma_{1t}\, dz + \sigma_{2t}\, dz^*) \end{aligned} \tag{A6}$$

From the flow budget constraint, for example see Merton (1971), we get:

$$\begin{aligned} dW(t) = [& \theta_t^S S_t \mu + \theta_t^{S^*} S_t^* \mu_{S^*} + \theta_t^B B_t r_t - c_t]\, dt \\ & + \theta_t^S S_t \sigma\, dz + \theta_t^{S^*} S_t^* [\sigma_{1t}\, dz + \sigma_{2t}\, dz^*] \end{aligned} \tag{A7}$$

Since, $W(t) = W(S_t, S_t^*)$, it must be the case that the diffusion terms in (A6) and (A7) are equal, and since dz is uncorrelated with dz^*, this can only be true if

$$\theta_t^S = W_S \tag{A8}$$

$$\theta_t^{S^*} = W_{S^*} \tag{A9}$$

Now, equating the coefficient of dt in equations (A6) and (A7), substituting in the results from (A8) and (A9), and using the self-financing constraint that $\theta_t^B B_t = W_t - [\theta_t^S S_t + \theta_t^{S^*} S_t^*]$, we get equation (10). □

Proof of Theorem 3: To show the second part, that is, $\{W_S, W_{S^*}\} = (d[W, \mathbf{K}]/dt)(d[\mathbf{K}, \mathbf{K}]/dt)^{-1} \Sigma_t \beta_t^{-1}$, note that from (A6) and (4):

$$\begin{aligned} & (d[W, \mathbf{K}]/dt)(d[\mathbf{K}, \mathbf{K}]/dt)^{-1} \Sigma_t \beta_t^{-1} \\ & \qquad = \{W_S, W_{S^*}\} \beta_t \Sigma_t^T (\Sigma_t^T)^{-1} \Sigma_t^{-1} \Sigma_t \beta_t^{-1} = \{W_S, W_{S^*}\}. \end{aligned}$$

The first part has already been proven in (A8) and (A9). These equations give us the optimal number of shares of the domestic and foreign capital stock that the home investor should hold. The portfolio weights are the value of these shares, divided by the investor's total wealth:

$$\pi_t^S = \frac{S_t \theta_t^S}{W(S_t, S_t^*)} = \frac{S_t W_S(S_t, S_t^*)}{W(S_t, S_t^*)}.$$

$$\pi_t^{S^*} = \frac{S_t^* \theta_t^{S^*}}{W(S_t, S_t^*)} = \frac{S_t^* W_{S^*}(S_t, S_t^*)}{W(S_t, S_t^*)}. \qquad \square$$

Proof of Lemma 2: The amount of borrowing required to meet the self-financing constraint, $\theta_t^B \mathrm{B}_t$, is given by the difference in the amount of wealth that the investor has, and that invested in the risky assets:

$$\theta_t^B B_t = W_t - (\theta_t^S S_t + \theta_t^{S^*} S_t^*).$$

Substituting for θ_t^S and $\theta_t^{S^*}$ from equations (A8) and (A9), we have that,

$$\theta_t^B B_t = W(S_t, S_t^*) - S_t W_S(S_t, S_t^*) - S_t^* W_{S^*}(S_t, S_t^*).$$

Now, using Euler's Theorem, and because $W(S, S^*)$ is homogeneous of degree one in S_t and S_t^*,[28] it follows that the right-hand side of the above expression is zero. Therefore, $\theta_t^B B_t = 0$. □

Transforming the Partial Differential Equation Into an Ordinary Differential Equation

To solve the partial differential equation (12), we first make the following transformation, using the fact that $W(S_t, S_t^*)$ is homogeneous of degree one in S_t and S_t^*.

