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Path Dependent Options: The Case of Lookback Options

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ABSTRACT

Lookback options are path dependent contingent claims whose payoffs depend on the extrema of a given security's price over a certain period of time. Using probabilistic tools, we derive explicit formulas for various European lookback options, and provide some results about their American counterparts.

LOOKBACK OPTIONS ARE PATH dependent options whose payoffs depend on the maximum or the minimum attained over a certain period of time by a security's price. Some of these options are already traded on specialized markets (such as foreign exchange markets), while others are still to be developed.

Since these options have not yet been baptized by the market, we have decided to give them names of our own choice.

A "standard lookback call" is the right to buy at the historical lowest price over a certain period. Work on standard lookback options was first done by Goldman, Sosin, and Gatto (1979) and Goldman, Sosin, and Shepp (1979).

A "call on maximum" looks like the ordinary call except that the security's price is replaced by its maximum. The advantage of this option is that it behaves like a standard call exercised at the optimal date (i.e., its intrinsic value is always non decreasing).

A "limited risk call" has the same payoff as a call on the security, except when the historical maximum is greater than a certain level, in which case the payoff becomes zero. It enables an investor to sell a call without being exposed to a huge upward movement of the security. Merton (1973) called a related option a "down and out" option.

A "partial lookback call" is the right to buy at some percentage over the minimum. It is a less expensive version of the "standard lookback call."

This paper shows that in the framework of the Black and Scholes model of option valuation, it is possible to obtain closed form solutions for the value of the above European lookback options (both for the calls and the correspond-

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ing puts). Moreover, using probabilistic tools (the Snell envelope), we obtain some results for their American counterparts. This work is very similar in spirit to Conze and Viswanathan (1989) on geometric average options.

The paper is organized as follows. The first section summarizes theoretical results which prove that it is possible to have an exact replication of these lookback options. In the second section, we compute the various relevant probability distributions. The third section presents the calculation of the options. In the fourth section, we present some results for American options pricing, and we use them to get closed form bounds on their values. The fifth section provides numerical examples. The sixth and last section gives extensions to futures and foreign exchange markets.

I. Theoretical Results About Option Pricing

We consider a horizon date T. As in the Black and Scholes model, the economy consists of two assets, a security of price S_t, and a zero-coupon bond maturing at T, of constant interest rate r. As usual, S_t is assumed to be a stochastic process on a probability space (Ω, F, p), with a filtration $F = \{F_t, 0 \leq t \leq T\}$ satisfying the usual requirements, and p a probability measure.[1] The filtration describes the way information is revealed to investors. We assume that this information at time t consists only of past prices. The technical conditions collectively known as "usual requirements for the filtration" insure us that the flow of information is continuous.

The security's price S_t satisfies the usual stochastic differential equation

$$dS_t = \alpha S_t \, dt + \sigma S_t \, dW_t, \; S_0 > 0, \tag{1}$$

where W_t is a standard Brownian motion on (Ω, F, p), and α and σ are two constants.

As in Harrison and Pliska (1981), a contingent claim (an option) is defined as any F_T = measurable nonnegative L^2 random variable. The market is said to be arbitrage free if there exists a probability p^* equivalent to p under which the process $S_t e^{-rt}$ is a martingale. The market is said to be complete if and only if this probability p^* is unique.

We know from Harrison and Kreps (1979) and Harrison and Pliska (1981) that if the market is complete, every contingent claim may be perfectly replicated with the stock and the zero-coupon bond, using a self-financing strategy. Their results also imply that the price of any option is the discounted expectation of its payoff under p^*, the unique measure mentioned above. Denoting by Π_t the value of the option at time t, and by V the payoff of the option at time T, this may be written as

$$\Pi_t = e^{-r(T-t)} E_{p^*}[V \mid F_t], \tag{2}$$

where E_{p^*} is the expectation under p^*.

[1] F_t is right continuous, and contains all the p-null sets. It is implicitly admitted that F is the natural filtration generated by S_t.

In our case, an application of the Girsanov theorem (see for instance Métivier (1982) for the Girsanov theorem and Harrison and Pliska (1981) for its application to option pricing) shows that the Black and Scholes market is complete, and that under p^*, the security's price follows the stochastic differential equation

$$dS_t = rS_t\,dt + \sigma S_t\,dW_t,\ S_0 > 0, \tag{3}$$

with W_t a new standard Brownian motion (keeping the same notation for the sake of simplicity) under p^*. Also for simplicity, we will now write p instead of p^*.

