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Positive Prices in CAPM

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ABSTRACT

Some equilibrium prices in CAPM may be negative because of nonmonotonicity of preferences. We identify several sets of sufficient conditions for prices to be positive. The central conditions impose bounds on the investors' risk aversion. These bounds do not need to hold globally but only in a relevant range of portfolios or combinations of mean and standard deviation. The relevant range is specified on the basis of exogenous parameters and variables, and it must contain any endogenously determined equilibrium. The bounds on risk aversion ensure that the preferences for assets are sufficiently well-behaved within the relevant range.

THE TWO-PERIOD MEAN-VARIANCE capital asset pricing model (CAPM) has the peculiar property that preferences may not be monotone. The expected return to an investor's portfolio increases as he holds more and more shares of the assets, but so does the variance of return. It may be that at some point, the additional expected return gained from adding more shares to the portfolio is not sufficient to compensate for the increase in variance. If so, then the induced preferences for assets are not monotone. Nonmonotonicity of preferences in the mean-variance model is analyzed in Nielsen (1987).

Because portfolio preferences are not necessarily monotone, equilibrium asset prices may be negative or zero. In fact, it is possible that the value of the market portfolio of risky assets is negative. This is demonstrated in Nielsen (1985, 1990a) and in two examples in the present paper.

When the CAPM is used as a model of the prices of stocks, which in reality have limited liability, it is of course disturbing that it may predict negative prices. It can be seen as a symptom of the fact that mean-variance preferences are not always a good approximation to reality. This paper analyzes various conditions on the exogenous parameters which ensure that the preferences are sufficiently realistic to yield positive prices. The exogenous parameters in question are the expectations, variances and covariances of total returns, the initial allocations of the assets (endowments), and the investors' utility functions for mean and variance of return. The analysis applies to the classical version of the CAPM with a riskless asset, as developed by Sharpe (1964), Lintner (1965) and Mossin (1966), as well as the CAPM with risky assets only, which was introduced by Black (1972), and has been much used in empirical work under the name the "zero-beta" or "two-factor" CAPM.

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To identify conditions which ensure positivity of prices, we begin by computing the gradient of the derived utility function for portfolios, and exploit the fact that at an equilibrium, all investors' gradients point in the direction of the price vector. This leads to sufficient conditions in the form of inequalities which involve expected returns, an asset's covariance with the return either on the market portfolio or on a particular investor's portfolio, and a measure of the risk aversion of either a particular investor or a representative investor. The nature of the inequalities is to impose a bound on risk aversion, the bound being related to expected returns, covariances, etc.

A first approach would be to impose these restrictions globally, that is, to require that they hold at every portfolio. For example, if it is assumed that some investor's risk aversion, as measured by the slope of his indifference curve for combinations of standard deviation and mean, is everywhere less than the ratio of mean to standard deviation available from each of the assets, then his preferences for assets are strictly monotone. In that case, all equilibrium prices must be positive. However, this condition is unduly restrictive because it is not satisfied by for example utility functions which are linear in mean and variance. These utility functions correspond to the popular case of negative exponential von Neumann-Morgenstern utility functions and normal returns distributions.

Even though utility functions such as these lead to nonmonotonicity, they may still be useful if they are treated as an approximation. The approximation will be reasonable for some values of the exogenous parameters, but not for others. Thus, we search for conditions that are weaker than the requirement of global strict monotonicity, yet ensure that the approximation is good enough to yield positive asset prices.

It may be that the unreasonable properties of preferences occur only for portfolios which are irrelevant for the process of establishing an equilibrium, for example because their mean return is larger than the mean return on the total market portfolio, or because they offer a combination of standard deviation and mean of return which is so unattractive that all investors prefer their endowment portfolio. There is a "relevant range" of portfolios such that the shape of preferences outside that range does not matter.

Specifically the relevant range of allocations is the "individually rational" allocations, which are allocations that are consistent with the total supply of assets and are preferred to the initial allocation by all investors. For each investor, the relevant range of portfolios is those portfolios that form part of individually rational allocations. This idea has been used in Nielsen (1989, 1990b, 1992) to study the existence of equilibrium.

Exploiting this idea, we observe that it is enough to require the inequality conditions to hold in the relevant range of portfolios, specifically at all individually rational allocations, or at all portfolios that form part of individually rational allocations. That is significantly weaker than the global requirement, because the set of individually rational allocations is bounded.

The set of individually rational allocations is difficult to identify in practice because it is defined by joint reference to the preference maps of all the investors. For this reason, it is desirable to find some conditions for positive prices that can be imposed in a manner which is independent across investors. These conditions are expressed as bounds on risk aversion over a range of combinations of standard deviation and mean of return.

Under the assumption that the investors have initial endowment portfolios which are preferable to zero, in an individually rational allocation every investor must hold a portfolio with mean return less than the expected return on the market, and at least one investor must hold a portfolio with mean return less than the mean return of his initial portfolio. Thus, we identify two relevant ranges of combinations of standard deviation and mean (for each investor). The "large relevant range" consists of combinations preferred to zero and with mean less than the expected return on the market; and the "small relevant range" consists of combinations preferred to zero and with mean less than the expected return on the investor's initial portfolio.

These relevant ranges are defined for a particular investor and are specified in a manner which does not involve reference to the other investors' preferences.

One sufficient condition for the equilibrium price of an asset to be positive is that its expected return is positive and either it has nonpositive covariance with the market portfolio, or else all investors' risk aversion is bounded in the large relevant range by a bound which depends on the asset's expected return and its covariance with the market.

A second sufficient condition is that some investor's risk aversion, measured as the marginal rate of substitution between mean and standard deviation, is less, over the large relevant range, than the ratio of the mean to standard deviation available from the asset. This condition means that the investor would want to add some of the asset to his portfolio if his portfolio had standard deviation and mean in the large relevant range, and even if it were perfectly positively correlated with the asset. The condition has the advantage of not involving a covariance term.

A third sufficient condition is that every investor's marginal rate of substitution between mean and standard deviation is less, over the small relevant range, than the ratio of the mean to standard deviation available from the asset. This means that the investor would want to add some of the asset to his portfolio if his portfolio had standard deviation and mean in the small relevant range and even if it were perfectly positively correlated with the asset. This condition also has the advantage of not involving a covariance term.