$$T(\psi_t) = \frac{W(S_t, S_t^*)}{S_t + S_t^*} \tag{A10}$$

This transformation allows us to reduce the partial differential equation in (12) from (S_t, S_t^*) space to an ordinary differential equation in terms of ψ_t, where $\psi_t = S_t/(S_t + S_t^*)$ and has a domain that is bounded between 0 and 1. The linear homogeneity of wealth can be verified by direct substitution of the transformed function into the differential equation for $W(S_t, S_t^*)$. Applying the transformation in (A10) to the differential equation in (12), and collecting terms, we obtain:

$$\begin{aligned} &T(\psi_t)\{\psi_t[r(\psi_t) - \hat{c}(\psi_t)] + (1 - \psi_t)[r(\psi_t) - \hat{c}^*(\psi_t)] - r(\psi_t)\} \\ &\quad + T'(\psi_t)[\psi_t(1 - \psi_t)]\{[r(\psi_t) - \hat{c}(\psi_t)] - [r(\psi_t) - \hat{c}^*(\psi_t)]\} \\ &\quad + T''(\psi_t)[\psi_t(1 - \psi_t)\sigma_{2t}]^2 + \psi_t \hat{c}(\psi_t) = 0 \\ &\qquad\qquad \text{for} \quad \frac{1}{1 + \lambda s} < \psi < \frac{\lambda s}{1 + \lambda s} \end{aligned} \tag{A11}$$

where $\hat{c}(\psi_t) = c(S_t, S_t^*)/S_t$ and $\hat{c}^*(\psi_t) = p_t c^*(S_t, S_t^*)/S_t^*$.

The transformed boundary conditions, where $\psi_t = 1/(1 + \lambda s)$ and $\psi_t = \lambda s/(1 + \lambda s)$, are:

$$T'(\psi_t) = 0 \tag{A12}$$

The differential equation, and the boundary conditions in (A12), when written in their finite difference form, give a system of linear equations for $T(\psi_t)$. This system of equations is sparse, having a tridiagonal band, and is solved using the efficient Cholesky LU-decomposition scheme.[29] To express the

[28] That $W(S, S^*)$ is homogeneous of degree one in S_t and S_t^*, can be seen by substituting (A10) in (12) to get (A11).

[29] See Dahlquist and Bjork (1974) for details of the Cholesky LU-decomposition technique.

portfolio weights as a function of ω instead of ψ, it is sufficient to note that the portfolio weights, $\left\{\frac{SW_S(S_t, S_t^*)}{W(S_t, S_t^*)}, \frac{S^* W_{S^*}(S_t, S_t^*)}{W(S_t, S_t^*)}\right\}$, are homogeneous of degree zero in S and S^*. This follows from the fact that $W(S_t, S_t^*))$ is homogeneous of degree one in S_t and S_t^*.

REFERENCES

Adler, M., and B. Dumas, 1983, International portfolio choice and corporation finance: A synthesis, *Journal of Finance* 38, 925–984.

Black, F., 1973, Transportation costs, tariffs, and taxes in a one-sector model of trade, Working paper, M.I.T.

———, 1974, International capital market equilibrium with investment barriers, *Journal of Financial Economics* 1, 337–352.

Branson, W., and D. Henderson, 1985, The specification and influence of asset markets, in R. Jones and P. Kenen, eds.: *Handbook of International Economics*, Vol. II (North-Holland, Amsterdam).

Constantinides, G., 1986, Capital market equilibrium with transaction costs, *Journal of Political Economy* 94, 842–862.

Cooper, I., and E. Kaplanis, 1991, What explains the home bias in portfolio investment?, Working paper, London Business School.

Cox, C., and J. Huang, 1989, Optimal consumption and portfolio policies when asset prices follow a diffusion process, *Journal of Economic Theory* 49, 33–83.

Dahlquist, G., and A. Bjork, 1974, *Numerical Methods* (Prentice Hall, Englewood Cliffs, N.J.).

Davis, M. and A. Norman, 1990, Portfolio selection with transaction costs, *Mathematics of Operations Research* 15, 676–713.

Dixit, A., 1991, A simplified treatment of the theory of optimal regulation of Brownian motion, *Journal of Economic Dynamics and Control* 15, 657–673.