In the following the payoffs of the options will be nonnegative L^2 random variables depending only on the trajectories of W. Hence, they will be F_T measurable, where $T > 0$ is the exercise date of the options.

Throughout this paper $N(x)$ will denote the cumulative distribution function of a standard Gaussian variable, i.e., $N(x) = \int_{-\infty}^{x} e^{-u^2/2} du / \sqrt{2\pi}$.

II. Densities and Distribution Functions

From the stochastic differential equation (3), we have

$$S_t = S_0 \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right\}. \tag{4}$$

Let us introduce some notation:

$$M_{t_1}^{t_2} = \max\{S_s / s \in [t_1, t_2]\}, \tag{5}$$

$$m_{t_1}^{t_2} = \min\{S_s / s \in [t_1, t_2]\}, \tag{6}$$

We also set:

$$X_t = \ln \frac{S_t}{S_0} = \left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t, \tag{7}$$

$$Y_t = \ln \frac{M_0^t}{S_0} = \max\{X_s / s \in [0, t]\}, \tag{8}$$

$$y_t = \ln \frac{m_0^t}{S_0} = \min\{X_s / s \in [0, t]\}. \tag{9}$$

Option prices will be computed at time 0, and it is assumed that the option contracts have been initiated at time $T_0 \le 0$. We set $\mu = r - \frac{1}{2}\sigma^2$. The following lemmas follow from the reflection principle. Their proofs are in Harrison (1985), p. 13, to which we refer the reader.

Lemma 1: *Take* (x, y) *such that* $y \ge 0$ *and* $y \ge x$. *Then*

$$p(X_t \le x, Y_t \le y) = N\left(\frac{x - \mu t}{\sigma\sqrt{t}}\right) - e^{\frac{2\mu y}{\sigma^2}} N\left(\frac{x - 2y - \mu t}{\sigma\sqrt{t}}\right).$$

Lemma 2: *Take* (x, y) *such that* $y \leq 0$ *and* $y \leq x$. *Then*

$$p(X_t \geq x, y_t \geq y) = N\left(\frac{-x + \mu t}{\sigma\sqrt{t}}\right) - e^{\frac{2\mu y}{\sigma^2}} N\left(\frac{-x + 2y + \mu t}{\sigma\sqrt{t}}\right).$$

With $y = x$ in the above lemmas, we get

Lemma 3: *Take* $y \geq 0$. *Then*

$$p(Y_t \leq y) = N\left(\frac{y - \mu t}{\sigma\sqrt{t}}\right) - e^{\frac{2\mu y}{\sigma^2}} N\left(\frac{-y - \mu t}{\sigma\sqrt{t}}\right).$$

and

Lemma 4: *Take* $y \leq 0$. *Then*

$$p(y_t \geq y) = N\left(\frac{-y + \mu t}{\sigma\sqrt{t}}\right) - e^{\frac{2\mu y}{\sigma^2}} N\left(\frac{y + \mu t}{\sigma\sqrt{t}}\right).$$

III. Valuation of European Lookback Options

Using the results in section I and the distributions provided in section II, it is straightforward to evaluate the European lookback options below by computing the expectation of their payoffs. This can be achieved by differentiating the cumulative distribution functions of lemmas 1 to 4 in order to obtain the corresponding density functions.

However, another approach can be followed that leads to easier computations. Using the Girsanov theorem, we can transform expectations of complex quantities into expectations of simpler quantities for a different probability measure. As an illustration of this technique, we provide in the appendix a simple derivation of the pricing formula for the "limited risk" call.