The two latter conditions can be simplified if the investors have decreasing risk aversion, which means that the higher the mean return is, the less additional mean return is required to compensate for a given increase in the standard deviation of return. In that case, the bound on the marginal rate of substitution needs to be imposed only at the upper right hand corner of the relevant range. This yields a particularly appealing version of the last condition: all investors have a marginal rate of substitution at their initial portfolio holdings which is less than the ratio of the mean to standard deviation available from the asset.

If there is a riskless asset, one can exploit the "separation theorem" to truncate the "large relevant range." In an individually rational allocation, every investor will hold a portfolio with a standard deviation of return less than the standard deviation of return to the market portfolio. Thus, the bounds on risk aversion need to be imposed only over a smaller range than in the case where all assets are risky.

The plan of the paper is as follows. Section I sets up the model. Section II discusses preference restrictions which are "joint" in the sense that they involve all investors' preference maps jointly. Section III discusses preference restrictions which are independent across investors. Section IV studies the case where there is a riskless asset. Section V contains some concluding remarks.

I. Prerequisites

Portfolios are represented by n-vectors x, where the jth entry x_j indicates the number of shares of the jth asset included in the portfolio. Short-selling is allowed, so that the number x_j of shares of asset j held in a portfolio may be negative.

All investors have the same beliefs about the total (gross) returns per share of the assets, which they summarize in a mean vector $\overline{R}$ and a covariance matrix Ω. The mean return to a portfolio x is $x^{\mathsf{T}}\overline{R}$, and the standard deviation is $\sigma(x) = (x^{\mathsf{T}}\Omega x)^{1/2}$.

A portfolio x is *riskless* if $\sigma(x) = 0$. It will be assumed that either there is no riskless asset or else the first asset is riskless while the remaining assets are risky. More specifically, in the first case (no riskless asset), it is assumed that the full covariance matrix Ω is positive definite. In the second case (where the first asset is riskless), the total return to the first (riskless) asset is assumed to be positive, and the covariance matrix of returns to the remaining assets is assumed to be positive definite. These assumptions imply that there are no *redundant assets or portfolios*: all portfolios $e \neq 0$ have $(\sigma(e), e^{\mathsf{T}}\overline{R}) \neq (0, 0)$.

Investor i has a utility function $W_i(v, \mu)$ which is a function of the variance and mean of total portfolio return. It is defined for $v \geq 0$ and for all values of μ. The corresponding utility function for standard deviation and mean is $U_i(\sigma, \mu) = W_i(\sigma^2, \mu)$.

We assume that W_i is continuously differentiable (also at $v = 0$) with $W'_{iv} < 0$ and $W'_{i\mu} > 0$, and U_i is quasiconcave. Continuous differentiability of W_i is not strictly necessary, but it is very convenient, and little interesting generality is lost by maintaining it. In fact, the main reason for introducing the function W_i instead of just working with U_i is the convenience of the differentiability assumption on W_i. It implies differentiability of U_i, but differentiability of U_i does not imply differentiability of W_i when $v = 0$.

The investor's utility function for portfolios is

$$V_i(x) = W_i\left(x^{\top}\Omega_i x,\, x^{\top}\overline{R}^i\right) = U_i\left(\sigma_i(x),\, x^{\top}\overline{R}^i\right).$$

It is quasiconcave and continuously differentiable.

Mean-variance behavior is consistent with expected utility maximization with general utility functions if the total returns follow the distributions described by Chamberlain (1983) and Owen and Rabinovitch (1983), which include the normal distributions. In the present paper, mean-variance behavior is treated not as a special case of expected utility maximization, but as an alternative, normal distributions being a special case of both.

Concavity of U_i is a stronger assumption than convexity of the induced preferences in (σ, μ)-space. For example, concavity of U_i implies that all the indifference curves in (σ, μ)-space have the same asymptotic slope at large values of σ. Kannai (1977) discusses the assumptions involved in representing a preference relation by a concave utility function. If U_i results from expected utility based on normal distributions and a risk averse von Neumann-Morgenstern utility function, then U_i is concave.

As discussed in Nielsen (1987), V_i may not be monotone, and it may exhibit satiation if there is no riskless asset. *Satiation* refers to the situation where V_i has an unconstrained global maximum at some portfolio x. Such a maximum is called a *satiation portfolio* for i. The possibility of satiation and the unboundedness of the investors' choice sets may lead to nonexistence of a general equilibrium (Nielsen (1990a)). Allingham (1991) and Nielsen (1985, 1989, 1990b, 1992) have provided several sets of sufficient conditions for existence of equilibrium in CAPM.

An *allocation* is an m-tuple $(x^i) = (x^1, \cdots, x^m)$ consisting of a portfolio x^i for each i. The investors are endowed with an *initial portfolio allocation* (ω^i). The *market portfolio* $\omega = \sum_i \omega^i$ indicates the total number of shares available of each asset. An *attainable allocation* is an allocation (x^i) such that $\sum_i x^i = \omega$.

A *general equilibrium* is a pair $(p, (x^i))$, where $p \neq 0$ is a price system (a k-vector) and (x^i) is an attainable allocation, such that for each i, $p^{\top}x^i \leq p^{\top}\omega^i$, and if y^i is a portfolio with $p^{\top}y^i \leq p^{\top}\omega^i$, then $V_i(y^i) \leq V_i(x^i)$. The initial allocation (ω^i) is exogenously given, while the asset price vector p and the equilibrium portfolio allocation (x^i) are endogenous.

Our goal is to find reasonable restrictions on the utility functions to ensure that prices are positive in all general equilibria. The restrictions that we identify actually have a slightly stronger implication than that: they ensure that prices are positive in all individually rational price equilibria. Every general equilibrium is an individually rational price equilibrium.

The relevant definitions are as follows. A *price equilibrium* is a pair $(p, (x^i))$ of a price system $p \neq 0$ and an attainable allocation (x^i) such that for each i, if y^i is a portfolio with $p^{\top}y^i \leq p^{\top}x^i$, then $V_i(y^i) \leq V_i(x_i)$. An *individually rational* allocation is an allocation x which is attainable and which every investor prefers to his initial endowment: $V_i(x^i) \geq V_i(\omega^i)$ for all i. A price equilibrium $(p, (x^i))$ is *individually rational* if the allocation (x^i) is individually rational.

