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An Option-Theoretic Approach to the Valuation of Dividend Reinvestment and Voluntary Purchase Plans

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ABSTRACT

Many firms with dividend reinvestment plans also allow their shareholders to voluntarily invest supplemental funds to purchase additional shares. The purchase price for newly-issued shares often is determined by the average stock price over a prespecified time period preceding the investment date. This gives the firm's shareholders an option to invest in additional shares only when the stock price exceeds the computed average. This paper uses both theoretical and numerical methods to analyze the value of these voluntary purchase options in theory and practice.

MOST FIRMS WITH DIVIDEND reinvestment plans also allow their shareholders to voluntarily invest additional funds through organized open market purchases of outstanding shares or the direct issuance of new shares, thereby facilitating investment in the firm by small shareholders and, in the latter case, providing an alternative mechanism for the firm to issue securities. Typically, shares acquired in the open market are sold to shareholders at the average price paid by the firm to acquire the shares, while newly issued shares are sold to shareholders at a price determined by a formula specified in the plan's prospectus. For example, in most instances the purchase price for newly issued shares is the average of daily closing (or daily high and low) stock prices over some prespecified period of time, such as the last five trading days of the month. These designs sometimes create valuable options. If the purchase price for shares is computed as an average of prices taken over a period of time that precedes (or partially precedes) the last date on which the firm will accept voluntary cash investments, the investor can use the available price information to decide whether to invest funds. The purpose of this paper is to analyze the value of these voluntary purchase options (VPOs) in theory and practice.

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The number of firms offering dividend reinvestment plans with an option to voluntarily invest additional funds is quite large. According to Evergreen Enterprises (1990), 1,045 publicly traded companies offered dividend reinvestment plans in 1990 and, of these, nearly all allowed shareholders to make voluntary cash investments. The types of shares issued under these plans varied considerably. Of the 1,045 companies with dividend reinvestment plans in 1990, 565 (or 54.1%) used only open market purchases of outstanding shares and 171 (or 16.4%) used only newly issued shares. The remaining 309 companies (or 29.6%) had combined plans, whereby the plan administrator was given the authority to use a combination of newly issued shares and open market purchases. The focus of this paper is on the valuation of the VPOs for those plans that sell newly issued shares at the average of market prices over some prespecified period of time.

The value of the VPO will depend upon a variety of plan-specific features. For example, a relatively small number of plans allow shareholders to purchase newly issued shares at a discount. These discounts, when they exist, are typically 5% or less. Most plans also limit the amount that each shareholder can invest annually, quarterly, or monthly. These investment limits, which vary widely across purchase programs, are typically less than $5,000 per quarter but in some cases can be much higher.[1] When they exist, these investment limits have a dramatic effect on the optimal exercise policy and the valuation of the VPO. We derive the optimal exercise policy in the presence of these investment limits and purchase discounts and provide the theoretical valuation formula for the VPO.

The method used to compute the purchase price for newly issued shares acquired under the plan plays an important role in valuing the VPO. The most common methods differ in two main respects: the length of the averaging period (ranging from one to twenty days) and the specific prices used to compute the average (daily closing prices or daily high and low prices).[2] Although most plans end the averaging period on (or near) the investment date (i.e., the date on which the firm purchases shares for the investor), the averaging period can end much earlier (see Section III for a specific example). Finally, the date on which the shareholder's funds must be received by the firm's transfer agent is also important for determining the value of the VPO. The earlier the funds must be received by the firm's transfer agent, the lower

[1] The investment limits are typically stated on a per account basis and, therefore, can be circumvented to some extent by increasing the number of accounts under which investments can be made. Occasionally, however, limits are imposed on the amount that can be invested under any particular social security or tax identification number. Detailed information on the investment limits and purchase discounts for dividend reinvestment and voluntary purchase plans is provided by Evergreen Enterprises (1990).

[2] Scholes and Wolfson (1989) report that in their sample of firms that offer purchase discounts, the most common averaging periods are 1 and 5 days, although a fair number of firms also use averaging periods that extend beyond 5 days. They conclude from conversations with corporate treasurers and investor relations personnel that the primary reason for using an average of prices over a number of days to determine the purchase price is to minimize the effect of short sales on the market price.

is the option's value.[3] We investigate the effects of these plan-specific features on the value of the VPO using Monte Carlo simulation methods.

Scholes and Wolfson (1989) provide a recent study of voluntary purchase plans that offer discounts. The purpose and focus of their paper differs from ours, however. They focus their attention on those plans that offer substantial purchase discounts and present detailed information describing the characteristics of these plans and how these discounts can be obtained in practice. While they recognize that some of these plans offer shareholders valuable options by allowing them to observe all or part of the price sequence used to compute the purchase price before deciding whether to invest, they do not formally analyze the optimal exercise policy or valuation of these options. The primary purpose of our paper is to provide an analysis of the valuation of the VPO and to explore the effects of various plan features on the option's value. Our results indicate that the incremental value of the VPO can be quite substantial. Under plausible conditions, the incremental value of the VPO is comparable to a direct purchase discount of 3%–5%.[4]

The paper is organized as follows. In Section I, we examine some of the qualitative features of the VPO, derive the optimal exercise policy, and provide the theoretical valuation formula using the Cox and Ross (1976) risk-neutral method. In Section II, we specialize the structure to assume geometric Brownian motion and use Monte Carlo simulation methods to obtain estimates of the value of the VPO. We also examine the effects of a variety of qualitative features on the value of the VPO. In Section III, we provide a specific real-world example illustrating the value of the VPO. Concluding comments and a discussion of some unresolved empirical issues are provided in Section IV.

