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Author(s): Eric Briys, Michel Crouhy, Rainer Schöbel
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The Pricing of Default-free Interest Rate Cap, Floor, and Collar Agreements

ERIC BRIYS, MICHEL CROUHY, and RAINER SCHÖBEL*

ABSTRACT

The paper focuses on the valuation of caps, floors, and collars in a contingent claim framework under continuous time. These instruments are interpreted as options on traded zero coupon bonds. The bond prices themselves are used as the underlying stochastic variables. This has the advantage that we end up with closed form solutions which are easy to compute. Special attention is devoted to the choice of the stochastic process appropriate for the price dynamics of the underlying zero coupon bonds.

THE INCREASED VOLATILITY OF interest rates and the introduction of numerous variable rate instruments have led to a more intensive use of highly sophisticated risk hedging tools. Among them, interest rate caps, floors, and collars have recently gained a growing popularity. An interest rate cap is an agreement between the provider of the cap and a borrower to limit the borrower's floating interest rate to a specific level for a given period of time. To be more precise, the borrower selects a reference rate to hedge (e.g., LIBOR), a period of time to hedge (e.g., 5 years), and the level of protection desired (e.g., a ceiling of 200 basis points above the current level of the LIBOR). If the market rate exceeds the ceiling rate, the cap provider will make payments to the buyer sufficient to bring his rate back to the ceiling. When the market rate is below the ceiling no payment is made.

Interest rate caps have been claimed to be quite expensive. To mitigate this effect, banks have introduced interest rate collars which can be thought of as interest rate swaps in a band. Indeed, the provider of a collar agrees to limit the borrower's floating interest rate to a band between a specific ceiling rate and a floor rate. In addition to a cap, a collar contains a floor agreement. An interest rate floor is a contract where the floating rate borrower forgoes the benefits of a decline in the interest rate below a contractual floor rate. For this type of contract, the borrower receives a payment for the lost

* Briys and Crouhy are from Groupe HEC, 78351 Jouy en Josas, France, and Schöbel is from Universität Lüneburg, 2120 Lüneburg, Federal Republic of Germany. The authors gratefully acknowledge helpful comments on earlier drafts by Suresh Sundaresan and an anonymous referee. The paper has also benefited from stimulating discussions with many of our colleagues at seminars and conversations, especially those with Dan Galai, Myron Scholes, Eduardo Schwartz, and Lee Wakeman. Of course, the authors are solely responsible for any remaining errors.

potential benefits. Hence, it turns out that a collar is cheaper than a cap alone.

The growing demand for such instruments by corporations has confronted both banks and researchers with the thorny topics of valuation and hedging strategies of interest rate-dependent instruments. This paper focuses on the valuation of default-free caps, floors, and collars in a contingent claim framework under continuous time.[1] The common approach by practitioners is to apply the Black and Scholes valuation procedure to the interest rate-dependent instruments.[2] On theoretical grounds this method is flawed since bond prices obviously do not follow a geometric Brownian motion.

Nevertheless an abundant academic literature has developed in which the stochastic interest rate is modeled in an equilibrium framework.[3] As shown by Courtadon (1982), Brennan and Schwartz (1983b), and Cox, Ingersoll, and Ross (1985) these equilibrium models can be used to evaluate options on bonds as well. Although superior, this approach requires the use of highly sophisticated numerical and statistical techniques. A recent body of literature attempts to avoid these difficulties by using modified partial equilibrium models in the spirit of Black and Scholes.[4]

The position taken in this paper is an intermediate one. Caps, floors, and collars are interpreted as options on traded zero coupon bonds. The bond prices themselves are used as the underlying stochastic variables. The advantage is that we can obtain closed form solutions which are very easy to compute. In addition, the number of parameters to estimate is parsimonious, and the estimation procedure remains tractable. Special attention is devoted to the choice of the stochastic process appropriate for the price dynamics of the underlying zero coupon bonds.