Note that $m_{T_0}^T = \min(m_{T_0}^0, m_0^T)$(resp. $M_{T_0}^T = \max(M_{T_0}^0, M_0^T)$) where, at time 0, $m_{T_0}^0$ and $M_{T_0}^0$ are known constants. In all formulas below, we set:

$$d = \left(\ln\frac{S_0}{K} + rT + \frac{1}{2}\sigma^2 T\right)/\sigma\sqrt{T}, \tag{10}$$

$$d' = \left(\ln\frac{S_0}{M_{T_0}^0} + rT + \frac{1}{2}\sigma^2 T\right)/\sigma\sqrt{T}, \tag{11}$$

$$d'' = \left(\ln\frac{S_0}{m_{T_0}^0} + rT + \frac{1}{2}\sigma^2 T\right)/\sigma\sqrt{T}. \tag{12}$$

A. "Standard Lookback" Options

The payoff of the call (resp. the put) is $S_T - m_{T_0}^T$(resp. $M_{T_0}^T - S_T$). The value of the call at time 0 is

$$\begin{aligned} C &= e^{-rT}E\left[S_T - \min\left(m_{T_0}^0, m_0^T\right)\right] \\ &= e^{-rT}E\left[S_T\right] - e^{-rT}E\left[\min\left(m_{T_0}^0, m_0^T\right)\right] \\ &= S_0 - e^{-rT}E\left[\min\left(m_{T_0}^0, m_0^T\right)\right] \end{aligned}$$

which gives

$$\begin{aligned} C = {} & S_0 N(d'') - e^{-rT} m_{T_0}^0 N(d'' - \sigma\sqrt{T}) \\ & + e^{-rT}\frac{\sigma^2}{2r} S_0\left[\left(\frac{S_0}{m_{T_0}^0}\right)^{-\frac{2r}{\sigma^2}} N\left(-d'' + \frac{2r}{\sigma}\sqrt{T}\right) - e^{rT}N(-d'')\right]. \end{aligned} \tag{13}$$

Similarly, we get for the value of the put

$$\begin{aligned} P = {} & -S_0 N(-d') + e^{-rT} M_{T_0}^0 N(-d' + \sigma\sqrt{T}) \\ & + e^{-rT}\frac{\sigma^2}{2r} S_0\left[-\left(\frac{S_0}{M_{T_0}^0}\right)^{-\frac{2r}{\sigma^2}} N\left(d' - \frac{2r}{\sigma}\sqrt{T}\right) + e^{rT}N(d')\right]. \end{aligned} \tag{14}$$

The above formulas can be seen as the sum of the value of a standard option given by the Black and Scholes formula plus another term. This characteristic may be useful for practical hedging purposes. In the following, we will continue to use this natural decomposition.[2]

B. Options on Extrema

The payoff of the call on maximum (resp. the put on minimum) is $(M_{T_0}^T - K)^+$ (resp. $(K - m_{T_0}^T)^+$) at time T where K is the strike price. The value of the call at time 0 is

$$C = e^{-rT}E\left[\left(\max\left(M_{T_0}^0, M_0^T\right) - K\right)^+\right]$$

which gives

$$\begin{aligned} C = {} & S_0 N(d) - e^{-rT} K N(d - \sigma\sqrt{T}) \\ & + e^{-rT}\frac{\sigma^2}{2r} S_0\left[-\left(\frac{S_0}{K}\right)^{-\frac{2r}{\sigma^2}} N\left(d - \frac{2r}{\sigma}\sqrt{T}\right) + e^{rT}N(d)\right] \end{aligned} \tag{15}$$

[2] The formulas (13) and (14) are easily shown to be equivalent to those in papers by Goldman et al.

in the case where $K \geq M^0_{T_0}$, and

$$C = e^{-rT}\left(M^0_{T_0} - K\right) + S_0 N(d') - e^{-rT} M^0_{T_0} N(d' - \sigma\sqrt{T})$$

$$+ e^{-rT}\frac{\sigma^2}{2r} S_0 \left[-\left(\frac{S_0}{M^0_{T_0}}\right)^{-\frac{2r}{\sigma^2}} N\left(d' - \frac{2r}{\sigma}\sqrt{T}\right) + e^{rT} N(d') \right] \quad (16)$$

otherwise.

Similarly, we get for the value of the put

$$P = -S_0 N(-d) + e^{-rT} K N(-d + \sigma\sqrt{T}) +$$

$$e^{-rT}\frac{\sigma^2}{2r} S_0 \left[\left(\frac{S_0}{K}\right)^{-\frac{2r}{\sigma^2}} N\left(-d + \frac{2r}{\sigma}\sqrt{T}\right) - e^{rT} N(-d) \right] \quad (17)$$

in the case where $K \leq m^0_{T_0}$, and

$$P = e^{-rT}\left(K - m^0_{T_0}\right) - S_0 N(-d'') + e^{-rT} m^0_{T_0} N(-d'' + \sigma\sqrt{T})$$

$$+ e^{-rT}\frac{\sigma^2}{2r} S_0 \left[\left(\frac{S_0}{m^0_{T_0}}\right)^{-\frac{2r}{\sigma^2}} N\left(-d'' + \frac{2r}{\sigma}\sqrt{T}\right) - e^{rT} N(-d'') \right] \quad (18)$$

otherwise.