I. Valuation of the Voluntary Purchase Option

For any particular month of the year, it is optimal to delay exercising the VPO until the last point at which funds must be committed to the firm. Early exercise within the month does not change either the exercise price that is paid for the stock or the date on which the stock is purchased for the investor, but does limit the amount of price information that the investor can collect before exercising and forces the investor to forgo the time value of the

[3] Some plans require investors to submit funds for voluntary purchases well in advance of the investment date, but permit the investors to withdraw their funds by sending a written request to the firm's transfer agent at least one or two days prior to the investment date. In such instances, the effective exercise date is the date investors must send notice to the transfer agent to return their funds.

[4] One valuable feature of the option to purchase stock at the average price determined under the plan is that investors receive capital gains treatment on the difference between the market value and the cost of the shares acquired (so that the profit can potentially be offset with capital losses). In contrast, direct purchase discounts (on both dividend reinvestments and voluntary purchases) are taxed as dividend income, which cannot be offset with capital losses. In our analysis we abstract from tax considerations.

exercise price. These conclusions apply even if the stock were to pay a dividend prior to the expiration of the option. The potential optimality of early exercise of conventional call options immediately prior to a dividend payment is a consequence of the right to acquire the stock cum dividend at that juncture. In contrast, the holder of the VPO only possesses the right to acquire the stock ex dividend.

In practice, most voluntary purchase plans limit the amount that can be invested monthly, quarterly, or annually. These investment limits have important implications for the optimal exercise decision and for the value of the VPO. For example, some companies offer their shareholders the option to make voluntary cash investments each month (at an average price determined monthly), but limit the total amount that can be invested for the year (or quarter). This means that if the option is exercised early in the year (quarter), the investor forgoes the opportunity to invest later in the year (quarter).[5] Since the opportunity cost of exercising the VPO is higher for the earlier months of the year (quarter), the investor will establish a higher exercise boundary for these months. The amount by which the option must be "in-the-money" to make it worthwhile for the investor to exercise the option is equal to the value of the future options given up by exercising. Thus, the value of the VPO can be solved recursively in the presence of an annual (or quarterly) investment limit, with the exercise price of the option for any particular month adjusted upward to include the value of the remaining options for the year (quarter).

Unlike conventional call options, the value of the VPO is "path-dependent" since its exercise price (i.e., the average of daily closing stock prices or daily high and low stock prices) is a function of the stock price path. Harrison and Kreps (1979) have shown that any contingent claim whose value is only a function of past stock prices is hedgable under certain regularity conditions.[6] Therefore, we can construct a portfolio whose value is perfectly locally correlated with the value of the VPO, thus allowing us to employ the Cox and Ross (1976) "risk-neutral" valuation methodology. Accordingly, the value of the VPO is found by discounting the option's expected payoff at the riskless

[5] In the absence of liquidity constraints, it is easy to show that if the VPO is exercised, it should be exercised to the limit allowed under the plan. Moreover, if multiple accounts are available, they should all be exercised at the same point in time. Consequently, under the optimal strategy, the investor will invest in the voluntary purchase plan at most once per year in the presence of an annual investment limit (assuming no liquidity constraints).

[6] Goldman, Sosin, and Gatto (1979) analyze the price dynamics and hedgability of a path-dependent option with an exercise price equal to the minimum stock price reached during its life. Another interesting analysis of a path-dependent equity option is Goldman, Sosin, and Shepp (1979), who study the option to buy at the minimum price and sell at the maximum price over a fixed future interval of time. Ritchken, Sankarasubramanian, and Vijh (1989), among others, analyze the valuation of a path-dependent option whose exercise price depends upon past stock prices and, in particular, examine the option to buy (sell) at the average of past stock prices. These options are obviously closely related to the VPO whose exercise price is the average of daily closing (or high and low) stock prices. We refer the reader to the above papers for the formal hedgability arguments.

interest rate, where the expectation is taken with respect to the "risk-neutral" probability measure.

Using the Cox and Ross (1976) "risk-neutral" valuation method, the value of the VPO with i months remaining in the year (quarter), $i = 1, \cdots, 12$ ($i = 1, 2, 3$), can be expressed by the difference equation:

$$V_i = \hat{E}\left[\max\left(V_{i-1}, \left[S_i^*(1 - c) - A_i(1 - d)\right] Z_i\right)\right] e^{-rt}, \tag{1}$$

where $V_0 = 0$ is the boundary condition, r is the annual riskless interest rate, t is the fraction of the year over which the option's payoff is discounted, c is the one-way transaction cost rate, d is the purchase discount rate, S_i^* is the closing stock price on the investment date in the ith month, A_i is the purchase (exercise) price computed for the ith month, Z_i is the number of shares the investor is allowed to purchase in the ith month, and $\hat{E}$ is the expectation operator taken with respect to the "risk-neutral" probability measure.[7] The value of Z_i depends upon the dollar investment limit, L, and is computed as

$$Z_i = \frac{L}{A_i(1 - d)}. \tag{2}$$

Notice that in the presence of a dollar investment limit, a purchase discount not only reduces the effective exercise price of the option, but also increases the number of shares that the investor is allowed to purchase through its effect on Z_i.