Let A denote the set of individually rational allocations,

$$A = \left\{ (x^i) : \sum_i x^i = \sum_i \omega^i, V_i(x^i) \geq V_i(\omega^i) \text{ for all } i \right\}.$$

It is worth noting that this set is compact. To see this, note first that it is closed and convex. It follows from Proposition 1 of Nielsen (1990b) that A has no nonzero directions of recession. It then follows from Rockafellar (1970) Theorem 8.4 that A is compact.

Say that a portfolio x is *Pareto attainable* for investor i if it is part of some allocation in A, i.e., it belongs to the set

$$A^i = \left\{ x : x = x^i \text{ for some } (x^1, \cdots, x^m) \in A \right\}.$$

Since A is compact, clearly A^i is compact as well.

II. Restrictions Over Allocations

The possibility that some of the prices associated with an equilibrium in the CAPM may be negative is a symptom of the fact that mean-variance preferences are not always a good approximation to reality. In particular, these preferences may not be monotone. In order for the model to be reasonable, the preference assumptions need to be approximately valid for a relevant range of portfolios. In this and the following section, we shall identify some restrictions on the parameters of the model which ensure that the utility functions yield a local approximation which is sufficiently realistic to make the equilibrium prices positive.

The restrictions derived in the present section are restrictions on the shape of the preferences over a range of portfolios. Specifically, they are restrictions on the induced utility functions on the set A of individually rational allocations or the sets A^i of Pareto attainable portfolios. The restrictions derived in the following two sections are restrictions on the shape of the indifference curves in (σ, μ)-space.

We shall compute the gradient of the induced utility function for portfolios and exploit the fact that at an equilibrium, all investors' gradients point in the direction of the price vector. Say that the induced utility function V_i is *locally strictly monotone* at a portfolio x in the direction of asset j if $V'_{ij}(x) > 0$. In a price equilibrium $(p, (x^i))$, the price of asset j must be positive if there is some investor i whose induced utility function is locally strictly monotone at x^i.

In order to find a useful expression for the gradient of V_i, it is convenient to introduce a measure of risk aversion. Set

$$\gamma_i(\sigma, \mu) = -2 W'_{iv}(\sigma^2, \mu) / W'_{i\mu}(\sigma^2, \mu).$$

Then $\gamma_i(\sigma, \mu) > 0$. Note that $\gamma_i(\sigma, \mu)$ equals two times the slope of the indifference curve in (v, μ)-space, or two times the marginal rate of substitution of mean for variance. Because of this, it can be interpreted as a measure of risk aversion. If x is a portfolio, let

$$\begin{aligned}\tilde{\gamma}_i(x) &= \gamma_i(\sigma(x), x^{\mathsf{T}}\overline{R}) \\ &= -2W'_{iv}(x^{\mathsf{T}}\Omega x, x^{\mathsf{T}}\overline{R})/W'_{i\mu}(x^{\mathsf{T}}\Omega x, x^{\mathsf{T}}\overline{R}).\end{aligned}$$

The gradient of V_i is:

$$\begin{aligned}V'_i(x) &= 2W'_{iv}\Omega x + W'_{i\mu}\overline{R} \\ &= W'_{i\mu}[\overline{R} - \tilde{\gamma}_i(x)\Omega x].\end{aligned}$$

This expression for the gradient implies that the induced utility function V_i is locally strictly monotone at x in the direction of asset j if:

$$\overline{R}_j > \tilde{\gamma}^i(x)\mathrm{cov}(R_j, x^{\mathsf{T}}R).$$

This inequality can be interpreted as follows. The higher the covariance between the investor's portfolio and asset j, and the higher the investor's risk aversion (as measured by $\tilde{\gamma}$), the less desirable is it for him to include more of asset j in his portfolio. The inequality says that the expected return to asset j is large enough to outweigh these two negative effects, so that the asset is desirable after all, and the investor's induced utility function is locally strictly monotone in the direction of asset j.

If there is an investor for whom this inequality holds at all portfolios, then the price of asset j must be positive in every price equilibrium. However, this condition requires that the investor's induced utility function is in fact globally strictly monotone in the direction of asset j. That is not true, for example, of utility functions that are linear in mean and variance. These utility functions can be quite useful if they are treated as a local approximation. Thus, imposing the inequality globally, at all portfolios, is unnecessarily restrictive.

It suffices to impose the inequality, and with it the requirement of local strict monotonicity, only over a range of portfolios that are relevant for the process of establishing an equilibrium. Specifically, since all general equilibria involve individually rational allocations, and since the portfolios that form part of such allocations are Pareto attainable, it suffices to impose the inequality and local strict monotonicity over the range of Pareto attainable portfolios. That is significantly weaker than the global requirement, because the set of individually rational allocations is bounded.

Proposition 1: *If there is an investor i such that*

$$\overline{R}_j > \max\{\tilde{\gamma}^i(x^i)\mathrm{cov}(R_j, x^{i\mathsf{T}}R) : x^i \in A^i\},$$

then $p_j > 0$ in every individually rational price equilibrium.

The function $\tilde{\gamma}_i(x^i)\mathrm{cov}(R_j, x^{i\mathrm{T}}R)$ does have a well defined maximum on the set A^i of Pareto attainable portfolios, because the function is continuous and the set A^i is compact. Similar observations apply to Proposition 2 below.

As is well known, because the "efficient frontier" in the mean-variance model is one-dimensional, it is possible to express CAPM equilibrium prices almost in closed form, except for one endogenous parameter. This expression for prices will lead us to another condition ensuring positive prices.

If $x = (x^i)$ is an allocation, set

$$\gamma(x) = \left[\sum_i \left(\tilde{\gamma}_i(x^i)\right)^{-1}\right]^{-1} > 0.$$

Suppose $(p, (x^i))$ is a price equilibrium. It follows from the first-order condition for utility maximization and the expression above for the gradient that for each investor i, there exists $\lambda_i \geq 0$ such that

$$\lambda_i p = \overline{R} - \tilde{\gamma}_i(x^i)\Omega x^i \tag{1}$$

Divide Equation 1 by $\tilde{\gamma}_i(x^i)$, sum over i, and multiply by $\gamma(x)$ to get

$$\lambda p = \overline{R} - \gamma(x)\Omega\omega, \tag{2}$$

where:

$$\lambda = \gamma(x)\sum_i \frac{\lambda_i}{\tilde{\gamma}_i(x^i)} \geq 0.$$

Equation 2 holds at all price equilibria. It follows that if:

$$\overline{R}_j > \gamma(x)\mathrm{cov}(R_j, \omega^{\mathrm{T}}R)$$

in a price equilibrium, then the price of asset j is positive. Consequently, if this inequality holds at all individually rational allocations, then the price of asset j must be positive in every individually rational price equilibrium. This can be most attractively stated under the innocuous assumption that the expected return to asset j is positive.