A curious feature of these options is that they can have a delta above 1, in conjunction with a large gamma.

C. "Limited Risk" Options

The "limited risk call" (resp. put) has a payoff $(S_T - K)^+ 1_{M^T_{T0} \leq m}$(resp. $(K - S_T)^+ 1_{m^T_{T0} \geq m}$) with K and m two constants. We get for the call

$$C = S_0 N(d) - e^{-rT} K N(d - \sigma\sqrt{T}) - S_0 N(d_m) + e^{-rT} K N(d_m - \sigma\sqrt{T})$$

$$+ \left(\frac{S_0}{m}\right)^{-\frac{2r}{\sigma^2}} \left[m N\left(2 d_m - d - \frac{2r}{\sigma}\sqrt{T} - \sigma\sqrt{T}\right) \right.$$

$$\left. - e^{-rT} K \frac{S_0}{m} N\left(2 d_m - d - \frac{2r}{\sigma}\sqrt{T}\right) \right]$$

$$- \left(\frac{S_0}{m}\right)^{-\frac{2r}{\sigma^2}} \left[m N\left(d_m - \frac{2r}{\sigma}\sqrt{T} - \sigma\sqrt{T}\right) \right.$$

$$\left. - e^{-rT} K \frac{S_0}{m} N\left(d_m - \frac{2r}{\sigma}\sqrt{T}\right) \right] \quad (19)$$

if $M_{T_0}^0 \leq m$, and 0 otherwise, with

$$d_m = \frac{\ln \frac{S_0}{m} + rT + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}. \tag{20}$$

Similarly, the value of the put is

$$\begin{aligned} P = &-S_0 N(-d) + e^{-rT} K N(-d + \sigma\sqrt{T}) + S_0 N(-d_m) \\ &- e^{-rT} K N(-dm + \sigma\sqrt{T}) - \left(\frac{S_0}{m}\right)^{-\frac{2r}{\sigma^2}} \left[mN\left(-2d_m + d + \frac{2r}{\sigma}\sqrt{T} \right.\right. \\ &\left.\left. + \sigma\sqrt{T}\right) - e^{-rT} K \frac{S_0}{m} N\left(-2d_m + d + \frac{2r}{\sigma}\sqrt{T}\right)\right] \\ &+ \left(\frac{S_0}{m}\right)^{-\frac{2r}{\sigma^2}} \left[mN\left(-d_m + \frac{2r}{\sigma}\sqrt{T} + \sigma\sqrt{T}\right) \right. \\ &\left. - e^{-rT} K \frac{S_0}{m} N\left(-d_m + \frac{2r}{\sigma}\sqrt{T}\right)\right] \end{aligned} \tag{21}$$

if $m_{T_0}^0 \geq m$, and 0 otherwise.

The reader can check that the value of the call is equal to the Black and Scholes formula in the case where $m = +\infty$, and equal to 0 if $m = K$. Corresponding results hold for the put.

These options are similar in spirit to the "down and out" options described in Merton (1973). Since the calculations are quite simple, it is straightforward to extend the pricing formulas to any mixture of ordinary option-type payoffs with indicator functions depending on the maximum or the minimum.

These options have attracted a lively amount of interest in the foreign exchange market, and recently in the equity index markets. "Limited risk" calls and puts and variants on this idea have been traded for example in Nikkei linked issues in the Euromarkets.

A number of recent papers, for instance Briys, Crouhy, and Schöbel (1988) and Schöbel (1986) develop the concept of anti-options, which is essentially similar to the option described in this section. The anti-option arises from the fact that the process representing the bond price is absorbed at 1 to prevent the interest rate from being negative, and the call option, instead of being equal to 0 when the barrier is reached, is equal to the intrinsic value at the barrier.