The payoff on the VPO is given by the max operator in Equation (1). If the investor exercises the option in the ith month, for example, he forfeits all future options for the remainder of the year (quarter) and receives the payoff given by the second term inside the max operator. This payoff is equal to the product of two terms. The first term, $[S_i^*(1 - c) - A_i(1 - d)]$, gives the payoff per share of stock purchased and the second term, Z_i, gives the number of shares that can be purchased given the investment limit of L dollars. If the option is not exercised in the ith month, however, the investor maintains the option to make voluntary purchases in the future. The value of this option is V_{i-1}. Therefore, the investor will choose to exercise the option in the ith month only if the value of the proceeds from exercising exceeds V_{i-1}. The

[7] Notice that the net proceeds from exercising the option are reduced by the transaction costs incurred in selling the acquired shares in the open market. These transaction costs are included to give a more realistic view of the value of the VPO in practice, although the existence of transaction costs themselves may draw into question the validity of the Cox and Ross (1976) risk-neutral method for valuation purposes. A literal interpretation of our analysis is that it provides the valuation of the VPO for a truly risk-neutral agent in the presence of transaction costs. An alternative interpretation is that since transaction costs only affect the final payoff on the VPO, its "shadow price" for a risk-averse (or risk-neutral) agent should be equal to the value of a dynamically adjusted hedge portfolio that can be traded *without* transaction costs and which mimics the final payoff on the VPO. The value of such a portfolio is given by the Cox and Ross (1976) risk-neutral approach. Moreover, given the success of the Cox and Ross (1976) risk-neutral method in valuing standard stock options, we are confident that our approach provides a reasonable approximation to the value of the VPO in practice.

expected payoff (under the "risk-neutral" probability measure) for the ith month is then discounted at the riskless interest rate to determine the option's value.

Interestingly, with a dollar investment limit the payoff on the VPO, and therefore its value, is independent of the beginning stock price (assuming a homogeneous stock price process). A higher stock price at the start of the month reduces the number of shares that can be purchased with an investment limit of L dollars, but increases the payoff per share purchased by exactly the amount necessary to leave the total payoff unaffected. While many voluntary purchase plans impose annual or quarterly investment limits, it is not uncommon to observe monthly investment limits as an alternative mechanism to control the issuance of new shares. In the case of a monthly investment limit, the exercise decision for one month has no effect on the investor's future investment opportunities. Therefore, with a monthly investment limit, the value of the VPO for any particular month is independent of the number of months remaining in the year. The value of this option will be equal to the value of the VPO with one month remaining in the year and an annual investment limit equal to the monthly investment limit. Setting $i = 1$ in Equation (1) yields:

$$V_1 = \hat{E}\left[\max\left(0, \left[S_1^*(1 - c) - A_1(1 - d)\right]Z_1\right)\right]e^{-rt}, \tag{3}$$

where Z_1 now represents the number of shares the investor is allowed to purchase given a monthly investment limit of L dollars. With a monthly investment limit, Equation (3) gives the value of the VPO for each month of the year.

II. Monte Carlo Simulations of the Voluntary Purchase Option

In this section, we estimate the value of the VPO using Monte Carlo simulations. The Monte Carlo simulation method is particularly useful for valuing options with path dependencies, such as the VPO.[8] The method simulates a large number of stock price paths and values the VPO using the Cox and Ross (1976) risk-neutral pricing method. Under the risk-neutral approach, the mean of the distribution of option values serves as an unbiased estimate of the option's value.

To value the VPO, we assume that the stock price follows a geometric Brownian motion process. Consequently, under the risk-neutral assumption, the stock price at date $t + \Delta t$ is given by

$$S(t + \Delta t) = S(t)\exp\left[(r - .5\sigma^2)\Delta t + \sigma(\Delta t)^{1/2}w\right], \tag{4}$$

where r is the annual riskless interest rate, σ is the standard deviation of the annual rate of return on the stock, and w is a realization of a standard normal random variable. The Monte Carlo method uses Equation (4) to

[8] See Boyle (1977) for a detailed discussion of the use of the Monte Carlo method for estimating option values.

simulate the stock price at discrete points along its continuous path, providing a simple way of simulating daily closing stock prices. However, the maximum and minimum stock prices taken from the discrete sample path will be biased estimates of the maximum and minimum stock prices generated by the continuous process. In particular, the estimate of the daily maximum stock price will be too low and the estimate of the daily minimum stock price will be too high. To mitigate this problem, we simulate 50 stock price changes per trading day.[9]

A. The Case of a Monthly Investment Limit

We begin with an investigation of the value of the VPO for a single month, assuming a monthly investment limit of $L = \$100$. With a monthly investment limit, as opposed to an annual (or quarterly) investment limit, the valuations and exercise decisions are independent across months of the year. Given this independence, the value of the VPO is the same for each month of the year (or quarter) and is given by Equation (3). The Monte Carlo estimate of the value of this option is given by:

$$V_1 = \frac{1}{N} \sum_{n=1}^{N} e^{-rt} \max\left(0, \left[S_1^*(n)(1-c) - A_1(n)(1-d)\right] Z_1(n)\right), \quad (5)$$

where n indexes the results for the nth simulation trial, N is the total number of simulation trials, and all other variables are defined previously.