Proposition 2: *If* $\overline{R}_j > 0$ *and*

$$\overline{R}_j > \left[\max_{x \in A} \gamma(x)\right]\mathrm{cov}(R_j, \omega^{\mathrm{T}}R),$$

then $p_j > 0$ *at every individually rational price equilibrium.*

Proposition 2 can be interpreted as follows. The market portfolio ω is the portfolio held by a representative investor, and $\gamma(x)$ is a measure of his risk aversion. Note that the representative investor's risk aversion does not only depend on the portfolio ω that he holds. It depends on how ω is distributed among the actual investors, which is why it is a function of x. The higher the covariance between the representative investor's portfolio ω and asset j, and the higher the representative investor's risk aversion (as measured by $\gamma(x)$), the less desirable it is for him to include more of asset j in his portfolio. The assumption in the proposition says that the expected return to asset j is large enough to outweigh these two negative effects, so that the asset is desirable after all. Therefore, the price of the asset must be positive in equilibrium. Allingham (1991) used this assumption in his equilibrium existence result.

Example 1: Utility linear in mean and variance, no riskless asset. Suppose $W_i(v, \mu) = \mu - a_i v/2$. In this case, $\tilde{\gamma}_i(x^i) = a_i$ independently of the portfolio x^i, and

$$\gamma(x) = \gamma = \left[\sum_i a_i^{-1}\right]^{-1}$$

independently of the allocation (x^i), so that the prices associated with a price equilibrium must, apart from a positive multiplicative constant, be given by the price system

$$q = \overline{R} - \left(\sum_i a_i^{-1}\right)^{-1} \Omega\omega.$$

If Ω is regular (no riskless asset), then each investor has the unique satiation portfolio $s^i = a_i^{-1}\Omega^{-1}\overline{R}$. It is shown in Nielsen (1989) that a general equilibrium exists if and only if $q^{\mathsf{T}}\omega^i \leq q^{\mathsf{T}}s^i$ for all i. The associated equilibrium price system q will be positive if and only if $\overline{R} >> \gamma\Omega\omega$. This condition is sufficient as well as necessary, because γ is independent of the allocation x. The inequality is expressed entirely in terms of exogenous variables and parameters. If it does not hold, then some of the equilibrium prices are zero or negative.

The harmonic mean of the risk aversion parameters $\tilde{\gamma}_i(x^i)$ is less than their maximum. Since the harmonic mean is $m\gamma(x^i)$, it follows that

$$\frac{1}{m} \max_i \tilde{\gamma}_i(x^i) \geq \gamma(x).$$

So, in Proposition 2, $\max_{x \in A} \gamma(x)$ can be replaced by $\dfrac{1}{m} \max_i \max_{x^i \in A^i} \tilde{\gamma}_i(x^i)$.

III. Restrictions Over (σ, μ)-Combinations

The sufficient conditions for positive prices developed in the previous section are expressed in terms of the functions $\tilde{\gamma}_i(x^i)$, which measure the risk aversion of the investor's induced utility function for portfolios. In Proposition 1, these functions are multiplied by a measure of covariance risk, and a bound is imposed on the resulting value over the set of Pareto attainable portfolios. In Proposition 2, they are aggregated to yield a measure of the representative investor's risk aversion, and a bound is imposed over the set of individually rational allocations.

The set of individually rational allocations can be difficult to identify because it is defined by joint reference to the preferences of all the investors.

For this reason, we shall now seek conditions for positive prices that can be expressed in a manner which is independent across investors. These conditions will be expressed in terms of the utility functions $U_i(\sigma, \mu)$ for mean and standard deviation. Again, they will involve bounds imposed on measures of risk aversion, but the bounds will not be imposed over a range of allocations or portfolios, but over a range of (σ, μ)-combinations.

We shall assume that every investor has an endowment which makes him at least as well off as if his endowment was zero:

$$U_i\left(\sigma\left(\omega^i\right), \omega^{i\top}\overline{R}\right) \geq U_i(0,0)$$

for all i. This assumption allows us to delineate two limited relevant ranges of (σ, μ)-combinations, as follows. In an individually rational allocation every investor must hold a portfolio with mean return less than or equal to the expected return on the market, and at least one investor must hold a portfolio with mean return less than or equal to the mean return of his initial portfolio. Thus, the "large relevant range" consists of combinations preferred to zero and with mean less than the expected return on the market; and the "small relevant range" consists of combinations preferred to zero and with mean less than the expected return on the investor's initial portfolio.

Formally, let (x^i) be an individually rational allocation. Then:

$$U_i\left(\sigma\left(x^i\right), x^{i\top}\overline{R}\right) \geq U_i\left(\sigma\left(\omega^i\right), \omega^{i\top}\overline{R}\right) \geq U_i(0,0),$$

and hence $0 \leq x^{i\top}\overline{R}$ for all i. Since $\sum_i x^i = \omega$, it follows that $0 \leq x^{i\top}\overline{R} \leq E_M$ for all i, where E_M is the expected total return to the market portfolio: $E_M = \omega^\top\overline{R}$. Furthermore, since $\sum_i x^{i\top}\overline{R} = \sum_i \omega^{i\top}\overline{R}$, there is at least one investor i such that $x^{i\top}\overline{R} \leq \omega^{i\top}\overline{R}$.

So, all investors hold a portfolio with standard deviation and mean in the range given by $U_i(\sigma, \mu) \geq U_i(\sigma(\omega^i), \omega^{i\top}\overline{R})$, $\mu \leq E_M$. Call this range the *large relevant range* for (σ, μ). At least one investor holds a portfolio with standard deviation and mean in the smaller range given by $U_i(\sigma, \mu) \geq U_i(\sigma(\omega^i), \omega^{i\top}\overline{R})$, $\mu \leq \omega^{i\top}\overline{R}$. Call this the *small relevant range* for (σ, μ). These relevant ranges are illustrated in Figure 1.

Note that the relevant ranges for an investor are specified in a manner which does not involve reference to the other investors' preferences.