D. "Partial Lookback" Options

These options are designed to exhibit behavior similar to "standard lookback" options, while being less expensive. The payoffs of the "partial lookback" call (resp. put) is $(S_T - \lambda m_{T_0}^T)^+$ with $\lambda > 1$ (resp.$(\lambda M_{T_0}^T - S_T)^+$ with

$0 < \lambda < 1$) where λ is a constant. After calculations, we get for the call

$$C = S_0 N\left(d'' - \frac{\ln \lambda}{\sigma\sqrt{T}}\right) - \lambda m_{T_0}^0 e^{-rT} N\left(d'' - \frac{\ln \lambda}{\sigma\sqrt{T}} - \sigma\sqrt{T}\right)$$
$$+ e^{-rT}\frac{\sigma^2}{2r}\lambda S_0\left[\left(\frac{S_0}{m_{T_0}^0}\right)^{-\frac{2r}{\sigma^2}} N\left(-d'' + \frac{2r}{\sigma}\sqrt{T} - \frac{\ln \lambda}{\sigma\sqrt{T}}\right)\right.$$
$$\left. - e^{rT}\lambda^{\frac{2r}{\sigma^2}} N\left(-d'' - \frac{\ln \lambda}{\sigma\sqrt{T}}\right)\right] \tag{22}$$

Similarly, we get for the put

$$P = -S_0 N\left(-d' + \frac{\ln \lambda}{\sigma\sqrt{T}}\right) + \lambda M_{T_0}^0 e^{-rT} N\left(-d' + \frac{\ln \lambda}{\sigma\sqrt{T}} + \sigma\sqrt{T}\right)$$
$$- e^{-rT}\frac{\sigma^2}{2r}\lambda S_0\left[\left(\frac{S_0}{M_{T_0}^0}\right)^{-\frac{2r}{\sigma^2}} N\left(d' - \frac{2r}{\sigma}\sqrt{T} + \frac{\ln \lambda}{\sigma\sqrt{T}}\right)\right.$$
$$\left. - e^{rT}\lambda^{\frac{2r}{\sigma^2}} N\left(d' + \frac{\ln \lambda}{\sigma\sqrt{T}}\right)\right] \tag{23}$$

The reader can check that the value of the "partial lookback call" (resp. put) is equal to the value of the "standard lookback call" (resp. put) in the case where $\lambda = 1$.

IV. American Lookback Options

In this section, we are interested in providing approximations of the value of some of the American versions of the above options. Although it is generally impossible to have closed form expressions for the value of American options, theoretical considerations can help provide rather tight bounds for their prices.

A. American Options and Snell Envelopes

Proofs of the results mentioned here may be found in Karatzas (1987).

Theorem 1: *Let V_t be the intrinsic value of the option at time t, i.e., its payoff if it is exercised at time t. Let $\tilde{V}_t$ be the Snell envelope of its discounted intrinsic value process, i.e., the smallest supermartingale dominating $V_t e^{-rt}$. Then the time 0 value of the option is $\tilde{V}_0$.*

Theorem 2: *If $\tilde{V}_t$ is a martingale, then the American option has the same value as the corresponding European Option.*

A lower bound for the value of the American option is trivially the European one. To provide an upper bound, we will construct a new American option dominating the American option we are interested in, and use theorem 2 to prove that this new option is in fact European (i.e., need not be exercised prior to maturity).

As an example, we can use theorem 2 to recover two well-known results: the American call on a non-dividend-paying stock is European, and the American put on a non-dividend-paying stock whose strike price increases as Ke^{rt} is also European. This last result was found by Margrabe in 1978 using financial methods.

To prove the first result, let $V_t = (S_t - K)^+$. Since $S_t e^{-rt} - Ke^{-rt}$ is a submartingale, by the Jensen inequality $V_t e^{-rt}$ is a submartingale, so $\tilde{V}_t$ is a martingale. To prove the second result, let $V_t = (Ke^{rt} - S_t)^+$. Then by the same argument $V_t e^{-rt} = (K - S_t e^{-rt})^+$ is a submartingale, hence the result.

In the following, C (resp. P) denotes the time-zero value of the European call (resp. put), while C^a (resp. P^a) is the time zero value of the American call (resp. put). Using the above-mentioned technique, we will give bounds for all our lookback options, except the "limited risk." "Limited risk" options are more complicated to deal with because their payoff is not convex.