Since many VPOs have either a 5- or 10-day averaging period that ends on (or near) the investment day, we simulate stock price movements over a 10-day period ending on the investment day.[10] We value the VPO at the start of the month (i.e., $t = 1/12$) using both a 5- and 10-day averaging period (ending on the investment day) and two alternative methods for computing the purchase (exercise) price: the average of daily closing stock prices and the average of daily high and low stock prices. We simulate the value of these options assuming both 0% and 1% transaction costs and both a 0% and 5% discount. The riskless interest rate is assumed to be 8% per year.

The results for both the daily closing stock price average (Cl) and the daily high and low stock price average (HL) are reported in Table I. These estimates are based on $N = 100{,}000$ simulation trials and represent the value (at the start of the month) of the one-month VPO with a monthly investment limit of \$100. The results are reported for six different standard deviations ranging from 0.1 to 0.6. It is assumed that the investor can observe the entire 10-day price sequence before deciding whether to invest, but cannot delay the investment decision beyond the investment day.

[9] Our stimulation results for the valuation of the VPO were virtually the same when the number of stock price changes per day was assumed to be 25. This suggests that simulating 50 stock price changes per day is adequate.

[10] Many voluntary purchase plans state that the average will be computed over the last 5 to 10 business days ending on the investment date. Thus, in generating stock price changes and discounting the payoffs on the option we use a 260-day year.

Table I

Estimates of the Value (at the Beginning of the Month) of the One-Month Voluntary Purchase Option (VPO), Assuming a Monthly Investment Limit of $100

The purchase price for shares (gross of any purchase discount) is computed as the average of daily closing stock prices (C1) or daily high and low stock prices (HL) over the last 5 business days (Panel A) or last 10 business days (Panel B) of the month. These estimates are averages taken over 100,000 simulation trials and are based on the assumption that the investor makes his investment decision immediately after observing the closing stock price for the month. The annual riskless interest rate is assumed to be 8%. The standard deviation of the annual rate of return on the stock ranges between 10% and 60%. TC is the one-way transaction cost rate, and PD is the rate of purchase discount.

TC (%)	PD (%)	Type	Standard Deviation 0.10	0.20	0.30	0.40	0.50	0.60
			Panel A: 5-day average					
0	5	C1	5.288	5.276	5.262	5.267	5.308	5.384
		HL	5.302	5.288	5.279	5.310	5.391	5.516
1	5	C1	4.241	4.231	4.229	4.268	4.352	4.472
		HL	4.256	4.244	4.260	4.339	4.472	4.642
0	0	C1	0.298	0.560	0.820	1.077	1.330	1.581
		HL	0.352	0.659	0.963	1.264	1.560	1.852
1	0	C1	0.025	0.189	0.409	0.645	0.887	1.129
		HL	0.047	0.267	0.538	0.822	1.108	1.393
			Panel B: 10-day average					
0	5	C1	5.365	5.347	5.374	5.487	5.670	5.899
		HL	5.379	5.363	5.411	5.556	5.774	6.037
1	5	C1	4.318	4.313	4.397	4.576	4.815	5.090
		HL	4.332	4.336	4.448	4.662	4.937	5.245
0	0	C1	0.482	0.884	1.281	1.672	2.056	2.434
		HL	0.523	0.956	1.384	1.805	2.218	2.625
1	0	C1	0.116	0.456	0.831	1.210	1.588	1.961
		HL	0.143	0.520	0.928	1.339	1.747	2.149

The values reported in Table I are an increasing function of the standard deviation and level of the purchase discount and a decreasing function of the level of transaction costs.[11] In the presence of a 5% discount, most of the value of the VPO is attributable to the value of the discount itself (i.e.,

[11] With a 5% discount, the value of the VPO is decreasing in the volatility at the low end of the volatility spectrum. This behavior is due entirely to sampling variation. With a 5% discount, the value of the VPO is relatively insensitive to the volatility of the stock, making comparisons across volatilities much more sensitive to sampling variation. Note also that in the absence of transaction costs and purchase discounts the value of the VPO increases approximately linearly in the volatility of the stock. This result reflects the well-known fact that the Black-Scholes (1973) formula is approximately linear in the volatility for "at-the-money" options (see, for example, Cox and Rubinstein (1985), page 218, Figures 5-6).

\$100[.05/.95] = \$5.263), and relatively little value is attributable to the volatility of the stock. In fact, it is apparent from Table I that the value of the VPO is relatively insensitive to changes in the volatility of the stock in the presence of a 5% discount. This behavior reflects the fact that the VPO is essentially a deep "in-the-money" option in the presence of a 5% discount, making volatility much less important in determining the option's value. Note also that the value of the VPO is highly sensitive to the level of transaction costs in the presence of a substantial purchase discount. This reflects the fact that the VPO is nearly always exercised in the presence of a 5% discount, even if the price of the stock declines over the averaging period.

