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Transformed Securities and Alternative Factor Structures

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ABSTRACT

Grinblatt and Titman (1985) reformulate a result of Chamberlain and Rothschild (1983) to show that the approximate factor structure of Chamberlain and Rothschild is asymptotically equivalent to the strict factor structure of Ross (1976) as long as investors can always repackage securities into an equal number of arbitrary portfolios. This paper uses a Procrustes rotation methodology that is compatible with the repackaging interpretation of Grinblatt and Titman to show that the empirical structure of stock prices is consistent with the convergency hypothesis.

TESTS OF ROSS' (1976) ARBITRAGE pricing theory (APT) generally used the statistical tool of maximum likelihood factor analysis (MLFA) to estimate the assumed underlying factor structure with uncorrelated residuals. The arbitrage pricing theorem that equates expected returns to (approximately) linear functions of factor loadings is then obtained by pairing the postulated strict factor structure (SFS) with the condition of no arbitrage.

In an important contribution, Chamberlain and Rothschild (1983) (CR) generalized the definition of a SFS to allow for correlated disturbances.[1] CR formally develop the concept of an approximate factor structure (AFS) in which residual returns may be weakly correlated across securities and derive the APT pricing relation in an arbitrage-free economy. They prove that an economy has an approximate K-factor structure if and only if exactly K eigenvalues of the covariance matrix of returns explode without bound and all remaining eigenvalues are bounded as the number of assets approach infinity. They also show that, under weak regularity conditions, the eigenvectors associated with the K largest eigenvalues converge to factor loadings as the number of assets grow large. This result is significant since it implies that either asymptotic principal components analysis (APCA) or MLFA can be used to obtain estimates of factor responses. In the limit, as the number of securities increase without bound, both APCA and MLFA provide estimates

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[1] The analyses of Ingersoll (1984) and Connor and Korajczyk (1986, 1988) also permit correlated residuals.

that converge to the true values. Given the computational complexity of MLFA, the simplicity of APCA has enormous appeal.

Grinblatt and Titman (1985) (GT) provide a portfolio interpretation of the two alternative factor structures. They demonstrate that any economy which satisfies a SFS is equivalent to one that satisfies an AFS by restructuring the original assets into new portfolios, and vice versa. The repackaging of the initial investments does not alter the underlying distribution of returns.[2] Thus, the two dependency structures are economically equivalent as investors can replicate an AFS from a SFS and a SFS from an AFS.

This paper investigates the testable implication of the economic equivalency of the SFS and the AFS by sequentially testing the convergency toward one another of the two alternative estimates of factor responses provided by APCA and MLFA as the number of assets increase. By testing for convergency as opposed to the equality of the estimates from the two statistical methods, the problem of specifying a sufficiently large number of assets a priori is avoided.[3] Although this problem is prevalent in any tests of hypotheses that are based on asymptotic distribution theory, it is of particular importance here given the enormous computational expenses associated with using MLFA for large samples. It is also important to emphasize that these are tests of the assumptions of the APT, not its implications. Nevertheless, the compatibility of the various factor structures is related directly to the issue of the appropriate statistical technique for estimating the arbitrage pricing relation. Our results indicate that the differences in the rotated factor structures of APCA and MLFA are consistent with the asymptotic equivalency of the SFS and the AFS.

The paper proceeds as follows. In Section I, the two factor structures are defined and the null hypothesis is specified. This is followed in Section II by a description of the statistical procedures. Section III presents the empirical results. The paper ends with a conclusion in Section IV.

I. Alternative Factor Structures and Testable Implications

The APT assumes that asset returns are generated by the following linear return generating model:

$$r_N = \mu_N + B_N F + U_N, \tag{1}$$

where r_N is an N vector of asset returns, $\mu_N = E(r_N)$, B_N is an $N \times K$ matrix of factor loadings, F is a K vector of common factors normalized for

[2] The notion of transforming portfolios to yield equivalent portfolios without altering their characteristics is often applied in finance. For example, it is the rationale behind Modigliani and Miller's capital structure irrelevancy theorems and the option pricing theory. Shanken (1982) has also employed the concept to question the testability of the APT. His argument is that portfolios can be repackaged in such a way as to conceal relevant factors. However, Dybvig and Ross (1985) argue that the degeneracy noted by Shanken cannot occur in large samples.