B. "Standard Lookback" American Options

The intrinsic value of the call is $S_t - m_{T_0}^t$. Since $e^{-rt} m_{T_0}^t$ is a decreasing process, and since $e^{-rt} S_t$ is a martingale, we find that the discounted intrinsic value $e^{-rt} S_t - e^{-rt} m_{T_0}^t$ is a submartingale. Hence, its Snell envelope is a martingale, which proves theorem 2 that

$$C^a = C. \tag{24}$$

The intrinsic value of the put is $M_{T_0}^t - S_t$. We have $M_{T_0}^t - S_t \leq e^{rt} m_{T_0}^t - S_t$. We consider the new option with intrinsic value $e^{rt} M_{T_0}^t - S_t$. When discounted, it is a submartingale equal to $M_{T_0}^t - e^{-rt} S_t$. Hence, by theorem 2, it is European. Moreover, the value of this new option at time zero is given by $(P + S_0)e^{rT} - S_0$, where P is the "standard lookback" European put. Hence, we have[3]

$$P \leq P^a \leq Pe^{rT} + S_0\left(e^{rT} - 1\right). \tag{25}$$

C. American Options on Extrema

The intrinsic value of the call is $(M_{T_0}^t - K)^+$, which is dominated by $V_t = e^{rt}(M_{T_0}^t - K)^+$. The process $e^{-rt} V_t$ is a submartingale, which implies that $\tilde{V}_t$ is a martingale. Hence, we get

$$C \leq C^a \leq Ce^{rT}. \tag{26}$$

[3] To be precise, given the price of the European option, these bounds are independent of any assumption on the stochastic process S_t. The only thing required is that S_t has the martingale representation property. This remark holds for all the bounds provided in this paper.

Similarly, we get for the American put of intrinsic value $(K - m_{T_0}^t)^+$ that

$$P \leq P^a \leq Pe^{rT}. \tag{27}$$

D. "Partial Lookback" American Options

The intrinsic value of the call is $V_t = (S_t - \lambda m_{T_0}^t)^+$. The process $e^{-rt}S_t - e^{-rt}\lambda m_{T_0}^t$ is the sum of a martingale and an increasing process, so the Jensen inequality implies that $e^{-rt}V_t$ is a submartingale. Hence, the American call is equal to the European one, i.e.,

$$C^a = C. \tag{28}$$

The intrinsic value of the "partial lookback" put with parameter λ is $(\lambda M_{T_0}^t - S_t)^+$. It is dominated by $V_t = (e^{rt}\lambda M_{T_0}^t - S_t)^+$. We have $e^{-rt}V_t = (\lambda M_{T_0}^t - e^{-rt}S_t)^+$ which is the positive part of the submartingale $\lambda M_{T_0}^t - e^{-rt}S_t$. Hence, by the Jensen inequality, it is a submartingale and $\tilde{V}_t$ is a martingale. We denote by $P(\lambda)$ (resp. $P^a(\lambda)$) the value of the European (resp. American) put of parameter λ. We get

$$P^a(\lambda) \geq P(\lambda) \text{ for all } \lambda \tag{29}$$

and

$$P^a(\lambda) \leq P(\lambda e^{rT}) \text{ for } \lambda e^{eT} \geq 1. \tag{30}$$

V. Numerical Examples

Tables I to IV provide prices for all the lookback options priced in this paper. Tables I to III also provide prices for European ordinary options (for the latter, the payoff is $(S_T - K)^+$ for the call and $(K - S_T)^+$ for the put).

Table I

"Standard Lookback" Options

This table gives prices for European "standard lookback" options with maturities 3 and 6 months and upper bounds for their American counterparts. It also provides prices for at-the-money European ordinary options. The short term rate r is 10%, the spot price of the security S_0 is 100, the volatility σ is 20%, and the option contracts have been initiated at time $T_0 = 0$. As a rule of thumb, the "standard lookback" is worth roughly twice the value of an at-the-money ordinary option.