Table I indicates that the one-month VPO is more valuable when the purchase (exercise) price is based upon the 10-day average rather than the 5-day average. The intuition underlying this result is straightforward. Since the price of the stock tends to increase over the averaging period, especially in those states of the world in which it is optimal to exercise the option, it is more advantageous to include earlier stock prices in computing the purchase (exercise) price. Similar reasoning also explains why the VPO is slightly more valuable when the purchase (exercise) price is based upon the daily high and low stock prices rather than the daily closing stock prices. Of course, this is an ex ante argument. On an ex post basis, it is possible that for a particular stock price path the relative values of these options can be reversed.

It is evident from Table I that the one-month VPO is valuable even in the absence of a purchase discount. For example, in the absence of transaction costs and purchase discounts, the value of the VPO ranges from a low of \$0.298 (or 0.298% of the monthly investment limit) for a stock with an annual standard deviation of 0.1 and a purchase price equal to the average of the daily closing stock prices over the last 5 days of the month (see Panel A), to a high of \$2.625 (or 2.625% of the monthly investment limit) for a stock with an annual standard deviation of 0.6 and a purchase price equal to the average of the daily high and low stock prices over the last 10 days of the month (see Panel B). In the presence of 1% transaction costs these values are \$0.025 and \$2.149, respectively.[12]

[12] The magnitude of the option values reported in Table I raises some interesting general equilibrium issues. Most companies allow their shareholders to participate in the voluntary purchase plan provided they own at least one share of stock. Moreover, the investment limit specified in the plan prospectus is typically independent of the number of shares owned by the shareholder. Consequently, if every shareholder held only one share of stock, and exercised the VPO optimally and to the limit allowed under the plan, then there would be no effect on the wealth of any of the firm's shareholders. In this case, the option to purchase shares at a discount would be equivalent to a pro rata dividend distribution or rights offering. If some shareholders hold more than one share of stock, or fail to exercise the VPO optimally and to the limit allowed under the plan, then there will be a net wealth transfer from nonparticipating shareholders to participating shareholders. The values reported in Table I assume that participation in the plan has a negligible effect on the equilibrium market price of the firm's shares. This assumption is not unrealistic given the low participation rates observed in practice (see Scholes and Wolfson (1989) for a discussion of the dollar amounts raised under these plans).

B. The Case of an Annual or Quarterly Investment Limit

We next investigate the value of the VPO in the presence of an annual (or quarterly) investment limit, as opposed to a monthly investment limit. In the presence of an annual (or quarterly) investment limit, with the option to invest monthly, the valuations and exercise decisions are interrelated across months of the year. The theoretical value of the VPO in the presence of an annual (or quarterly) investment limit is given by Equation (1). The Monte Carlo estimate of this option's value is given by the following recursive formula:[13]

$$V_i = \frac{1}{N} \sum_{n=1}^{N} e^{-rt} \max\left(V_{i-1}, \left[S_i^*(n)(1-c) - A_i(n)(1-d)\right] Z_i(n)\right), \quad (6)$$

where V_i is the value of the VPO with i months remaining in the year (or quarter). The boundary condition for Equation (6) is $V_0 = 0$.

Table II provides estimates of the values of the VPO for each of the 12 months of the year for a stock with an annual standard deviation of 0.4. The annual riskless interest rate is assumed to be 8%. In the presence of a quarterly investment limit, as opposed to an annual investment limit, only the first three columns of Table II are relevant.[14] Panel A reports the results for the 5-day closing price average and Panel B reports the results for the 10-day closing price average. These estimates are based on $N = 100{,}000$ simulation trials and are reported for both 0% and 1% transaction costs and both a 0% and 5% purchase discount. The option values reported in Table II have been discounted to the beginning of the month ($t = 1/12$) and represent the values of the option to invest a total of \$100 at any time during the remainder of the year (or quarter) at the average price determined for the month in which the option is exercised. For the moment, we assume that the investor has complete price information and thus makes his exercise decision

[13] Notice that the Monte Carlo estimate of the value of the VPO for month i involves the sample estimate of the option value for month $i - 1$. In small samples, this induces a small upward bias in the estimate of the value of the option for month i because of the convexity of the max operator. However, this bias vanishes asymptotically and the estimator is consistent. To investigate this bias, we simulated the option values for each month of the year using 10 subsamples of 10,000 observations each and computed the average option values across the 10 subsamples. If the bias is present, there will be a difference between the average option values and the corresponding option values estimated from the full sample of 100,000. The maximum difference we detected was one-tenth of one cent, suggesting that our sample size of 100,000 is adequate.

[14] Table II assumes that the investor has the option to invest in any month of the year (or quarter), but occasionally firms will only permit their shareholders to invest in the last month of the quarter (i.e., typically the month in which the firm pays a dividend). In this case, the first four columns of Table II can be used to compute the value of the option to invest at the end of each quarter in the presence of an annual investment limit. The only adjustment that needs to be made to the option values reported in Table II involves an adjustment for quarterly (as opposed to monthly) discounting.