[3] Due to the well-known nonuniqueness problem of factor loading, it is also undesirable to test the equality of estimates.

convenience to have zero mean and unit variance and to be uncorrelated with each other, and U_N is an N vector of mean-zero nonsystematic residuals that are uncorrelated with the common factors. Define the covariance matrix of asset returns to be

$$V_N = E\left[(r_N - \mu_N)(r_N - \mu_N)^T\right], \tag{2}$$

where superscript T denotes a transposed matrix. It can be decomposed as

$$V_N = B_N B_N^T + Q_N, \tag{3}$$

where $Q_N = D_N$ is a diagonal matrix under Ross' (1976) SFS, and $Q_N = \Omega_N$ is a positive semidefinite matrix under CR's (1983) AFS. The diagonal elements in D_N are assumed to be uniformly bounded from above by some finite values as N goes to infinity. Similarly, the eigenvalues of Ω_N are assumed to be bounded from above by some positive number as N becomes infinite.

CR demonstrate that either APCA or MLFA can be used to obtain consistent estimates of the factor loadings when N is sufficiently large because of the asymptotic equivalency of the SFS and the AFS. This implication is of significance as APCA is computationally simpler and less expensive to implement than MLFA in tests of the APT. GT reformulate the CR's equivalency result to show that Ross' economy and CR's economy are equivalent in the sense that investors can replicate one economy from the other by merely repackaging their portfolios. They note that the idiosyncratic disturbance matrix Ω_N can be diagonalized by pre- and post-multiplying by an $N \times N$ matrix with columns equal to the eigenvectors of Ω_N. Therefore, by using portfolio weights corresponding to eigenvectors of the covariance matrix, the N original securities can be transformed into N new portfolios.

The portfolio repackaging interpretation of GT can be used to test the implication that APCA and MLFA estimates converge to one another for large number of securities. GT consider the repackaging in terms of assets that comprise the unsystematic risks. The asymptotic equivalency of the two factor structures also can be established by considering the transformation of the factor structures themselves. In the limit, as the number of assets increase without bound, an appropriately converted matrix of factor loadings will yield a matrix of eigenvectors.

Let G_N be an $N \times K$ matrix of eigenvectors computed using APCA and P be a $K \times K$ transformation matrix. When SFS and AFS coincide in the limit, $B_N P = G_N$. However, for any finite number of securities, the convergency is less than perfect. The process of repackaging will nonetheless minimize the differences between the two factor patterns that decrease with increasing sample size. This can be analyzed from the perspective of least-squares by considering the minimization of the sum of squared deviations between the two matrices. Let $B_N P = G_N + Z_N$ and consider the summary statistic used by Velicer, Peacock, and Jackson (1982):

$$S_N = \mathrm{tr}\left(Z_N^T Z_N\right)/NK, \tag{4}$$

where tr denotes trace. The statistic measures the degree of fit between the two patterns by calculating the sum of squares of the differences between the corresponding elements of G_N and $B_N P$. In essence, we are finding the mean-squared error between two matrices. Therefore, the transformation matrix P is obtained by minimizing the $\mathrm{tr}(Z_N^T Z_N)$. In the factor analytic literature, this is called the Procrustes method whereby the matrix B_N is rotated to a position of maximum similarity to a target matrix, here G_N, produced by the APCA.

By stating the relation between the two factor structures in terms of the systematic risks, the null hypothesis can now be stated in terms of S_N:

$$\begin{aligned} H_O&: \quad S_{N(i)} \leq S_{N(j)} \\ H_A&: \quad S_{N(i)} > S_{N(j)} \end{aligned}$$

where the sample size $N(i)$ is smaller than $N(j)$. Thus, a *rejection* of the null for a sequence of increasing sample sizes is consistent with the convergency in distribution of the two alternative factor structures.

II. Statistical Implementation

The sample data consist of all companies in the 1986 tape compiled by the Center for Research in Security Prices at the University of Chicago that have less than twenty monthly missing observations between January 1974 and December 1985. This results in a total of 612 New York Stock Exchange firms. To examine the convergency issue, we rank the securities alphabetically and construct 20 groups with 30 securities each, 10 groups with 60 securities each, 6 groups with 90 securities each, 5 groups with 120 securities each, and 4 groups with 150 securities each.