In the first section of the paper we show why cap, floor, and collar agreements can be interpreted as options on discount bonds. In the second section, we discuss our choice of the dynamics of the underlying zero coupon bond prices. In the third section, we derive the pricing formula for caps and collars. A conclusion summarizes the main findings and suggests avenues for further research.

I. Caps, Floors, and Collars as Zero Coupon Bond Options

In order to show that a cap can be interpreted as a series of put options on zero coupon bonds, let us examine a representative example. Consider a borrower whose interest payments are linked to a floating rate basis over the next 5 years to the 3-month LIBOR. To cap his interest rate risk exposure the

[1] Ho and Lee (1986) and Stapleton and Subrahmanyam (1988) provide valuation models for interest rate contingent claims in discrete time.

[2] An early attempt in this direction is the paper by Leibowitz and Kopprasch (1981).

[3] Vasicek (1977), Courtadon (1980), Brennan and Schwartz (1982), and Cox, Ingersoll, and Ross (1985).

[4] Ball and Torous (1983), Schöbel (1986), Bühler (1988), and Schaefer and Schwartz (1987).

borrower buys a 5 year, 10% cap on the 3-month LIBOR. If, for example, on a given 3-month interval settlement date indexed by t_j, the actual 3-month LIBOR, $\hat{r}$, were to exceed the 10% ceiling $\hat{c}$, by 2 percentage points, then for the following reset interval, $\Delta t = 3/12$, the interest rate difference yields the amount payable to the borrower on a \$1 loan; namely, \$(0.12 − 0.10) × 0.25 = \$0.005. If the nominal value of the cap agreement is \$1 million, the compensation at the end of the reset interval, which is the next settlement date, say $t_j + \Delta t$, is \$5,000.

Hence, at time $t_j + \Delta t$ the borrower will receive from the issuer of the cap a cash payment equal to $\max\{(\hat{r} - \hat{c})\Delta t, 0\}$ per \$1 of nominal value of the cap. At the settlement date t_j, the value of this claim is given by:

$$P_j = \max\{(\hat{r} - \hat{c})\Delta t/(1 + r\Delta t), 0\} \tag{1}$$

which gives \$4,854 in our example for a 200 basis points difference between LIBOR and the cap rate. Current practice of large banks in New York and London, specialized in market making in caps and floors, is to make the actual payments at the beginning of the reset period, and not at the end, according to formula (1).[5]

Therefore, to simplify considerably the whole valuation problem, we approximate this usual quotation for short-term discount bonds with the exact definition of the present value of such instruments. With $E = e^{-c\Delta t}$ and $B = e^{-r\Delta t}$, where c and r denote the continuously compounded zero coupon rate equivalent to the corresponding proportional rates $\hat{c}$ and $\hat{r}$, respectively,[6] it is easy to show that the payoff value at t_j is almost exactly the same as:[7]

$$P_j = \max(E - B, 0) \tag{2}$$

For realistic values of the reset interval and the interest rate spread $(\hat{r} - \hat{c})$ the approximation quality of formula (2) is more than sufficient; in our example the difference corresponds to 5 basis points, which is less than the usual bid-ask spread.

[5] The reference rate for the LIBOR is taken two New York days before the reset, and Δt is computed based on the actual number of days until the next reset date divided by 360. As it was stressed by the referee, this definition abstracts from the contract nonperformance risk. This paper, therefore, addresses only the pricing of default-free interest interest cap, floor, and collar agreements. We further assume that the spread between LIBOR and the implied rate for comparable T-bills is zero.

[6] These definitions imply the following relationships: $e^{r\Delta t} \equiv 1 + r\Delta t$ and $e^{c\Delta t} \equiv 1 + c\Delta t$.