The propositions from the previous section can be restated fairly easily in terms of these relevant ranges of (σ, μ)-combinations. For example, the following proposition is a variant of Proposition 2:

Proposition 3: *Consider a particular asset j. Assume that $\overline{R}_j > 0$ and for all i,*

$$\overline{R}_j > \frac{1}{m}\gamma_i(\sigma, \mu)\operatorname{cov}\left(R_j, \omega^\top R\right),$$

in the large relevant range. Then $p_j > 0$ at all individually rational price equilibria.

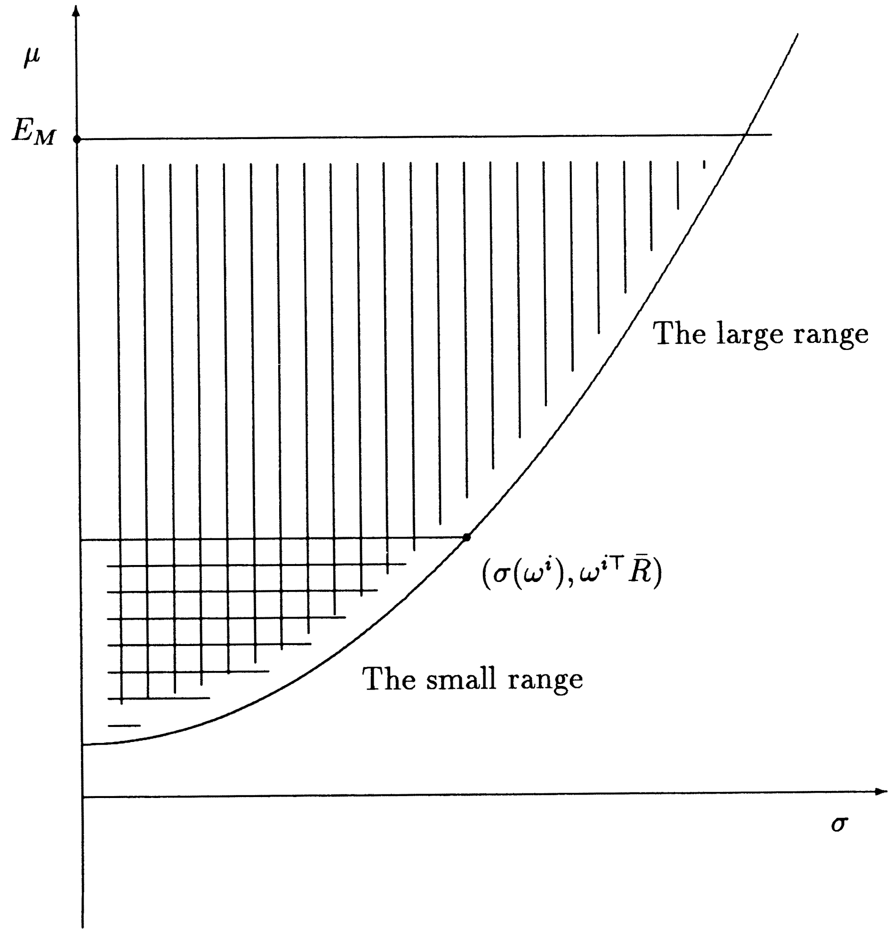

Figure 1. The Relevant Ranges for (σ, μ).

Proof: Suppose $(p, (x^i))$ is an individually rational price equilibrium. Then each x^i is in the large relevant range, so:

$$\overline{R}_j > \frac{1}{m}\tilde{\gamma}_i(x^i)\mathrm{cov}(R_j, \omega^{\top}R).$$

Since $\overline{R}_j > 0$ and $\dfrac{1}{m}\max\tilde{\gamma}_i(x^i) \geq \gamma(x)$, it follows that $\overline{R}_j > \gamma(x)\mathrm{cov}(R_j, \omega^{\top}R)$. The result follows from Equation 2.

The proposition says that a sufficient condition for the equilibrium price of an asset to be positive is that its expected return is positive and either it has nonpositive covariance with the market portfolio or else all investors' risk aversion is bounded in the large relevant range by a bound which depends on the asset's expected return and its covariance with the market.

If Proposition 1 were to be restated in terms of relevant ranges of (σ, μ)-combinations, it would still involve a covariance risk term $\operatorname{cov}(R_j, x^{i\mathsf{T}}R)$, which depends on the portfolio x^i. We shall now find some conditions which rely only on the utility functions, not on the covariance term.

Let

$$S_i(\sigma, \mu) = -U'_{i\sigma}(\sigma, \mu)/U'_{i\mu}(\sigma, \mu)$$

denote the slope of investor i's indifference curve in (σ, μ)-space at (σ, μ). Like $\gamma_i(\sigma, \mu)$, $S_i(\sigma, \mu)$ is a measure of the investor's risk aversion. It indicates the marginal rate of substitution of mean return for standard deviation of return. A high value of S_i implies high risk aversion.

If x is a portfolio, let

$$\tilde{S}_i(x) = S_i(\sigma_i(x), x^{\mathsf{T}}\overline{R}^i)$$

be the slope of the indifference curve at the standard deviation and mean of return to x.

Since $U'_{i\sigma} = 2\sigma W'_{iv}$, and $U'_{i\mu} = W'_{i\mu}$, it follows that:

$$\sigma(x)\tilde{\gamma}_i(x) = \tilde{S}_i(x).$$

For each asset j, let σ_j denote the standard deviation of total return per share to asset j (so that σ_j^2 is the jth diagonal entry in the covariance matrix Ω). Let α_j denote the ratio of mean to standard deviation of total return per share of asset j. Formally, $\alpha_j = \overline{R}_j/\sigma_j$ (if $\sigma_j = 0$ then $|\alpha_j| = \infty$).

The following proposition indicates that the best condition we can hope for which is sufficient for V_i to be locally strictly monotine in the direction of asset j, but does not involve a covariance term, is $\tilde{S}_i(x) < \alpha_j$.