We use the APCA similar to one proposed by Connor and Korajczyk (1986, 1988). Following this procedure, the factor sensitivities are the first K eigenvectors of $R^T R/N$ where R is the $N \times \tau$ matrix of de-meaned asset returns and τ is the number of time-series observations. These eigenvectors are identical to the factor loadings of (3) up to a nonsingular $K \times K$ transformation. This is because a pre- and post-multiplication of (3) by $(Q_N)^{-1/2}$ leads to

$$V_N^* = B_N^*\left(B_N^*\right)^T + I, \tag{5}$$

where $B_N^* = (Q_N)^{-1/2} B_N$ and $V_N^* = (Q_N)^{-1/2} V_N (Q_N)^{-1/2}$ is the covariance matrix of the transformed asset returns $r^* = (Q_N)^{-1/2} r$.

Whereas APCA provides unique estimates in the limit, it is well-known that the use of factor analysis involves the choice of rotation methodology because of the factor indeterminancy problem. We report the results based on a transformation matrix P that yields an orthogonal rotation scaled to yield $PP^T = I$.[4] The pertinent programming procedures are provided in Schönemann (1966). We also conduct the tests using the oblique Procrustes rotation option of the SAS MLFA routines.[5] Both the orthogonal and oblique rotations yield similar conclusions.

We rely on a distribution-free nonparametric test as the distribution of each element in the difference matrix S_N may not always meet the assumptions underlying parametric tests. Specifically, the Wilcoxon (1945) signed-rank test is quite appropriate as it uses information about the magnitudes of the differences and their direction. Since it also uses more information than the sign test, it is often more powerful. We apply the Wilcoxon signed-rank test to differences in summary statistics between 30 and 60 securities, 60 and 90 securities, 90 and 120 securities, and 120 and 150 securities. To use the Wilcoxon procedure, we first rank the differences in order of the absolute size. We then assign the original signs of the differences to the ranks and compute two sums: the sum of ranks with negative signs and the sum of ranks with positive signs. If the sum for one sign is too small, the null hypothesis is rejected.[6] Daniel (1990) provides the associated critical values.

Since portfolios are selected alphabetically, the first two groups in the $N = 30$ set cover the same 60 firms in the first $N = 60$ set. Similarly, the first three (four and five) groups in the $N = 30$ set cover the same 90 (120 and 150) firms in the first $N = 90$ ($N = 120$ and $N = 150$) set. Using the average of S_N for the first two (three, four, and five) groups in the $N = 30$ set as the pair from the first $N = 60$ ($N = 90$, $N = 120$, and $N = 150$) set is a natural way to pair the two populations and avoids the problem of arbitrary selection.

III. Empirical Results

Before testing the null hypothesis, we first provide some descriptive statistics to indicate how close the two sets of factor estimates are for each cross-section. Specifically, we follow Roll (1988) and report the distribution of R^2, adjusted for degrees of freedom, for regressions of the type

$$r_{j,t} = a_{0,j} + a_{1,j} f_{1,t} + \cdots + a_{K,j} f_{K,t} + e_{j,t}, \tag{6}$$

where the a's are the regression coefficients, $e_{j,t}$ is the residual term for the jth security at time t, and $f_{i,t}$ is the ith factor loading when MLFA is used and ith eigenvector when APCA is used.[7]

[4] The derivation of the P matrix is available from the authors upon request.

[5] SAS routines rely on the Howe's solution that maximizes the determinant of the partial correlation matrix.

[6] As the data are paired (i.e., 30 and 60, 60 and 90, 90 and 120, and 120 and 150 securities), the Wilcoxon test is a nonparametric equivalent of the paired t-test as it assumes a continuous variable but does not require the probability density function to be symmetric.

[7] The adjusted R^2's are calculated as $R^2 = 1 - [(\tau - 1)/(\tau - K - 1)][\sigma^2(e)/\sigma^2(r)]$, where σ^2 is the sample variance.

The average R^2's obtained from estimating regression (6) are reported in Table I. The numbers are shown for the entire sample in Panel A, the first half of the sample period in Panel B, and the second half in Panel C. The results indicate that for each sample size, the adjusted R^2's increase as more factors are introduced. This is observed for all three sample periods. The adjusted R^2's also generally increase when more assets are added. Perhaps,

Table I

Adjusted R^2's for Regressions of Asset Returns on Factor Loadings and Eigenvectors

The table reports the average R^2's, adjusted for degrees of freedom, for regressions of the type

$$r_{j,t} = a_{0,j} + a_{1,j} f_{1,t} + \cdots + a_{K,j} f_{K,t} + e_{j,t},$$

where the a's are the regression coefficients, $r_{j,t}$ is the return for the jth security at time t, e is the residual term, and f_i is the ith factor loading when maximum likelihood factor analysis is used and ith eigenvector when asymptotic principal components analysis is used. N denotes the number of securities and K the number of factors. The numbers without parentheses are average adjusted R^2's from regressing asset returns on estimated factor loadings that are obtained using maximum likelihood factor analysis. The numbers in parentheses are average adjusted R^2's from regressing asset returns on estimated eigenvectors that are obtained using asymptotic principal components analysis. There are 20 groups for $N = 30$, 10 groups for $N = 60$, 6 groups for $N = 90$, 5 groups for $N = 120$, and 4 groups for $N = 150$.