[7] In fact:

$$\begin{aligned}
\{(\hat{r} - \hat{c})\Delta t\}/(1 + r\Delta t) &= \{(\hat{r} - \hat{c})\Delta t\}e^{-r\Delta t} \\
&= \{(r - c)\Delta t + o(\Delta t^2)\}e^{-r\Delta t} \\
&\approx \{1 + (r - c)\Delta t - 1\}e^{-r\Delta t} \\
&= \{e^{(r-c)\Delta t} + o(\Delta t^2) - 1\}e^{-r\Delta t} \\
&\approx \{e^{-c\Delta t} - e^{-r\Delta t}\} \\
&= E - B \qquad \text{Q.E.D.}
\end{aligned}$$

Thus, this simplified payoff formula corresponds exactly to the price difference between two zero coupon bonds E and B, the first one selling at a price of 0.97561 and the second at 0.97087, both bonds maturing at time $t_j + \Delta t$, the next settlement date. Therefore, the payoff structure at the settlement date t_j is equivalent to the payoff structure of a European put option P_j maturing at time t_j which is written on a zero coupon bond B_{j+1} maturing at time $t_j + \Delta t$ with a strike price of E. Figure 1 illustrates this point.

A 5-year cap can thus be interpreted as a string of $n = 19$ put options written at time t_0 by the bank. Each option has the same strike price of E^c (with c for a cap). The first option matures at the first settlement date in 3 months from time t_0 and offers protection between 3 and 6 months, the second one matures in 6 months, and so on...Hence, the underlying instrument for the first option is a zero coupon bond maturing at the second settlement date. The underlying instrument for the last put option is a zero coupon bond which matures exactly 5 years from the issuance date of the cap.

The string is just composed of n independent puts. To find the price of the cap agreement C_p we just have to sum up the nineteen premia, P_j[8]:

$$C_p = \sum_{j=1}^{19} P_j(B_{j+1}, B_j, \tau_j) \tag{3}$$

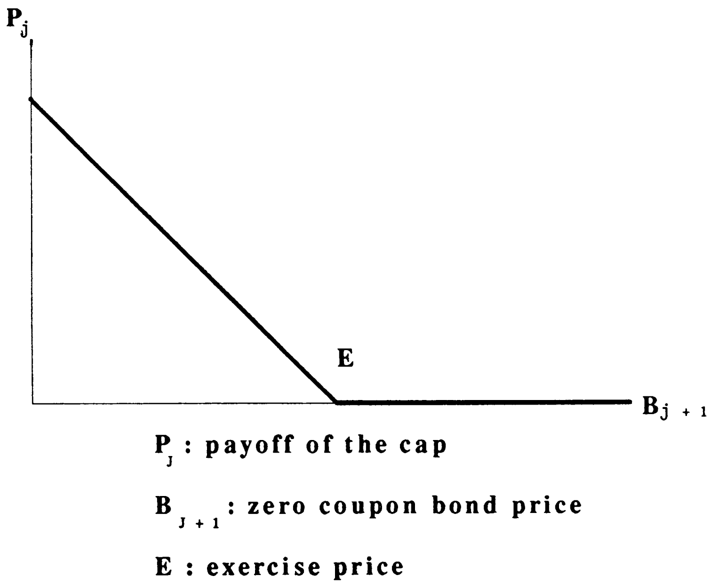


Figure 1. Payoff structure of a cap at a settlement date. The solid line represents the payoff of a cap at settlement date t_j. Since it is equivalent to the value of a put option, then it can be written as $P_j = \max(0, E - B_{j+1})$, where P_j is the payoff of the cap, B_{j+1} is the price at time t_j of a zero coupon bond maturing at time $t_j + \Delta t$, and E is the strike price.

[8] Contrary to the value of options written on portfolios, the value of a portfolio of independently exercisable options is just the sum of the single options values.

where B_j is the reference bond with time to maturity τ_j; B_{j+1} is the underlying instrument, i.e., a zero coupon bond with time to maturity $\tau_{j+1} = \tau_j + \Delta t$; and Δt is the reset interval.

A floor agreement can be interpreted analogously. A floor is equivalent to a string of call options with the same underlying instruments and settlement dates as the corresponding cap. Each call option of the floor has the same strike price of E^f (with f for a floor) which corresponds to the value of a zero coupon bond with time to maturity equal to the reset interval and yield to maturity equal to the floor rate.