Proposition 4: *Consider a particular asset j and an investor i.*

1. *If $\tilde{S}_i(x) < \alpha_j$ at some particular portfolio x, then V_i is locally strictly monotone at x in the direction of asset j.*
2. *If V_i is globally strictly monotone in the direction of asset j, then $\tilde{S}_i(x) < \alpha_j$ at all portfolios x.*

Proof: The first statement is obvious if $\sigma_j = 0$. If $\sigma_j > 0$, then the statement follows from the following computation:

$$\begin{aligned}\tilde{\gamma}_i(x)\operatorname{cov}(R_j, x^{\mathsf{T}}R) &\leq \tilde{\gamma}_i(x)|\operatorname{cov}(R_j, x^{\mathsf{T}}R)| \\ &\leq \tilde{\gamma}_i(x)\sigma_j\sigma(x) \\ &= \tilde{S}_i(x)\sigma_j \\ &< \alpha_j\sigma_j \\ &= \overline{R}_j.\end{aligned}$$

To prove the second statement, let e^j be a portfolio consisting of one share of asset j. For $t > 0$ increasing, the curve $(\sigma(x + te^j), (x + te^j)^{\mathsf{T}}\overline{R})$ in (σ, μ)-space cuts across ever higher indifference curves. The slope of the curve decreases strictly toward the limit α_j. It stays above the indifference curve through $(\sigma(x), x^\top \overline{R})$, whose slope increases strictly as σ and μ increase. Hence, $\tilde{S}_i(x) < \alpha_j$.

We can use the proposition above to identify two sufficient conditions for positive prices which have the advantage that they do not involve a covariance term. One condition is that some investor's risk aversion, measured as the marginal rate of substitution between mean and standard deviation, is less, over the large relevant range, than the ratio of the mean to standard deviation available from the asset. This means that the investor' would want to add some of the asset to his portfolio if his portfolio had standard deviation and mean in the large relevant range, and even if it were perfectly positively correlated with the asset. The other sufficient condition is that all investors' marginal rate of substitution between mean and standard deviation, is less, over the small relevant range, than the ratio of the mean to standard deviation available from the asset. This means that every investor would want to add some of the asset to his portfolio if his portfolio had standard deviation and mean in the small relevant range, and even if it were perfectly positively correlated with the asset. These two conditions are formally stated in the following theorem. Note that the conditions are considerably weaker than requiring the induced utility function to be globally strictly monotone.

Theorem 1: *Consider a particular asset j. Assume that one of the following conditions are satisfied*:

1. *For all investors, $S_i(\sigma, \mu) < \alpha_j$ in the small relevant range.*
2. *There is some investor for whom $S_i(\sigma, \mu) < \alpha_j$ in the large relevant range.*

Then $p_j > 0$ at all individually rational price equilibria.

Proof: Let $(p, (x^i))$ be an individually rational price equilibrium. We know that $0 \leq x^{i\top} \overline{R} \leq E_M$ for all i. First, suppose Condition (1) holds. There is at least one investor i such that $x^{i\top} \overline{R} \leq \omega^{i\top} \overline{R}$. For that investor, $(\sigma(x^i), x^{i\top} \overline{R})$ is in the small relevant range. Hence, $\tilde{S}_i(x^i) < \alpha_j$. Next, suppose Condition (2) holds for some investor i. Since $(\sigma(x^i), x^{i\top} \overline{R})$ is in the large relevant range, $\tilde{S}_i(x^i) < \alpha_j$. In both cases, it follows from Proposition 4 that V_i is locally strictly monotone at x^i in the direction of asset j. Hence, $p_j > 0$.

The conditions in Theorem 1 limit the investor's risk aversion $S_i(\sigma, \mu)$ to be less than α_j when (σ, μ) is in one of the relevant ranges. As a matter of fact, it suffices to limit the risk aversion when (σ, μ) is on the upper boundary of the relevant range in question. The reason is simple. For every point (σ, μ) in the relevant range, there is a (unique) point $(\tilde{\sigma}, \tilde{\mu})$ which lies on the same indifference curve and on the upper boundary of the range. The slope of the indifference curve at $(\tilde{\sigma}, \tilde{\mu})$ must be larger than the slope at (σ, μ). So, the first condition in Theorem 1 can be replaced by the condition that for all investors, $S_i(\sigma, \omega^{i\top} \overline{R}) < \alpha_j$ for all σ with $0 \leq \sigma \leq \sigma(\omega^i)$. The second condition in Theorem 1 can be replaced by the condition that there is some investor for whom $S_i(\sigma, E_M) < \alpha_j$ for all $\sigma \geq 0$ such that $U_i(\sigma, E_M) \geq U_i(\sigma(\omega^i), \omega^{i\top}\overline{R})$.

Furthermore, if the investor has decreasing risk aversion, then it suffices to limit his risk aversion at the upper right hand corner of the relevant range. This will yield a particularly appealing version of the first condition in Theorem 1: all investors have a marginal rate of substitution at their initial portfolio holdings which is less than the ratio of the mean to standard deviation available from the asset.

Decreasing risk aversion means that the higher the mean return is, the less additional mean return is required to compensate for a given increase in the standard deviation of return. Formally, say that U_i exhibits *decreasing risk aversion* if S_i is a nonincreasing function of μ. This means that the higher the investors' mean return is, the less additional mean return does he need in order to accept a given increase in standard deviation of return. If U_i exhibits decreasing risk aversion, then and S_i is a nondecreasing function of σ. If U_i is derived from expected utility based on normal distributions and a von Neumann-Morgenstern utility function u_i, then U_i exhibits decreasing risk aversion (in the sense defined here) if u_i exhibits decreasing absolute risk aversion (in the ordinary sense). For an example of an application of the concept of decreasing risk aversion in the mean-variance model, see Nielsen (1988).

In the context of Theorem 1, if the investor has decreasing risk aversion, then there is a maximum slope of his indifference curves over each of the relevant ranges of (σ, μ)-combinations. The maximum slope is achieved at the upper right hand corner of the range.

The upper right hand corner of the small relevant range is $(\sigma(\omega^i), \omega^{i\top}\overline{R})$. So, the first condition in Theorem 1 can be replaced by the condition that all investors have decreasing risk aversion and $S_i(\omega^i) < \alpha_j$. This inequality says that every investor would like to add to his initial portfolio a small amount of some security which has the same mean and standard deviation as asset j, even if that security were perfectly positively correlated with the initial portfolio.

The upper right hand corner of the large relevant range is the unique point of the form (σ, E_M) which lies on the indifference curve through the endowment point. So, the second condition in Theorem 1 can be replaced by the condition that there is some investor i with decreasing risk aversion whose indifference curve has slope less than α_j at that corner point: $S_i(\sigma, E_M) < \alpha_j$ at the unique σ where $U_i(\sigma, E_M) = U_i(\sigma(\omega^i), \omega^{i\top}\overline{R})$.