Table II

Estimates of the Value (at the Beginning of Each Month) of the Voluntary Purchase Option (VPO) for Each of the 12 Months of the Year, Assuming an Annual Investment Limit of $100

The purchase price for shares (gross of any purchase discount) is computed as the average of daily closing stock prices over the last 5 business days (Panel A) or last 10 business days (Panel B) of the investment month. ID refers to the number of days prior to the end of the month that the investor must make his investment decision (i.e., ID = 0 means that the investor can observe the entire stock price sequence for the month before deciding whether to invest and ID = 2 means that the investor can observe all but the last two days of stock price movements for the month before deciding whether to invest). These estimates are averages taken over 100,000 simulation trials and are based on the assumption that the standard deviation of the annual rate of return on the stock is 40%. The riskless interest rate is assumed to be 8%. TC is the one-way transaction cost rate and PD is the rate of purchase discount.

TC (%)	PD (%)	ID	Months Remaining in the Year											
			1	2	3	4	5	6	7	8	9	10	11	12
Panel A: 5-day average														
0	5	0	5.266	6.374	7.003	7.436	7.762	8.018	8.226	8.399	8.550	8.676	8.788	8.886
		2	5.233	5.683	5.925	6.092	6.225	6.322	6.397	6.455	6.505	6.546	6.582	6.614
1	5	0	4.267	5.344	5.962	6.389	6.712	6.987	7.174	7.347	7.499	7.625	7.739	7.838
		2	4.187	4.636	4.880	5.047	5.182	5.281	5.359	5.420	5.472	5.513	5.551	5.584
0	0	0	1.080	1.709	2.138	2.462	2.721	2.935	3.115	3.269	3.407	3.525	3.632	3.728
		2	0.440	0.698	0.866	0.994	1.107	1.199	1.271	1.333	1.387	1.432	1.474	1.511
1	0	0	0.648	1.096	1.428	1.691	1.910	2.095	2.254	2.391	2.516	2.624	2.723	2.813
		2	0.104	0.194	0.266	0.330	0.391	0.446	0.493	0.537	0.571	0.605	0.637	0.665
Panel B: 10-day average														
0	5	0	5.487	7.113	8.065	8.730	9.233	9.635	9.963	10.238	10.480	10.688	10.869	11.030
		2	5.296	6.418	7.052	7.484	7.812	8.069	8.276	8.450	8.601	8.732	8.844	8.939
1	5	0	4.575	6.124	7.045	7.694	8.187	8.583	8.908	9.180	9.420	9.626	9.807	9.968
		2	4.298	5.388	6.009	6.436	6.761	7.017	7.223	7.397	7.549	7.680	7.793	7.890
0	0	0	1.672	2.645	3.314	3.820	4.224	4.559	4.841	5.082	5.299	5.488	5.656	5.807
		2	1.103	1.736	2.175	2.501	2.764	2.976	3.157	3.312	3.449	3.572	3.678	3.772
1	0	0	1.210	2.000	2.572	3.018	3.382	3.689	3.950	4.175	4.379	4.558	4.718	4.864
		2	0.663	1.118	1.459	1.723	1.946	2.131	2.292	2.433	2.556	2.669	2.768	2.855

immediately after observing the closing stock price on the monthly investment day.[15] The results for this case are denoted by ID = 0 in Table II.

As the results for the complete price information case indicate, the value of the VPO increases as the number of months remaining in the year increases. This is not surprising since an increase in the number of months remaining in the year increases the option's life. The incremental value of an additional month declines, however, as the number of months remaining in the year increases. The incremental value of an additional month declines slightly as the level of transaction costs increases, or the level of the purchase discount decreases. It also appears that the incremental value of an additional month is higher for the 10-day average than for the 5-day average.

With complete price information the VPO can have considerable value. As an illustration, consider the VPO with a 5% discount and 0% transaction costs. If the purchase (exercise) price is equal to the closing stock price on the investment day (i.e., if the firm does not use an averaging period to compute the purchase price for shares), then the investor optimally exercises the option in the first month of the year and collects the straight discount income of \$5.263 (i.e., \$100[.05/.95] = \$5.263). However, if the exercise price is determined by the average of the closing stock prices over the last 10 days of the month (see Panel B), then the VPO is worth much more than \$5.263 at the start of the year. With a full 12 months remaining in the year (column 12) the option is worth \$11.030, which is more than twice the value of the straight 5% discount! The reason the option is so valuable in this case is that the investor, without the risk of losing the 5% discount, can delay his exercise decision and wait for a month in which the average purchase price for shares is especially favorable. In the early part of the year, the option to delay the exercise decision has considerable value. As the end of the year approaches, however, the value of the VPO approaches the value of the straight 5% discount. With one month remaining in the year the VPO is worth \$5.487.