Although these cap and floor agreements are mainly issued by commercial banks, we abstract here from the default risk dimension. Indeed, we treat caps and floors as if they were default-free. Such an abstraction may not be severe for at least two reasons: (i) caps and floors exclude principal payments, and as consequence eliminate one important aspect of default; (ii) the cap and floor contracts may be designed to minimize the impact of default risk. For example, by requiring more collateral, more frequent resets and payments (i.e., "marking to the market" as frequently as possible), the cap and floor agreements may become a lot safer.[9]

In any case, capturing correctly the default risk dimension is quite tricky as witnessed in Cooper and Mello (1991), Sundaresan (1990), Ramaswamy and Sundaresan (1986).

A default-free collar is just a long position on a default-free cap and a short position on a default-free floor with the same settlement dates and reset interval. Figure 2 illustrates the payoff structure of a collar at a settlement date.

As a consequence, the price of a collar is equal to the difference between the price of the put string with strike price E^c and the price of a call string with strike E^f.

$$C_0 = \sum_{j=1}^{19} \left[P_j(B_{j+1}, B_j, \tau_j) - C_j(B_{j+1}, B_j, \tau_j) \right] \tag{4}$$

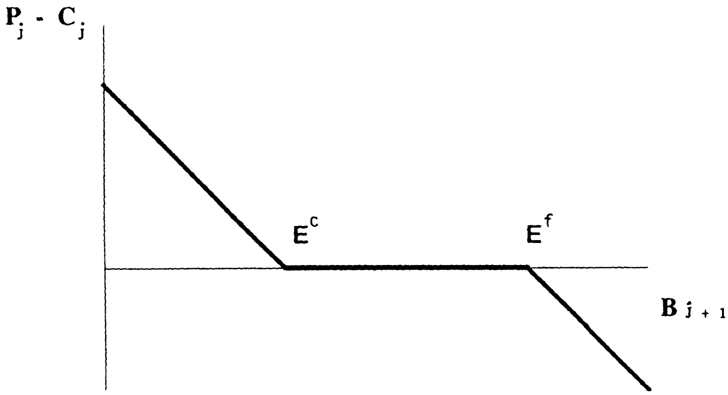
II. Zero Coupon Bond Price Dynamics

In the preceding section the price of caps and floors has been modeled as the value of a string of independent options on zero coupon bonds. Before developing any pricing model, the dynamic behavior of zero coupon bond prices has to be discussed. Consider the general stochastic process for bond prices:

$$dB = \mu(B, \tau)\, dt + \sigma(B, \tau)\, dZ \tag{5}$$

where Z is a standard Brownian motion process. The coefficients $\mu(B, \tau)$ and $\sigma(B, \tau)$ have to be chosen so as to offer an adequate and parsimonious description of the dynamics of a zero coupon bond $B(\tau)$ with maturity τ.

[9] See footnote 14 in Wakeman (1990).



$P_j - C_j$: payoff of the collar

B_{j+1} : zero coupon bond price

E^C : exercise price of the cap

E^F : exercise price of the floor

Figure 2. Payoff structure of a collar at a settlement date. The solid line represents the payoff of a collar at settlement date t_j. Since it is equivalent to the difference between the value of a put option (value of a default-free cap) and the value of a call option (value of a default-free floor), then it can be written as $P_j - C_j = \max(0, E^c - B_{j+1}) - \max(0, B_{j+1} - E^f)$, where P_j and E^c are, respectively, the payoff and the strike price of the cap; C_j and E^f are, respectively, the payoff and the strike price of the floor, with $E^f \geq E^c$; and B_{j+1} is the price at time t_j of a zero coupon bond maturing at time $t_j + \Delta t$.

Compared with the behavior of stock prices, several peculiarities of bond price dynamics should be stressed. First, at the maturity date of a default-free discount bond its price is no longer stochastic, but reaches a deterministic boundary value, the face value of the bond with probability one. Second, because of this boundary the volatility of bond prices cannot be assumed to remain constant over time, and should converge to zero at the maturity date. Finally, if one looks at a series of independent options on bonds as it is the case here, one should recognize that bond price dynamics for all bonds with different maturities should be mutually linked in an arbitrage-free manner.