	N				
K	30	60	90	120	150
			Panel A: Whole period		
1	0.1537(0.1308)	0.1634(0.1422)	0.1771(0.1618)	0.1814(0.1750)	0.1795(0.1781)
2	0.1819(0.1625)	0.1942(0.1796)	0.2103(0.1996)	0.2086(0.1957)	0.2073(0.1938)
3	0.1852(0.1683)	0.1991(0.1848)	0.2112(0.1992)	0.2210(0.2231)	0.2322(0.2297)
4	0.1989(0.1744)	0.2367(0.2157)	0.2289(0.2113)	0.2437(0.2291)	0.2650(0.2546)
5	0.2398(0.2032)	0.2667(0.2347)	0.2726(0.2441)	0.2972(0.2799)	0.3069(0.2921)
10	0.2907(0.2454)	0.3190(0.2771)	0.3361(0.3018)	0.3225(0.2900)	0.3514(0.3207)
			Panel B: First half-period		
1	0.1219(0.0987)	0.1428(0.1198)	0.1616(0.1405)	0.1605(0.1497)	0.1601(0.1555)
2	0.1473(0.1156)	0.1696(0.1401)	0.1782(0.1548)	0.1816(0.1602)	0.1805(0.1704)
3	0.1515(0.1288)	0.1745(0.1520)	0.1836(0.1640)	0.1940(0.2028)	0.2041(0.1944)
4	0.1588(0.1423)	0.1793(0.1637)	0.2203(0.1759)	0.2203(0.2110)	0.2312(0.2172)
5	0.2095(0.1611)	0.2163(0.1714)	0.2714(0.2391)	0.2640(0.2418)	0.2747(0.2582)
10	0.2795(0.2270)	0.2768(0.2338)	0.2957(0.2471)	0.3111(0.2752)	0.3293(0.2999)
			Panel C: Second half-period		
1	0.1745(0.1404)	0.2093(0.1783)	0.2225(0.1980)	0.2301(0.2165)	0.2446(0.2381)
2	0.2159(0.1912)	0.2282(0.2151)	0.2527(0.2358)	0.2435(0.2311)	0.2638(0.2465)
3	0.2386(0.2004)	0.2461(0.2137)	0.2570(0.2387)	0.2678(0.2462)	0.2782(0.2659)
4	0.2433(0.2103)	0.2738(0.2422)	0.2654(0.2448)	0.2846(0.2702)	0.2812(0.2636)
5	0.2727(0.2279)	0.2834(0.2544)	0.2927(0.2694)	0.3127(0.2937)	0.3036(0.2851)
10	0.3147(0.2631)	0.3242(0.2753)	0.3204(0.2777)	0.3560(0.3184)	0.3475(0.3118)

most interestingly, the absolute differences in R^2's obtained under the two methods generally decrease with increasing sample size.[8]

Table II presents arithmetic averages of the summary statistic, S_N. The statistic S_N is computed for each group of assets and the numbers reported in Table II are an arithmetic average of S_N for all groups with equal sample size. For the entire period and for the two subperiods, the results show that in general, the statistic S_N decreases with increasing N. However, some S_N's increase when more assets are included in the sample. Although the exceptions for the subperiods are double those of the entire period, they only number three or four for each subperiod. Moreover, the differences in the factor structures of APCA and the rotated factor structures of MLFA are

Table II

The Degree of Fit Between Factor Loadings and Eigenvectors

The table reports the averages of the summary statistic $S_N = \text{tr}(Z_N^T Z_N)/NK$, where tr denotes trace, Z_N is the difference between the transformed matrix of factor loadings and the matrix of eigenvectors, N denotes the number of securities, and K is the number of factors. There are 20 groups for $N = 30$, 10 groups for $N = 60$, 6 groups for $N = 90$, 5 groups for $N = 120$, and 4 groups for $N = 150$.