The VPO also can be quite valuable in the absence of a discount. For example, in the absence of a purchase discount and transaction costs, the VPO with a purchase price equal to the 10-day average of closing stock prices (see Panel B) is worth \$5.807 at the start of the year. This is approximately equal to the value of a straight purchase discount of 5.5% (i.e., \$100[.055/.945] = \$5.820). Thus, although purchase discounts have considerable value, the option provided by the averaging techniques used to compute the purchase price also can be quite valuable to the investor and should not be overlooked. Of course, the value of the purchase discount is independent of

[15] Since Table II uses closing stock prices to compute the average purchase price for shares, we simulated only a single stock price change per day, as compared to the 50 stock price changes per day in Table I. For this reason, the values reported in the first column of Table II differ slightly from those reported in Table I. The same sequences of stock prices were used to construct both panels in Table II. For each month, however, a new set of random draws is used to simulate the stock price changes. This ensures that the stock price sequences are independent across months.

the volatility of the stock, whereas the value of the option to purchase at the average stock price increases in the volatility of the stock.

Table II also illustrates the effect of transaction costs on the value of the VPO in the presence of an annual (or quarterly) investment limit. With a 5% purchase discount, the value of the VPO is highly sensitive to transaction costs across all months of the year. This reflects the fact that the VPO is nearly always exercised in the presence of a 5% discount. In contrast, in the absence of a purchase discount the value of the VPO is more sensitive to transaction costs at the start of the year than at the end of the year. This reflects the fact that the probability of exercising the VPO declines as the end of the year approaches.

C. The Effect of Incomplete Price Information

One real-world complication that reduces the value of the VPO relates to the timing of the cash investments. Many companies require that the proceeds of voluntary purchases be received by the company's transfer agent one or two days in advance of the investment date. Thus, it may be necessary to make an exercise decision without knowing the complete stock price sequence. In this case, the VPO should be exercised if the *expected* terminal stock price exceeds the *expected* exercise price by more than the value of the remaining options for the year (or quarter), conditional on the price information known at the time the exercise decision must be made. (Remember, we are in a risk-neutral economy so that expected values are all that matter.) However, because the exercise decision must be made with incomplete information, there will be instances in which the investor will regret his exercise decision ex post.

Table II also provides estimates of the value of the VPO for a stock with an annual standard deviation of 0.4 in the presence of incomplete price information. In particular, the option values for the case in which the investor must make his exercise decision 2 days before the investment day are denoted by ID = 2 in Table II. For comparability, the same sample of 100,000 simulation trails that were used to construct the complete price information results (ID = 0) were also used to construct the incomplete price information results (ID = 2). Intuitively, the earlier the exercise date in relation to the investment date, the less price information the investor is able to collect and the lower will be the value of the VPO. This is clearly evident from the results reported in Table II. Nevertheless, with incomplete price information the VPO retains much of its value. For example, in the absence of a purchase discount and transaction costs, the VPO based on the 10-day closing price average (see Panel B) is worth \$3.772 with 12 months remaining in the year and \$1.103 with one month remaining in the year. The corresponding values for the VPO based on the 5-day closing price average (see Panel A) are \$1.511 and \$0.440, respectively.

The effect of incomplete price information on the value of the VPO depends upon the averaging period used to compute the purchase price for shares. In

general, the shorter the averaging period used to compute the purchase price for shares, the more sensitive is the value of the VPO (in terms of the percentage of value lost) to incomplete price information. For example, the results reported in Table II indicate that the percentage of value lost is higher for the 5-day closing price average than it is for the 10-day closing price average. The explanation is intuitive. With a shorter averaging period, the amount by which the option is in the money at the end of the month depends more heavily on the stock price movements over the last few days of the month. Without the ability to observe these stock price movements, the investor is more likely to regret his exercise decision ex post the shorter the averaging period.

The effect of incomplete price information on the value of the VPO also depends upon the level of the purchase discount and the level of transaction costs. In the presence of a 5% purchase discount, the effect of incomplete price information on the value of the VPO is relatively small and tends to be less important at the end of the year than it is at the beginning of the year when price variability is relatively more important. Incomplete price information has a larger effect on the value of the VPO (in terms of the percentage of value lost) for higher levels of transaction costs. This is because the end-of-month stock price movements are relatively more important in determining the option's payoff in the presence of transaction costs. Thus, losing the opportunity to observe the stock price movements over the last few days of the month is relatively more costly in the presence of transaction costs.

III. A Real-World Example: The Case of Southwestern Bell Corporation

An interesting real-world example involves the voluntary purchase plan offered by Southwestern Bell Corporation (SBC). Effective January 1, 1988, the voluntary purchase plan offered by SBC allowed its shareholders to make monthly voluntary cash investments for the purchase of additional shares of common stock, subject to an annual investment limit of $50,000 per shareholder account. Owners of record of at least one share of common stock of the company were qualified to participate in the plan.[16] The purchase price for shares was determined by the average of the daily high and low transaction prices for the company's common stock over the period from the 10th through the 20th (or the next business day thereafter if the stock market was closed on the 20th) of each month. This averaging period represents approximately 8 trading days. Although the purchase price for shares could be computed at the close of trading on the 20th of each month, the company accepted all voluntary cash investments that were received on or before the last business day of the month. This allowed the company's shareholders to observe the

[16] The annual investment limit of $50,000 per shareholder account is independent of the number of shares held. Moreover, the annual investment limit could be circumvented easily through the use of multiple shareholder accounts.

stock price movement for approximately 6 additional business days from the end of the averaging period before making their exercise decisions. The method used to compute the purchase price, the timing of the voluntary cash investments in relation to the averaging period and the large annual investment limit made this VPO particularly attractive despite the absence of a direct purchase discount and the low volatility of the stock.

