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ABSTRACT

Geske and Johnson (1984) develop an equation for the American put price and obtain accurate prices using a method requiring quadrivariate normal integrals evaluated over an interval containing four equally spaced exercise points. We show that a modification of their method which uses optimal placement of exercise points yields in most cases accurate values using nothing more than bivariate normals. In the more difficult (deep-in-the-money) cases, trivariate normals suffice.

USING THE COMPOUND OPTION approach of Geske (1979), Geske and Johnson (1984) develop an analytical expression for the American put price, and give a method for calculating put prices that uses Richardson extrapolation on a sequence of hypothetical puts, where each put has a finite number of exercise points located at equally spaced time intervals. Evaluating the puts requires calculation of multivariate normal integrals, as described below. To obtain acceptable accuracy, Geske and Johnson use a four-point extrapolation that is expensive in terms of computer time. Some researchers have attempted to address this problem by developing more efficient integration schemes, e.g., by exploiting the special structure of the integrals to obtain equivalent expressions involving integrals of reduced dimension (see Schroder (1989)). In addition, Selby and Hodges (1989) develop a useful integral identity which reduces a partial sum of integrals to a single integral. Unfortunately, even with these improvements, four-point extrapolations are still quite slow.

This note improves the efficiency of the Geske-Johnson approach by optimally locating the exercise points. We recommend a procedure that gives accurate values using fewer exercise points, thus improving the solution's efficiency. We compare our calculated values with those in Cox and Rubinstein (1985) and give some estimates of the computer time required.

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We make use of Geske and Johnson's analytical formula

$$P = Xw_2 - Sw_1 \tag{1}$$

where P is the American put price, X is the exercise price, and S is the underlying stock price. The terms w_1 and w_2 in (1) are infinite series involving multivariate normal integrals, and may be expressed conveniently using the following notation from Selby and Hodges (1989).

Let $N_i(\mathbf{d}_i, R_i)$ denote the i-dimensional multivariate normal integral with upper limits of integration given by the i-dimensional vector $\mathbf{d}_i$ and correlation matrix R_i. Let $D_i = \text{diag}(1,\cdots,1,-1)$ be the i-dimensional matrix that changes the sign of the last element of an i-dimensional vector, and define $\mathbf{d}_i^* = D_i\mathbf{d}_i$ and $R_i^* = D_i R_i D_i$. Let σ be the instantaneous standard deviation of the return on the stock, and r be the risk-free rate. Then w_1 and w_2 in the Geske-Johnson formula are given by

$$w_1 = \sum_{i=1}^{\infty} N_i(\mathbf{d}_{i1}^*, R_i^*) \tag{2}$$

$$w_2 = \sum_{i=1}^{\infty} e^{-rt_i} N_i(\mathbf{d}_{i2}^*, R_i^*), \tag{3}$$

where

$$\mathbf{d}_{i1}^* = (d_{11}, d_{21}, \cdots, d_{i1})',$$

$$\mathbf{d}_{i2}^* = \mathbf{d}_{i1}^* - (\sigma\sqrt{t_1}, \cdots, \sigma\sqrt{t_i})',$$

$$d_{j1} = \frac{\ln(S/S_{tj}) + (r + 0.5\sigma^2)t_j}{\sigma\sqrt{t_j}}, \quad \text{for } j = 1,\cdots,i.$$

The terms t_j and S_{tj} represent the jth "time instant" and the jth critical stock price, respectively.

Now define $P_k(\mathbf{t}_k)$ to be the value of a put which can only be exercised at the k points in time given by the vector $\mathbf{t}_k = (t_1, t_2, \cdots, t_k)$. This value is expressed by the partial sum

$$P_k(\mathbf{t}_k) = X\sum_{i=1}^{k} e^{-rt_{ik}} N_i(\mathbf{d}_{i2}^*, R_i^*) - S\sum_{i=1}^{k} N_i(\mathbf{d}_{i1}^*, R_i^*) \tag{4}$$

Let T be the expiration date of the put and define the vector of k *equally spaced* time points by $\mathbf{t}_k^{\text{GJ}} = (T/k, 2T/k, \cdots, T)$. The Geske-Johnson (GJ) method for evaluating (1) computes the sequence $P_1(\mathbf{t}_1^{\text{GJ}})$, $P_2(\mathbf{t}_2^{\text{GJ}}), \cdots$, $P_N(\mathbf{t}_N^{\text{GJ}})$, and then uses Richardson extrapolation[1] to obtain the American put value.