	N				
K	30	60	90	120	150
			Panel A: Whole period		
1	.00046	.00039	.00007	.00005	.00004
2	.00085	.00034	.00058*	.00028	.00019
3	.00139	.00065	.00026	.00020	.00020
4	.00168	.00079	.00028	.00024	.00022
5	.00876	.00109	.00048	.00046	.00037
10	.00624	.00394	.00087	.00120*	.00108
			Panel B: First half-period		
1	.00040	.00017	.00006	.00004	.00002
2	.00085	.00034	.00024	.00023	.00016
3	.00194	.00143	.00118	.00110	.00105
4	.00199	.00092	.00169*	.00114	.00080
5	.00582	.00463	.00125	.00284*	.00089
10	.00748	.00125	.00168*	.00156	.00134
			Panel C: Second half-period		
1	.00059	.00089*	.00016	.00009	.00008
2	.00113	.00046	.00178*	.00143	.00024
3	.00165	.00121	.00018	.00023*	.00016
4	.00135	.00088	.00022	.00025*	.00013
5	.00916	.00076	.00061	.00045	.00035
10	.00287	.00167	.00148	.00134	.00113

*Cases where the summary statistic increases with a larger sample size.

[8] However, MLFA tends to produce higher R^2's than APCA.

relatively small. Indeed, it appears that for cross-sections of 90 or more assets, the statistics for $K + 1$ factor structures are very close to that for K-factor structure. It is also interesting that as more factors are added, the 'fit' between two matrices is less accurate. For example, with ten factors and 150 observations, the 'fit' is 0.00108 which is not as good as that for 1 factor and 30 observations.

The p-values produced by applying the Wilcoxon signed-rank test to paired summary statistics are reported in Table III. In the pairwise comparison of $N = 30$ and $N = 60$, all the p-values for different K-factor structures are significant at the 5% level for all three sample periods. This means that S_N of 30 securities is reliably larger than that of 60 securities. For $N = 120$ against $N = 150$, the p-values generally are less significant than when

Table III

Test Results of Differences in the Degree of Fit Between Factor Loadings and Eigenvectors

The degree of fit between factor loadings and eigenvectors is measured by $S_N = \mathrm{tr}(Z_N^T Z_N)/NK$, where tr denotes trace, Z_N is the difference between the transformed matrix of factor loadings and the matrix of eigenvectors, N denotes the number of securities, and K is the number of factors. The table reports the p-values obtained from applying Wilcoxon signed-rank test to differences in S_N. There are 20 groups for $N = 30$, 10 groups for $N = 60$, 6 groups for $N = 90$, 5 groups for $N = 120$, and 4 groups for $N = 150$.

	Comparison of			
K	$N=30$ and $N=60$	$N=60$ and $N=90$	$N=90$ and $N=120$	$N=120$ and $N=150$
		Panel A: Whole period		
1	.002	.016	.031	.062
2	.002	.016	.031	.062
3	.005	.047	.062	.062
4	.007	.047	.031	.125
5	.005	.078	.094	.125
10	.010	.078	.094	.312
		Panel B: First half-period		
1	.002	.016	.062	.062
2	.003	.031	.031	.062
3	.007	.078	.094	.125
4	.007	.047	.062	.125
5	.010	.078	.094	.188
10	.014	.109	.156	.312
		Panel C: Second half-period		
1	.002	.016	.031	.062
2	.002	.016	.031	.062
3	.005	.047	.062	.062
4	.007	.078	.031	.062
5	.005	.078	.094	.125
10	.007	.078	.094	.188

smaller cross-sections are compared. However, due to the fewer degrees of freedom, the statistical power is relatively weak. Overall, there is strong evidence that differences in the rotated factor structures of APCA and MLFA become smaller for larger cross-sections.

IV. Conclusion

This paper has examined the speed of convergence toward one another of the strict factor structure of Ross (1976) and the approximate factor structure of Chamberlain and Rothschild (1983). An analysis of the implication of Grinblatt and Titman's (1985) interpretation for the systematic risks associated with the two structures suggests the use of a Procrustes rotation for the maximum likelihood factor analysis with the eigenvectors produced by the asymptotic principal components analysis as the target matrix. The empirical results are consistent with the hypothesis that the strict factor structure and the approximate factor structure become increasingly similar as the number of securities grow. This suggests that the asymptotic principal components analysis can be used with virtually no loss in information and that it is not necessary to compute a complete maximum likelihood factor analysis to estimate the factor loadings.

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