From these requirements it is obvious that the geometric Brownian motion used by Black and Scholes (1973) for stock options is not relevant for bond price dynamics. Indeed, the geometric Brownian motion process with constant coefficients implies a log-normally distributed bond price at maturity. In addition, the variance of relative price changes is independent of time to maturity under such a process.

However, one should be careful in the search for alternatives. Not all processes are suitable candidates for pricing by arbitrage.

For example, the Brownian bridge process, often used to represent the bond price dynamics, has a drift which ensures that the bond price reaches the face value at maturity. This feature is appealing since it could help to solve for the boundary problem. But, from Harrison and Kreps (1979) pricing by arbitrage requires the ability to create an equivalent martingale measure for the stochastic process, with no drift. Such a transformation can be achieved using Girsanov's theorem.[10] For the Brownian bridge process such a transformation is not possible as was shown recently by Cheng (1989). It is quite intuitive that the drift term cannot be eliminated since for this process the drift is essential for the bond price to converge to the face value.

To keep the simple structure of the process (5) while avoiding the above mentioned theoretical inconsistencies, we assume:

$$dB/B = r\,dt + \sigma(\tau)\,dZ \tag{6}$$

where r is the spot rate of interest on instantaneously maturing default-free bonds. In other words, we assume that the local expectations hypothesis holds: the expected rate of return per unit of time on any bond is equal to the short-term rate of interest per unit of time. The local expectations hypothesis corresponds to the case where the risk premium is equal to zero. As shown by Cox, Ingersoll, and Ross (1981) and Ingersoll (1987) this is clearly a permissible form of equilibrium since no arbitrage opportunities are available.[11]

At least two Gaussian stochastic processes for r are consistent with our assumptions. First, the Ornstein-Uhlenbeck process[12]

$$dr = k(\alpha - r)\,dt + \sigma\,dZ \tag{7}$$

where k is the speed of adjustment toward the long run mean α, and σ is the instantaneous standard deviation of the short-term rate. This leads to

$$\sigma_B(\tau) = -(\sigma/k)(1 - e^{-k\tau}) \tag{8}$$

Second, the pure random walk[13]

$$dr = \sigma\,dZ \tag{9}$$

for which

$$\sigma_B(\tau) = -\sigma\tau \tag{10}$$

This second term structure model is fully compatible with the use of Macaulay's duration as shown by Ingersoll, Skelton, and Weil (1978).[14]

[10] See Duffie (1988) for a presentation of Girsanov's theorem.

[11] See also Cheng (1989) for an interesting discussion of bond price processes consistent with arbitrage-based pricing.

[12] See Vasicek (1977).

[13] For the sake of simplicity we assume a zero drift term for the short term rate. Note that under these two assumptions the bond prices reach their face value at maturity with probability one (see Merton (1973), Vasicek (1977)).

[14] The approach adopted in this paper used bond prices as the underlying state variables of the model. A second avenue to address the pricing problem of caps and collars consists in defining the state space only in terms of interest rates. In addition to the two interest rate processes we have already considered, the alternative approach would also allow us to value directly the string of options as in (3) or (4) with the closed form solution based on the square root process of Cox, Ingersoll, and Ross (1985).

However, these two representations of the term structure of interest rates permit the spot and the forward rates to become negative with a nonzero probability.[15] In other words, bond prices can become unbounded increasing functions of maturity.[16] This means for instance that call options on bonds will be overvalued if nothing is done to neutralize this source of mispricing. To overcome this difficulty, we have chosen to constrain the derivative asset price state space itself rather than the underlying asset price state space. As shown below, a specific boundary condition (13c) is introduced using a dominance argument. This boundary condition plays the role of an absorbing barrier and the model is solved as if the forward rates would remain non-negative.