Because $P_k(\mathbf{t}_k^{\text{GJ}})$ involves calculating a sequence of k multivariate normal integrals of increasing dimension (i.e., dimension $= 1, 2, \cdots, k$), the GJ method can be burdensome both in programming complexity and in computer

[1] See Geske and Johnson (1984) for a discussion of Richardson extrapolation.

time requirements when N is greater than three or four. In addition, $k - 1$ critical stock prices are required; each of these requires solution of a nonlinear equation involving calculations of $P_j(\mathbf{t}_j^{GJ})$, for $j = 1, \cdots, k - 1$.

It turns out that the GJ extrapolation procedure with two points often works reasonably well, and many users of the technique may find this simple method quite adequate. Approximation errors do occur, however, because only two points are being used to extrapolate to the limit. Geske and Johnson solved this problem by including more intermediate dates, i.e., by doing four-point extrapolations, but at additional cost. An alternative approach described in the next section is to choose the exercise dates in (4) with a bit more care, and to use fewer points when the put is easier to evaluate.

I. A Modified Geske-Johnson Method

There is nothing in the Geske-Johnson approach that requires the exercise points to be equally spaced. Hence, a natural way to improve on the GJ procedure is to choose the sequence of exercise points optimally, i.e., to choose them as a wealth maximizer would. We first consider the case of a put that is not in the money. Suppose that a wealth-maximizing put buyer were allowed to buy a put that could be exercised at only one point in time (as with the European put) but that he could *choose* the point's location at the time the put is purchased. In virtually all realistic cases the exercise point would be placed at expiration (for out of the money puts), and the European put, i.e., $P_1(T)$, would be optimal.[2] Next suppose that the put buyer has the opportunity to choose *two* points. It turns out that locating the two points (t_1, t_2) so as to maximize $P_2(t_1, t_2)$ frequently places one of the points at T (the optimal one-point location). This simplifies the search, since one point (t_2) can be placed at T and the remaining point can be found by performing a one-dimensional maximization of $P_2(t_1, T)$ over the interval $[0, T)$. That is, we choose the intermediate point, t^*, $0 \le t^* < T$, as the one which yields the highest value, P_2^*. To evaluate the American put, use Richardson extrapolation:

$$P^{\max} = 2P_2^* - P_1^*. \tag{5}$$

We call this approach the "maximum method," to denote that P_i is maximized for $i = 1, 2$. We expect this method to work better than the simple method of equally spaced points because each value we compute, P_1^*, P_2^*, $\cdots$ is in general higher, and therefore closer to the limit of the sequence; hence, the extrapolated value from our method should have less error.

Table I compares values from both this modified GJ (maximum) method and the two-point GJ method to those in Cox and Rubinstein (1985) (CR) for puts that are not in the money. We find the intermediate point, t^*, by examining seven trial values for t^*—$T/8$, $2T/8, \cdots, 7T/8$—and choosing the

[2] Very recently, longer-lived options have begun trading. It might not be optimal to put an exercise point at expiration for such puts. So far, such options have not proven very popular.

Table I

Valuation of American Puts That Are Not in the Money

Values for puts from Cox and Rubinstein (1985) that are not in the money are computed using methods [1] and [2], and compared to the Cox-Rubinstein (CR) values [3]. For these puts the risk-free rate $r = 0.0488$ and the stock price $S = \$40$; X, σ, and T are the exercise price (in dollars), the standard deviation of the rate of return on the stock, and the time to expiration (in years), respectively. Method [2] is the "maximum method" defined by equation (5).