IV. Riskless Asset

In this section we assume that the first asset is riskless. In that case, one can exploit the "separation theorem" to truncate the "large relevant range" of combinations of standard deviation and mean of return, thus further weakening some of the sufficient conditions for equilibrium prices to be positive. The idea is that in an individually rational allocation, every investor will hold a portfolio whose standard deviation of return is less than the standard deviation of return to the market portfolio.

We make a few changes in the notation to reflect the existence of the riskless asset. Now, Ω denotes the covariance matrix of returns per share only to the risky assets, and R and $\overline{R}$ denote the vector of total returns per share and the vector of expected total returns per share to those assets. The total return per share of the riskless asset, which is assumed to be positive, is denoted R_f (subscript "f" for risk-*free*). A portfolio will be written in the form (x, y), where x is the number of shares of the riskless asset in the portfolio, and where the entries in the vector y are the numbers of shares of the risky assets included in the portfolio. The total return to a portfolio (x, y) is $xR_f + y^{\mathsf{T}}R$, and the expected total return is $xR_f + y^{\mathsf{T}}\overline{R}$. The prices of the assets will be normalized in such a manner that the price per share of the riskless asset is one; and p will denote the vector of prices per share only of the risky assets. Let $\sigma_M = \sqrt{\omega^{\mathsf{T}}\Omega\omega}$ be the standard deviation of return to the market portfolio of risky assets.

When the first asset is riskless, equation 2 takes the following form for the system p of prices of the risky assets:

$$p = \frac{1}{R_f}\left[\overline{R} - \gamma\Omega\omega\right].$$

This equation holds at all price equilibria.

Example 2: Utility linear in mean and variance, first asset riskless. Suppose $W_i(v, \mu) = \mu - a_i v/2$, as in Example 1, but suppose the first asset is riskless. Then $\tilde{\gamma}_i(x^i, y^i) = a_i$ independently of the portfolio (x^i, y^i), and

$$\gamma(x, y) = \gamma = \left[\sum_i a_i^{-1}\right]^{-1}$$

independently of the allocation (x, y). The prices of risky assets associated with a price equilibrium must be

$$p = \frac{1}{R_f}\left(\overline{R} - \left(\sum_i a_i^{-1}\right)^{-1}\Omega\omega\right).$$

In fact, there is always a unique general equilibrium, and the associated prices are as above. The price system p will be positive if and only if $\overline{R} \gg \gamma\Omega\omega$. If this inequality does not hold, then some of the equilibrium prices are zero or negative. Note that the inequality is expressed entirely in terms of exogenous variables and parameters. It is possible that some prices are negative; in fact, it is possible that the value of the market portfolio ω of risky assets is negative. Since $p^{\mathsf{T}}\omega = (\omega^{\mathsf{T}}\overline{R} - \gamma\sigma_M^2)/R_f$, $p^{\mathsf{T}}\omega$ is negative if and only if $\omega^{\mathsf{T}}\overline{R}/\sigma_M^2 < (\sum_i a_i^{-1})^{-1}$. This is more likely to be the case the lower is the expected return to the market, the higher is the variance of return to the market, and the more risk averse are the investors.

The results of the previous sections can easily be translated to the present situation where there is a riskless asset. Better yet, it is possible to further limit the "large relevant range" for (σ, μ), as follows. If $(p, ((x^i, y^i)))$ is a price equilibrium, then each y^i has the form $y^i = \lambda_i \omega$ with $\lambda_i \geq 0$, by the "separation theorem." Here, $\lambda_i \leq 1$ for all i because $\sum_i \lambda_i = 1$, and so $\sigma(y^i) = \lambda_i \sigma_M \leq \sigma_M$. So, all investors hold a portfolio with standard deviation and mean in the range given by $U_i(\sigma, \mu) \geq U_i(\sigma_M, x^i R_f + y^{i\top} \bar{R})$, $\mu \leq E_M$, and $\sigma \leq \sigma_M$, where E_M denotes the total return to the entire market portfolio, including riskless as well as risky assets. This range can replace the "large relevant range" in Proposition 3 and Theorem 1 (when there is a riskless asset). The new truncated range is illustrated in Figure 2.

The condition in Proposition 3 and the second condition in Theorem 1 as amended here, limit the investor's risk aversion $S_i(\sigma, \mu)$ to be less than α_j

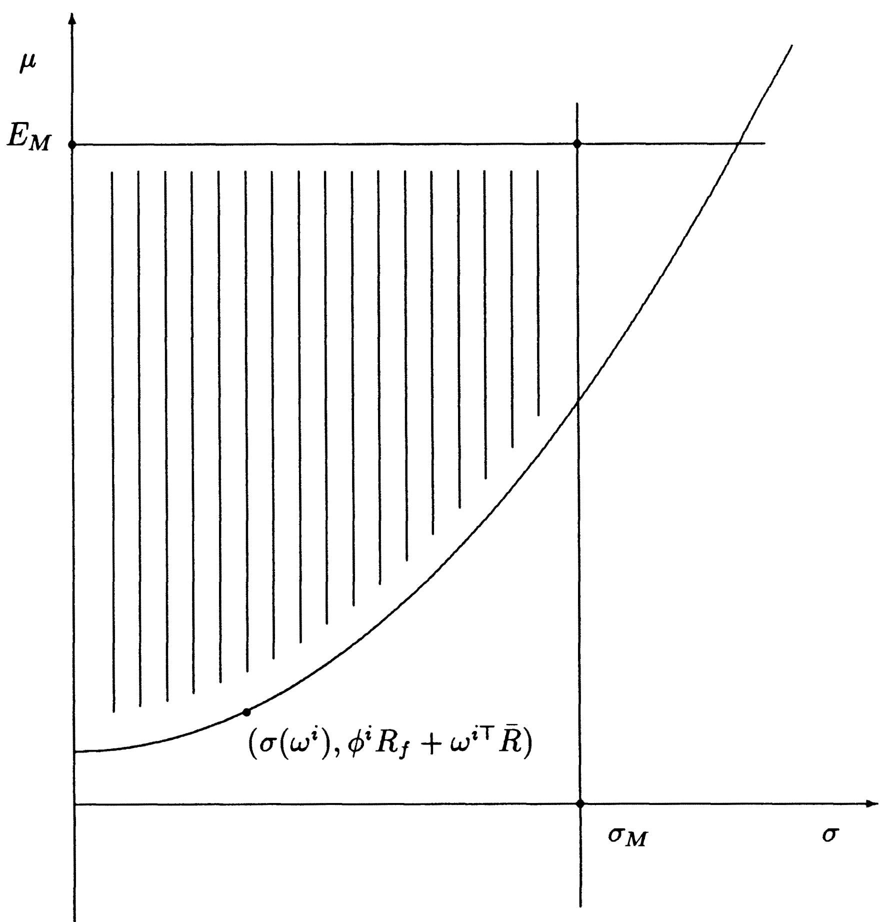

Figure 2. The Truncated "Large Relevant Range."