III. The Pricing of Cap, Floor, and Collar Agreements

Using the above specified stochastic process for the underlying zero coupon bond price, it is easy to show that Merton's (1973) partial differential equation applies. To simplify notation we rename the stochastic variables so that $B_{j+1} = x$ and $B_j = y$, and drop all subscripts j for P_j, E_j, and τ_j. From equation (6) the two relevant price dynamics now become:

$$dx = r\,x\,dt + \sigma_x\,x dZ \tag{11}$$

$$dy = r\,y\,dt + \sigma_y\,y dZ \tag{12}$$

with $\sigma_x = \sigma_x(\tau_{j+1})$ and $\sigma_y = \sigma_y(\tau_j)$. It is worthwhile to mention that the bond prices are perfectly positively correlated under the two term structure assumptions. Indeed, they do depend on the same source of uncertainty, i.e., the short-term rate.

We can now state the valuation problem by using Merton's fundamental partial differential equation:

$$1/2\,\sigma_x^2 x^2 P_{xx} + \sigma_x \sigma_y\, xy P_{xy} + 1/2\,\sigma_y^2 y^2 P_{yy} - P_\tau = 0 \tag{13}$$

subject to:

$$P(x, y, 0) = \max(E^c - x, 0) \tag{13a}$$

$$P(0, y, \tau) = E^c y \tag{13b}$$

$$P(y, y, \tau) = 0 \tag{13c}$$

The first two boundary conditions correspond to those usually met when valuing European put options. The third boundary condition deserves some more attention. Since the state space of a zero coupon bond price is not

[15] See Cox, Ingersoll, and Ross (1979) for a discussion of these undesirable properties.

[16] In a Vasicek world, this nevertheless would happen with a low probability provided α is high and σ is low. Indeed, the expected first passage time through the origin of the Ornstein-Uhlenbeck process is long away enough for realistic values of the parameters. In the arithmetic Brownian motion case this would happen when $\tau > \sqrt{2r}/\sigma$, i.e., at least three years with reasonable values for r and σ.

unbounded, the third boundary condition ensures that the chance of a zero coupon bond put option holder to sell it for more than $B_j - B_{j+1}$ has a zero probability. This can be shown using a dominance proof along the lines of Merton (1973). Let us consider three portfolios. Portfolio 1 contain one zero coupon bond B_{j+1} plus a put option P_j with a strike price E_j. Portfolio 2 contains E_j units of the zero coupon bond B_j, and portfolio 3 contain exactly one unit of the zero coupon bond B_j. In order to prevent dominance, it can be inferred from the state-dependent outcomes of these three portfolios at the expiration date of the put (see Table I) that portfolio 1 cannot have a lower market value than portfolio 2 and cannot have a higher market value than portfolio 3. Hence, the following relationship holds true:

$$E_j B_j \leq P_j + B_{j+1} \leq B_j \tag{14}$$

As for stock options, the market value P_j of a European put on a zero coupon bond can never fall below $\max(E_j B_j - B_{j+1}, 0)$. But, according to the second part of (14), P_j can also never exceed $\min(B_j - B_{j+1}, E_j B_j)$.[17] Finally, in the limiting case where the boundary $B_{j+1} = B_j$ is reached,[18] these two conditions hold true if and only if P_j is equal to zero. This dominance argument leads to the third boundary condition.

For these boundary conditions the solution to the partial differential

Table I

State Dependent Portfolio Values at Expiration of a European Zero Coupon Bond Put Option

Portfolio 1 contains one zero coupon bond B_{j+1} maturing at time $t_j + \Delta t$, plus a put option P_j on B_{j+1} maturing at time t_j with a strike price E_j. Portfolio 2 contains E_j units of the zero coupon bond B_j maturing at time t_j, and portfolio 3 contains exactly one unit of the zero coupon bond B_j. The current values of these three portfolios are given in the second column. The third and fourth columns show the values at time t_j when B_j reaches maturity and is worth 1; the third column ($B_{j+1} < E_j$) and the fourth column ($B_{j+1} \geq E_j$) correspond to the cases where the put expires in-the-money and out-of-the-money, respectively.