	European "Standard Lookback" Option		Upper Bound for the the American "Standard Lookback" Option		At-the-Money European Ordinary Option	
	3 months	6 months	3 months	6 months	3 months	6 months
Call	8.95	13.19	8.95	13.19	5.30	8.28
Put	6.98	9.29	9.69	14.89	2.83	3.40

Table II

"Options on Extrema"

This table gives prices for European "options on extrema" with various strikes and maturities 3 and 6 months, and upper bounds for their American counterparts. It also provides prices for European ordinary options with same strikes. The short term rate r is 10%, the spot price of the security S_0 is 100, the volatility σ is 20%, and the option contracts have been initiated at time $T_0 = 0$. European "options on extrema" have prices very close to the corresponding American upper bounds. They also have much higher prices than European ordinary options.

		European Option on Extrema		Upper Bound for the American Option on Extrema		European Ordinary Option	
	Strike (K)	3 months	6 months	3 months	6 months	3 months	6 months
Call	95	14.32	18.92	14.68	19.90	8.58	11.50
Put	95	2.76	4.44	2.83	4.55	1.23	1.87
Call	100	9.45	14.17	9.69	14.90	5.30	8.28
Put	100	6.48	8.32	6.64	8.75	2.83	3.40
Call	105	5.34	9.89	5.47	10.30	2.95	5.70
Put	105	11.36	13.07	11.65	13.74	5.35	5.57

Table III

European "Limited Risk" Options

This table gives prices for European "limited risk" options with various strikes and maturities 3 and 6 months, and also provides prices for European ordinary options. The short term rate r is 10%, the spot price of the security S_0 is 100, the volatility σ is 20%, the option contracts have been initiated at time $T_0 = 0$, and the cut-off value m is equal to $K + 20$ for the call and to $K - 20$ for the put, with K being the strike price. These options have much lower prices than ordinary options. Also prices of "limited risk" options are not increasing functions of maturities.

		European "Limited		European	
	Strike (K)	3 months	6 months	3 months	6 months
Call	95	4.23	2.39	8.58	11.50
Put	95	1.19	1.45	1.23	1.87
Call	100	3.23	2.35	5.30	8.28
Put	100	2.52	2.12	2.83	3.40
Call	105	2.15	2.03	2.95	5.70
Put	105	3.95	2.50	5.35	5.57

Standard lookback options may seem expensive compared to ordinary at-the-money options, but they always finish in-the-money. At initiation it should be noted that they are much less sensitive to the asset price than ordinary options. Partial lookback options are much cheaper while providing similar levels of protection in volatile markets.

"Limited risk" are a rare type of nonconvex payoff options. A consequence is that their prices are not increasing functions of maturities. An intuitive

Table IV

"Partial Lookback" Options

This table gives prices for European "partial lookback" options with various values for the parameter λ and maturities 3 and 6 months, and upper bounds for their American counterparts. The short term rate r is 10%, the spot price of the security S_0 is 100, the volatility σ is 20%, and the option contracts have been initiated at time $T_0 = 0$. These options have much lower prices than "standard lookback" options corresponding to the case $\lambda = 1$.

		European Option		Upper Bound for the American Option	
	λ	3 months	6 months	3 months	6 months
Call	1.10	2.59	6.20	2.59	6.20
Put	0.90	0.98	2.27	1.67	4.62
Call	1.05	5.09	9.27	5.09	9.27
Put	0.95	2.93	4.89	4.56	8.24
Call	1	8.95	13.19	8.95	13.19
Put	1	6.98	9.29	9.69	14.89

explanation is that the longer the maturity the higher the probability that the cut-off clause will be triggered.

The upper bounds for the "American options on extrema" are pretty tight. In all cases, the knowledge of these bounds is very useful: the use of binomial trees to price lookback options is made difficult by the path dependency which forbids "connected" trees.

VI. Extensions

All the formulas that give the values of the above European options may be written as

$$\Pi = e^{-rT} f(r, S_0)$$

where f is a function provided by the formulas.

One can easily check that the corresponding formulas for options on forwards are

$$\Pi = e^{-rT} f(0, F_0) \tag{31}$$

where F_0 is the forward price of the security.[4]

The corresponding formulas for options on foreign exchange are

$$\Pi = e^{-rT} f(r - r_f, X_0) \tag{32}$$

where r_f is the foreign currency interest rate and X_0 the spot foreign exchange rate.[5]

[4] More precisely, $\Pi = e^{-rT} \lim_{u \to 0} f(u, F_0)$.