when (σ, μ) is in the truncated large relevant range. As a matter of fact, it suffices to limit the risk aversion when (σ, μ) is on the upper boundary and the right hand boundary of this truncated range. The reason for this is that for every point (σ, μ) in the truncated large relevant range, there is a unique point $(\tilde{\sigma}, \tilde{\mu})$ which lies on the same indifference curve and either on the upper boundary or the right hand boundary of the range. The slope of the indifference curve at $(\tilde{\sigma}, \tilde{\mu})$ must be larger than the slope at (σ, μ).

Furthermore, if the investor has decreasing risk aversion, then it suffices to limit his risk aversion at the lower left hand corner of the truncated large relevant range.

V. Concluding Remarks

The possibility of negative equilibrium prices in the static CAPM is due to the assumption that the utility of a portfolio is a function of the mean and variance of its return. The intertemporal consumption-based capital asset pricing model (CCAPM) of Breeden (1979), which is based on Merton (1973) and has been further developed by Cox, Ingersoll, and Ross (1985), Huang (1987), and Duffie and Zame (1989), does not depend on any such assumption about utility. Instead, it requires asset prices to follow Itô processes, stochastic processes built by stochastic integration with respect to Brownian motion. The question then arises, whether negative equilibrium asset prices are possible in CCAPM.

Usually, the utility functions used in CCAPM are strictly monotone functions of the consumption process. For example, they can be time separable, with instantaneous utility (felicity) functions which are strictly monotone functions of the instantaneous flow of consumption. In that case, equilibrium asset prices must be positive, provided that their dividend processes are positive. A negative asset price would entail an arbitrage opportunity, which would be inconsistent with equilibrium.

However, some versions of the CCAPM employ utility functions that are not necessarily monotone. An example is the model of Detemple and Zapatero (1990), where utility is time separable, and the instantaneous utility is an increasing function of instantaneous consumption, but a decreasing function of an index of past consumption. The index of past consumption captures habit formation and could cause the utility to be a nonmonotonic function of the consumption process. Detemple and Zapatero impose a fairly complex assumption on the index and on the derivatives of the instantaneous utility function in order to ensure that the equilibrium prices are positive.

The static CAPM continues to be an important part of the standard equipment of financial economists because of its simplicity and its striking implications. The model may be questioned on several grounds. It is not easy to construct a reasonably powerful empirical test that does not reject the model. The assumption of unrestricted short sales may be unrealistic. Mean-variance behavior is consistent with expected utility only in special cases.

Despite the objections, the usefulness and popularity of the CAPM makes it important to understand exactly the nature of its explanation of asset prices. The aim of this paper has been to contribute to this understanding by suggesting a way of dealing with the possibility that the model may predict negative asset prices. Mean-variance preferences should be seen as only an approximation to reality. It matters little that the approximation may be bad for some portfolios which are irrelevant in the sense that they cannot occur as equilibrium portfolios. We have identified conditions which ensure that over a relevant range, the preferences are a sufficiently good local approximation to yield positive equilibrium prices.

REFERENCES

Allingham, M., 1991, Existence theorems in the capital asset pricing model, *Economentrica* 59, 1169-1174.

Black, F., 1972, Capital market equilibrium with restricted borrowing, *Journal of Business* 45, 444-455.

Breeden, D. T., 1979, An intertemporal asset pricing model with stochastic consumption and investment opportunities, *Journal of Financial Economics* 7, 265-296.

Chamberlain, 1983, A characterization of the distributions that imply mean-variance utility functions, *Journal of Economic Theory* 29, 185-201.

Cox, J., J. Ingersoll and S. Ross, 1985, An intertemporal general equilibrium model of asset prices, *Econometrica* 53, 363-384.

Detemple, J. B. and F. Zapatero, 1991, Asset prices in an exchange economy with habit formation, *Econometrica*, 59, 1633-1657.

Duffie, D. and W. Zame, 1989, The consumption-based capital asset pricing model, *Econometrica* 57, 1279-1297.

Huang, C., 1987, An intertemporal general equilibrium asset pricing model: The case of diffusion information, *Econometrica* 55, 117-142.

Kannai, Y., 1977, Concavifiability and constructions of concave utility functions, *Journal of Mathematical Economics* 4, 1-56.

Lintner, J., 1965, The valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets, *Review of Economics and Statistics* 47, 13-37.

Merton, R., 1973, An intertemporal capital asset pricing model, *Econometrica* 41, 867-888.

Mossin, J., 1966, Equilibrium in a capital asset market, *Econometrica* 34, 768-783.

Nielsen, L. T., 1985, Risk-taking and capital market equilibrium, Unpublished Ph.D. dissertation, Harvard University.

———, 1987, Portfolio selection in the mean-variance model: A note, *Journal of Finance* 42, 1371-1376.

———, 1988, Uniqueness of equilibrium in the classical capital asset pricing model, *Journal of Financial and Quantitative Analysis* 23, 329-336.

———, 1989, Asset market equilibrium with short-selling, *Review of Economic Studies* 56, 467-474.

———, 1990a, Equilibrium in CAPM without a riskless asset, *Review of Economic Studies* 57, 315-324.

———, 1990b, Existence of equilibrium in CAPM, *Journal of Economic Theory* 52, 223-231.

———, 1992, Existence of equilibrium in CAPM: Further results, *Nationaløkonomisk Tidsskrift* 130, 189-197.

Owen J. and R. Rabinovitch, 1983, On the class of elliptical distributions and their applications to portfolio choice, *Journal of Finance* 38, 745-752.

Rockafellar, R. T., 1970, *Convex Analysis*, (Princeton: Princeton University Press).

Sharpe, W. F., 1964, Capital asset prices: A theory of market equilibrium under conditions of risk, *Journal of Finance* 19, 425-442.