		Value at Expiration	
Portfolio	Current Value	$B_{j+1} < E_j$	$B_{j+1} \geq E_j$
1	$P + B_{j+1}$	E_j	B_{j+1}
2	$E_j B_j$	E_j	E_j
3	B_j	1	1

[17] Obviously, for any European put: $P_j \leq E B_j$ and $P_j \geq 0$.

[18] Indeed, in a world where negative forward interest rates are precluded the following inequality should hold:

$$B_{j+1} \leq B_j \quad \text{with} \quad \tau_{j+1} > \tau_j$$

equation (13) is given by the following analytic solution (see Appendix):

$$P_j(B_{j+1}, B_j, \tau_j) = -B_{j+1} N(-d_1^j) + E_j^c B_j N(-d_2^j) + B_j N(d_3^j) - E_j^c B_{j+1} N(d_4^j) \tag{15}$$

$$d_1^j = \left[\ln(B_{j+1}/B_j) - \ln E_j^c + 1/2\, T_j\right]/\sqrt{T_j}$$

$$d_2^j = \left[\ln(B_{j+1}/B_j) - \ln E_j^c - 1/2\, T_j\right]/\sqrt{T_j}$$

$$d_3^j = \left[\ln(B_{j+1}/B_j) + \ln E_j^c - 1/2\, T_j\right]/\sqrt{T_j}$$

$$d_4^j = \left[\ln(B_{j+1}/B_j) + \ln E_j^c + 1/2\, T_j\right]/\sqrt{T_j}$$

where $N(\cdot)$ denotes the unidimensional cumulative normal distribution, with

$$T_j = \int_0^{\tau_j} \Sigma_j^2(\Theta)\, d\Theta$$

and

$$\Sigma_j^2(\Theta) = \sigma_x^2(\Theta + \Delta t) - 2\sigma_x(\Theta + \Delta t)\sigma_y(\Theta) + \sigma_y^2(\Theta).$$

In the case of the Ornstein-Uhlenbeck process (7), formula (8) applies for $\sigma_x(\cdot)$ and $\sigma_y(\cdot)$, thus

$$\Sigma_j^2(\Theta) = (\sigma^2/k^2) A e^{-2k\Theta} \quad \text{with} \quad A = 1 - 2\, e^{-k\Delta t} + e^{-2k\Delta t} \tag{16}$$

For the duration based process (9) for which the standard deviation of the rate of return on the bond is proportional to time to maturity, we obtain a constant

$$\Sigma_j^2(\Theta) = \sigma^2(\Delta t)^2 \tag{17}$$

The price of a cap agreement is obtained by using both equations (3) and (15).

The pricing of a floor agreement is easily grasped by applying the put-call parity condition (cf. Stoll (1969)). Indeed, for a bank a floor agreement is equivalent to a long position on a string of calls, the value of which is:

$$C_j(B_{j+1}, B_j, \tau_j) = +B_{j+1} N(d_1^j) - E_j^f B_j N(d_2^j) + B_j N(d_3^j) - E_j^f B_{j+1} N(d_4^j) \tag{18}$$

The first two terms in equations (15) and (18) correspond to Merton's call and put formulas, the last two terms are due to the boundary condition (13c). Economically we can interpret these last two terms as a price correction. Provided that the bond price processes allow for negative forward rate, these correcting terms ensure a pricing as if the forward rate remained non-negative. As a consequence of the dominance theorem, the probability mass in (11) and (12) which would be responsible for this mispricing, is shifted to the limiting boundary. In other words, the boundary condition (13c) is an absorbing barrier.

A collar agreement can thus be priced by applying equation (4) and by carefully recognizing that the strike price of the put string, E^c is generally smaller than the strike price of the call string, E^f.

It is also worthwhile noting that a collar whose cap and floor strike prices. E^c and E^f, respectively, are equal to E, replicate the cash flow pattern of an interest rate swap with a fixed rate equal to $c = -\ln E/\Delta t$. The equilibrium level of c then is such that the premium of the collar is zero.