Duffie, D., and C. Huang, 1985, Implementing Arrow-Debreu equilibria by continuous trading of few long-lived securities, *Econometrica* 53, 1337–1356.

Dumas, B., 1991, Super contact and related optimality conditions, *Journal of Economic Dynamics and Control* 15, 675–685.

———, 1992, Dynamic equilibrium and the real exchange rate in a spatially separated world, *Review of Financial Studies* 5, 153–180.

———, and E. Luciano, 1991, An exact solution to a dynamic portfolio choice problem under transactions costs, *Journal of Finance* 46, 577–595.

Eldor, R., D. Pines, and A. Schwartz, 1988, Home asset preference and productivity shocks, *Journal of International Economics* 25, 165–176.

Errunza, V., 1983, Emerging markets: A new opportunity for improving global portfolio performance, *Financial Analysts Journal* 39, 51–58.

Eun, C., and B. Resnick, 1988, Exchange rate uncertainty, forward contracts, and international portfolio selection, *Journal of Finance* 43, 197–215.

French, K., and Porterba, 1991, Investor diversification and international equity markets, *American Economic Review* 81, 222–226.

Gagnon, J., 1989, Adjustment costs and international trade dynamics, *Journal of International Economics* 26, 327–344.

Glen, J., 1989, Exchange rate uncertainty, forward contracts, and the performance of global equity portfolios, Rodney L. White Working Paper 37–89, University of Pennsylvania.

Grubel, H., 1968, Internationally diversified portfolios: Welfare gains and capital flows, *American Economic Review* 58, 1299–1314.

Harrison J. 1985, *Brownian Motion and Stochastic Flow Systems* (Wiley, New York).

——— and M. Taksar, 1983, Instantaneous control of Brownian motion, *Mathematics of Operations Research* 8, 439–453.

Huang, C., 1987, An intertemporal general equilibrium asset pricing model: The case of diffusion information, *Econometrica* 55, 117–142.

Ingersoll, J., 1987, *Theory of Financial Decision Making* (Rowman and Littlefield, Totowa, N.J.).

Jorion, P., 1985, International portfolio diversification with estimation risk, *Journal of Business* 58, 259–278.

Krugman, P., 1981, Consumption preferences, asset demands, and distribution effects in international financial markets, NBER Working Paper No. 651.

Lessard, D., 1973, World, national and industry factors in equity returns, *Journal of Finance* 29, 379–391.

Levy, H., and M. Sarnat, 1970, International diversification of investment portfolios, *American Economic Review* 60, 668–675.

Merton, R., 1971, Optimum consumption and portfolio rules in a continuous time model, *Journal of Economic Theory* 3, 373–413.

———, 1973, Theory of rational option pricing, *Bell Journal of Economics and Management Science* 4, 141–183.

Sercu P., 1980, A generalization of the international asset pricing model, *Revue de l'Association Francaise de Finance* 1, 91–135.

Singleton, K., 1990, Specification and estimation of intertemporal asset pricing models, in B. Friedman and F. Hahn, eds: *Handbook of Monetary Economics* (North-Holland, Amsterdam).

Solnik, B., 1974a, Why not diversify internationally rather than domestically, *Financial Analysts Journal* 30, 48–54.

———, 1974b, An equilibrium model of the international capital market, *Journal of Economic Theory* 8, 500–524.

Stockman, A., and H. Dellas, 1989, International portfolio nondiversification and exchange rate volatility, *Journal of International Economics* 26, 271–289.

Stulz, R., 1981a, On the effects of barriers to international investment, *Journal of Finance* 36, 923–934.

———, 1981b, A model of international asset pricing, *Journal of Financial Economics* 9, 383–406.

———, 1983, The demand for foreign bonds, *Journal of International Economics* 15, 225–238.

———, 1992, International portfolio choice and asset pricing: An integrative survey, Working paper WPS 92–31, Ohio State University.