X	σ	T	[1] Two-Point GJ[a]	[2] Two-Point Modified GJ	[3] CR	[1] − [3]	[2] − [3]
35	0.2	0.0833	0.0062	0.0062	0.01	0.00	0.00
35	0.2	0.3333	0.1968	0.1996	0.2	0.00	0.00
35	0.2	0.5833	0.4231	0.4307	0.43	−0.01	0.00
40	0.2	0.0833	0.8483	0.8517	0.85	0.00	0.00
40	0.2	0.3333	1.5735	1.5798	1.58	−0.01	0.00
40	0.2	0.5833	1.9867	1.9925	1.99	0.00	0.00
35	0.3	0.0833	0.0770	0.0773	0.08	0.00	0.00
35	0.3	0.3333	0.6897	0.6956	0.7	−0.01	0.00
35	0.3	0.5833	1.2035	1.2169	1.22	−0.02	0.00
40	0.3	0.0833	1.3051	1.3094	1.31	0.00	0.00
40	0.3	0.3333	2.4711	2.4806	2.48	−0.01	0.00
40	0.3	0.5833	3.1572	3.1698	3.17	−0.01	0.00
35	0.4	0.0833	0.2457	0.2464	0.25	0.00	0.00
35	0.4	0.3333	1.3351	1.3438	1.35	−0.01	−0.01
35	0.4	0.5833	2.1341	2.1517	2.16	−0.03	−0.01
40	0.4	0.0833	1.7628	1.7675	1.77	−0.01	0.00
40	0.4	0.3333	3.3725	3.3856	3.38	−0.01	0.01
40	0.4	0.5833	4.3340	4.3507	4.35	−0.02	0.00

[a] GJ = Geske-Johnson value.

one which gives the highest value for P_2.[3] As in CR, the stock price S and the interest rate r are always 40 and 0.0488, respectively, and the exercise price X, the standard deviation of the rate of return σ, and the time to expiration T are varied.

The two-point GJ method does reasonably well, but it often underestimates the put value, with errors ranging from one to three cents. The maximum method is more accurate, yet is still calculated using only bivariate normal integrals; it is never off by more than a penny.

We next consider the case of puts that are in the money. These puts occasionally display markedly different behavior from that of the puts considered above, and can be more difficult to evaluate. This is illustrated by

[3] Other types of one-dimensional maximization methods could be used. Subroutines implementing all the methods in this paper were developed in Fortran. For information on obtaining copies, contact the first author. Univariate normals are calculated using ALNORM (Hill, 1973), and bivariate normals are calculated using a modified version of BIVNOR (Donnelly, 1973). The search for the critical stock price is performed using a safeguarded Newton-Raphson method. Higher-order integrals are calculated using a modified version of MULNOR (Schervish, 1984).

comparing the graphs of the European put value (p) given in Figures 1 and 2. In Figure 1, European put values are graphed as a function of time to expiration T (in years) for an *at-the-money* option (stock price = 40 and $X = 40$) with $r = 0.0488$, and $\sigma = 0.1$. The graph exhibits typical behavior for options that are either out of the money or at the money: the function is concave with an initial rapid increase for small T, and a flat area with a local maximum (at 1.40 years) for larger T. For the expiration date of most options p is either still rising or is in the flat area, as discussed previously. However, OTC option traders and traders of long-lived listed options might be interested in the fact that the maximum of the European put value function could occur *before* expiration.

In contrast, Figure 2 shows an *in-the-money* example (stock price = 40, $X = 45$, $r = 0.0488$, $\sigma = 0.20$). The function is rapidly changing and nonlinear with both a local minimum (at 0.23 years) and a local maximum (at 1.72 years). In addition, for this case p is less than $X - S = 5$ (graphed as the T axis) for all $T > 0$. Suppose that a wealth-maximizing put buyer must choose one exercise point for the maximum method: the highest value would be obtained by placing the exercise point for P_1 at zero (immediate exercise) because p is less than $X - S$. Furthermore, for the example in Figure 2, optimal assignment of *two* points will also lead to immediate exercise, and the extrapolated put value would be $X - S$. (This is not directly obvious from Figure 2, but arises from the extreme nonlinear behavior of the put.) This illustrates that for deep-in-the-money cases the two-point maximum method