[5] To illustrate and shed some light on a conjecture in Garman (1985), here are bounds for the foreign exchange standard lookback options: $C \le C^a \le C + X_0(1 - e^{-r_f T})$, and if $r_f \le r$ then $C^a \le Ce^{r_f T}$. Similarly, $P \le P^a \le Pe^{rT}$ if $r \le r_f$ and $P^a \le Pe^{rT} + e^{-r_f T} X_0(e^{rT} - 1)$.

Conclusion

In this paper we obtain closed form valuation formulas for European lookback options by using the joint distribution of the spot price and its historical maximum (or minimum). We also use results related to the Snell envelope approach of American options pricing to obtain bounds for some American lookback options.

The computations do not require the use of partial differential equations. Nevertheless, the curious reader could recover our results by writing the partial differential equations, provided by the classical arbitrage argument, along with the right boundary conditions, which are quite complex.

Path dependent options are becoming more actively traded. They provide payoffs that may be better suited to customers' needs than ordinary options. Among these, lookback options are very attractive options because they keep track of past events. As a consequence, they allow its holder to take advantage of anticipated market movements without knowing the exact date of their occurrences, and may provide psychological comfort by minimizing regrets.

Appendix: Pricing of The "Limited Risk" Call

The purpose of this appendix is to illustrate how powerful martingale based techniques can be in pricing options. Here we derive the valuation formula for the "limited risk" call. First note that if $M_{T_0}^0 > m$, then the cut-off clause applies and the value of the option is 0, so that we are only interested in the case $M_{T_0}^0 \leq m$. In this case the value of the option is given by $E[(S_T - K)^+ \mathbf{1}_{M_0^T \leq m}]$. We have

$$(S_T - K)^+ \mathbf{1}_{M_{T0}^T \leq m} = S_T \mathbf{1}_{S_T \geq K, M_{T0}^T \leq m} - K \mathbf{1}_{S_T \geq K, M_{T0}^T \leq m}.$$

Furthermore,

$$\mathbf{1}_{S_T \geq K, M_{T0}^T \leq m} = \mathbf{1}_{M_{T0}^T \leq m} - \mathbf{1}_{S_T < K, M_{T0}^T \leq m},$$

so that, from the lemmas of Section II,

$$\begin{aligned} E\left[\mathbf{1}_{S_T \geq K, M_{T0}^T \leq m}\right] &= p\left(M_{T_0}^T \leq m\right) - p\left(S_T < K,\, M_{T_0}^T \leq m\right) \\ &= N\left(-d_m + \sigma\sqrt{T}\right) + \left(\frac{S_0}{m}\right)^{-\frac{2r}{\sigma^2}} \frac{S_0}{m} N\left(d_m - \frac{2r}{\sigma}\sqrt{T}\right) \\ &\quad - N\left(-d + \sigma\sqrt{T}\right) + \left(\frac{S_0}{m}\right)^{-\frac{2r}{\sigma^2}} \frac{S_0}{m} \\ &\quad \times N\left(2d_m - d - \frac{2r}{\sigma}\sqrt{T}\right). \end{aligned} \tag{33}$$

Note that

$$E\left[S_T \mathbf{1}_{S_T \geq K, M_{T0}^T \leq m}\right] = S_0 e^{rT} E\left[L_T \mathbf{1}_{S_T \geq K, M_{T0}^T \leq m}\right],$$

where $L_t = \exp\{-\frac{1}{2}\sigma^2 t + \sigma W_t\}$ is an exponential martingale which can be used as a Radon-Nikodym derivative in the Girsanov theorem to change the probability measure. Hence,

$$E\left[L_T \mathbf{1}_{S_T \geq K, M_{T0}^T \leq m}\right] = \tilde{E}\left[\mathbf{1}_{S_T \geq K, M_{T0}^T \leq m}\right]$$

where $\tilde{E}$ is the expectation under a probability for which $\sigma^2 t$ is added to the Brownian motion. Therefore, the right side of this equation is equal to the right side of (33) in which r is replaced by $r + \sigma^2$.

A more detailed exposition of the Girsanov theorem may be found in Métivier (1982), p. 209. In simple terms, the Girsanov theorem can be thought of as the equivalent of a change of variable in ordinary integrals.

It is straightforward to recover Merton's (1973) Down and Out option formula with the above methodology.

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