Table III provides estimates of the value of the VPO offered by SBC for each of the 12 months of the year. These estimates are based on $N = 100{,}000$ simulation trials and are reported for both 0% and 1% transaction costs. The option values reported in Table III have been discounted to the beginning of each month ($t = 1/12$) using an annual riskless interest rate of 8% and represent the value of the option to invest a total of \$50,000 at any time during the year at the average price determined for the month in which the voluntary purchase is made. Although the voluntary purchase plan offered by SBC clearly stated that the purchase price for shares is the average of the daily high and low transaction prices over the 10th through the 20th of each month, the option values reported in Table III are based on the simplifying assumption that the purchase price for shares is the average of daily *closing* stock prices over an 8-day period. As we have seen, the use of daily closing stock prices instead of daily high and low stock prices will underestimate the value of the VPO. We assume that the investor has the opportunity to observe 6 additional days of stock price movements beyond the averaging period before making his exercise decision. The annual standard deviation of the rate of return on the common stock of SBC was assumed to be 0.2 in the simulations.

Table III indicates that the VPO offered by SBC is quite valuable despite the lack of a purchase discount and the low volatility of the stock. At the start of the year, the value of the VPO is \$2,463.23 in the absence of transaction costs and \$1,993.88 with 1% transaction costs. These values are equivalent to a direct purchase discount of approximately 4.7% (e.g., with $c = 0$, \$50,000[.047/.953] = \$2,465.90). With one month remaining in the year, the VPO still has a value of \$735.87 in the absence of transaction costs and \$498.67 with 1% transaction costs. It is interesting to note that the actual realized returns from exercising this option over the 17 months that it was available were quite substantial, exceeding 7% of the amount invested for some months. In late May 1989, the company notified its shareholders that in the future the purchase price for shares would be computed as the average of the high and low transaction prices on the investment day, thereby eliminating the value of the VPO.

IV. Concluding Comments

We have examined in detail the VPO offered by many voluntary purchase plans that use an average stock price computed over a period of time preceding the investment date to determine the purchase price for shares and have shown that these VPOs can have considerable value. We derived the

Table III

Estimates of the Value (at the Beginning of Each Month) of the Voluntary Purchase Option (VPO) Offered by Southwestern Bell Corporation (SBC) for Each of the 12 Months of the Year

The annual investment limit is $50,000. The purchase price for shares is computed as the average of daily closing stock prices over the 10th through the 20th of the month (taken to be 8 business days in the simulations) in which the investment is made. These estimates are averages taken over 100,000 simulation trials and are based on the assumption that the investor observes the entire stock price sequence for the month (taken to be 6 additional business days beyond the averaging period) before making his investment decision. The standard deviation of the annual rate of return on SBC's common stock is assumed to be 20% and the annual riskless interest rate is assumed to be 8%. TC is the one-way transaction cost rate.

TC	Months Remaining in the Year											
(%)	1	2	3	4	5	6	7	8	9	10	11	12
0	735.87	1152.45	1430.69	1644.81	1814.31	1952.88	2071.65	2174.56	2259.14	2336.11	2403.38	2463.23
1	498.67	824.87	1056.10	1241.44	1392.02	1517.19	1626.00	1721.35	1800.40	1872.92	1936.77	1993.88

optimal exercise policy in the presence of an investment limit and estimated the value of the VPO using Monte Carlo simulations. Under plausible conditions, the incremental value of the VPO beyond any purchase discount was shown to be as high as 3%–5% of the allowable investment limit.

Our analysis was based on the assumption that shareholder participation in the voluntary purchase plan has no effect on the stock price. The justification for such an assumption was that participation rates in practice tend to be quite small, even for plans that offer purchase discounts (see Scholes and Wolfson (1989)). The reason for this is still not well understood. It may be that the low participation rates are due to the fact that it is difficult for large institutional investors to exploit these plans given the investment limits per shareholder account. Clearly, these investment limits favor small shareholders and encourage disperse ownership of the company's stock. Also, perhaps the existence of a valuable VPO is not well understood or appreciated by small shareholders. An analysis of the exercise behavior of both small and large shareholders would shed additional light on this issue.

There are two other interesting empirical issues relating to the VPO. The first has to do with the stock price reaction to the announcement by the firm of the introduction of a voluntary purchase plan. For example, does the stock price react positively or negatively to the announcement and does this reaction depend upon the value of the VPO being offered and the percentage of the firm's shares held by large institutions? The second empirical issue has to do with the stock price reaction during the averaging period used to compute the purchase price for shares. Scholes and Wolfson (1989) have suggested that firms use an average of prices over a number of days to compute the purchase price for shares to minimize the effect of short sales. This raises the question of whether short-sale activity and trading volume increases during the averaging period and to what extent this trading activity is related to the exercise behavior of shareholders.

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