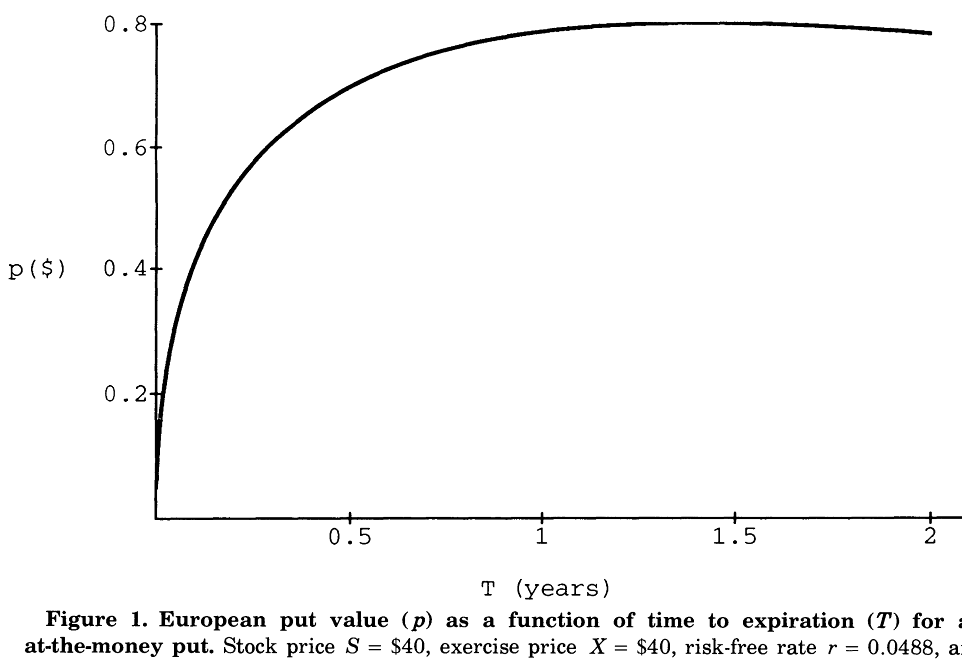

Figure 1. European put value (p) as a function of time to expiration (T) for an at-the-money put. Stock price S = \$40, exercise price X = \$40, risk-free rate $r = 0.0488$, and standard deviation of the rate of return on the stock $\sigma = 0.1$.

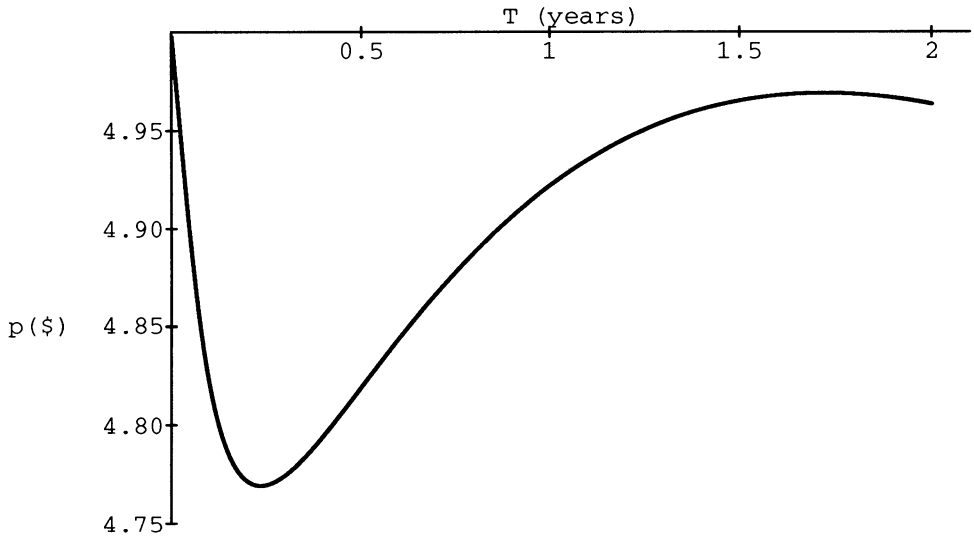

Figure 2. European put value (p) as a function of time to expiration (T) for an in-the-money put. Stock price S = \$40, exercise price X = \$45, risk-free rate $r = 0.0488$, and standard deviation of the rate of return on the stock $\sigma = 0.2$.

can sometimes be inadequate, requiring the use of an additional (third) point. In fact, it is remarkable that two-point methods work as well as they do, considering the difficulty in general of solving free boundary problems.

Results for some deep-in-the-money puts are given in Table II. Three methods are included: two-point Geske-Johnson, two-point modified Geske-Johnson ("maximum method"), and three-point Geske-Johnson. Our experience with deep-in-the-money puts leads to the following recommendation: if $p < X - S$, then use the three-point GJ approach, but if $p > X - S$ use the two-point maximum method. The results of this recommendation are illustrated in Table II. In the upper half of the table p is less than $X - S$ and the three-point GJ method is used, resulting in errors no larger than two cents. In the second part of the table p is greater than $X - S$ and the two-point maximum method is used. There is one error of four cents, one of two cents, and the remaining are at most one-cent errors. In relative terms, all errors are less than 1%.

The tradeoff is one of speed versus accuracy. Table III gives rough measures of CPU times for the methods, calculated using averages over the 27 puts in Tables I and II. Actual times will depend upon such things as the method of implementation, the specific computer hardware, the efficiency of the compiler, and the values for the put parameters, T, S, X, σ, and r. Times in Table III were obtained on a MicroVax II using Vax Fortran and the method recommended above. Conversion factors using a standard floating-point benchmark were used to get approximate comparable times on some other CPUs.

Table II

Valuation of In-the-Money American Puts

Values for in-the-money puts from Cox and Rubinstein (1985) are computed and compared to the Cox-Rubinstein (CR) values. The risk-free rate $r = 0.0488$ and the stock price $S = \$40.00$; X, σ, and T are the exercise price (in dollars), the standard deviation of the rate of return on the stock, and the time to expiration (in years), respectively. The value (A) of the put is the three-point Geske-Johnson (GJ) value if the European put value p is less than $X - S$; it is the two-point modified Geske-Johnson method, otherwise. The error in this procedure is given in the last column. The two-point Geske-Johnson value is included for comparison purposes.

X	σ	T	$p < X - S$	Two-Point GJ	Two-Point Modified GJ	Three-Point GJ	CR	A-CR
45	0.2	0.0833	Yes	4.9965	5.0000	4.9969	5.00	0.00
45	0.2	0.3333	Yes	5.1296	5.0018	5.1052	5.09	0.02
45	0.2	0.5833	Yes	5.3263	5.2443	5.2893	5.27	0.02
45	0.3	0.0833	Yes	5.0762	5.0667	5.0630	5.06	0.00
45	0.3	0.3333	No	5.7333	5.7333	5.7017	5.71	0.02
45	0.3	0.5833	No	6.2812	6.2812	6.2366	6.24	0.04
45	0.4	0.0833	No	5.2951	5.2951	5.2846	5.29	0.01
45	0.4	0.3333	No	6.5180	6.5197	6.5013	6.51	0.01
45	0.4	0.5833	No	7.3965	7.3976	7.3694	7.39	0.01

Table III

Comparison of Computer Time Requirements for Four Valuation Methods

Average CPU times on four computer systems are computed using the 27 puts from Tables I and II. Actual times were obtained on the MicroVax II; times for the remaining three systems are approximated using floating-point benchmark multipliers, and are included for comparison purposes. The last column gives relative CPU time for each method using two-point Geske-Johnson (GJ) as a standard of comparison; it is the fastest to compute, but it is also the least accurate.

	CPUs				Relative CPU Time versus 2-pt GJ
Method	MicroVax II	12 MHz AT	20 MHz 386	Mac SE/30	
2-pt GJ	0.02	0.08	0.01	0.02	1
2-pt modified GJ	0.13	0.54	0.06	0.15	6.5
3-pt GJ	1.18	4.94	0.52	1.38	59
3-pt modified GJ	27.9	116.80	12.32	32.72	1395

The two-point maximum method is roughly nine times faster than the three-point GJ method, but may occasionally be less accurate. One could perform a three-point maximum method, but this is much more expensive: the two-point maximum method is 215 times faster, and about as accurate. We found the four-point GJ method to be unacceptable in terms of time requirements, and have not included timing results for this method. The two-point and three-point methods should be considerably faster than the binomial or other alternative methods; for a comparison of alternative methods (not including Geske-Johnson), see Geske and Shastri (1985).

Our idea of optimally locating the exercise points for extrapolation may have applications to other methods. For example, Breen (1991) gives a method for "accelerating" the binomial option pricing model by applying the Geske-Johnson approach using equally-spaced exercise points. Presumably, the techniques described here could be used to improve this method as well.

II. Summary

The two-point maximum method developed in this paper works very well for puts that are not close to the point of early exercise. For the more difficult (deep-in-the-money) cases a three-point GJ extrapolation may be required to give sufficient accuracy. A combination of these methods yields a numerically efficient procedure for valuation of American puts that is useful even for personal computers.

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