Conclusion

This paper has proposed a methodology for pricing caps, floors, and collars. In fact, these instruments have been interpreted as strings of options on zero coupon bonds. Closed form solutions involving simple univariate normal distributions have been derived for two different term structure of interest rate assumptions, namely the Vasicek one and the arithmetic Brownian process. Although they both allow for negative forward rates, a dominance based boundary condition for the caps and floors ensures that everything goes as if forward rates remained non-negative. One other advantage of the proposed model is to yield in a straightforward manner the various parameters for hedging caps, floors, and collars. Deltas, gammas, thetas can be easily computed in order to determine the hedging portfolio of the issuer's position.

An obvious extension to our model would be to consider caps, floors, and collars on weighted averages of past interest rates. Indeed, in most financial markets the reference interest rate for these instruments is a floating weighted index, but this raises technical difficulties which were already identified by Ramaswamy and Sundaresan (1986) and Bouaziz, Briys, and Crouhy (1990). Nonstandard indices could also be considered.

Another possible extension would be to explicitly capture the default risk premium along the lines suggested by Sundaresan (1990). It should also be pointed out that this paper does not cover the intricacies entailed by the potential lack of liquidity on some currently traded caps. Indeed, nonstandard caps, e.g., caps with long maturities, often exhibit large bid-offer spreads.

Finally, an empirical investigation could be conducted to examine the consistency between our model and the swap quotations on the market place.

Appendix

In this Appendix we want to reexamine the solution procedure of the valuation problem (13).

$$\frac{1}{2}\sigma_x^2 x^2 P_{xx} + \sigma_x \sigma_y xy P_{xy} + \frac{1}{2}\sigma_y^2 y^2 P_{yy} - P_\tau = 0 \tag{13}$$

subject to

$$P(x, y, 0) = \max(E - x, 0) \tag{13a}$$

$$P(0, y, \tau) = Ey \tag{13b}$$

$$P(y, y, \tau) = 0 \tag{13c}$$

By the transformation of variables

$$z = -\ln(x/y) \quad \text{and} \quad T = \int_0^{\tau} \Sigma^2(\xi)\, d\xi \tag{A1}$$

$$\text{with} \quad P(x, y, \tau) = Ee^{-\frac{1}{2}z} e^{-\frac{1}{8}T} u(z, T) y$$

$$\text{and} \quad \Sigma^2(\tau) = \sigma_x^2 - 2\sigma_x \sigma_y + \sigma_y^2$$

we find the partial derivatives

$$P_{xx} = Ee^{-\frac{1}{2}z} e^{-\frac{1}{8}T} \left[u_{zz} - \frac{1}{4} u \right] \frac{y}{x^2} \tag{A2}$$

$$P_{xy} = Ee^{-\frac{1}{2}z} e^{-\frac{1}{8}T} \left[\frac{1}{4} u - u_{zz} \right] \frac{1}{x} \tag{A3}$$

$$P_{yy} = Ee^{-\frac{1}{2}z} e^{-\frac{1}{8}T} \left[u_{zz} - \frac{1}{4} u \right] \frac{1}{y} \tag{A4}$$

$$P_{\tau} = Ee^{-\frac{1}{2}z} e^{-\frac{1}{8}T} \left[u_T - \frac{1}{8} u \right] y \Sigma^2(\tau) \tag{A5}$$

This gives us

$$\frac{1}{2} u_{zz} - u_{\tau} = 0 \tag{A6}$$

subject to the transformed boundaries

$$u(z, 0) = e^{\frac{1}{2}z} \max(1 - Ke^{-z}, 0) \quad \text{with} \quad K \equiv 1/E \tag{A7}$$

$$u(0, T) = 0 \tag{A8}$$

$$\lim_{z \to +\infty} u(z, T) = e^{\frac{1}{2}z} e^{\frac{1}{8}T} \tag{A9}$$

Note that the last condition is of "slow" growth. Hence, this is the standard heat-transfer problem along the positive half of the real axis. Its general solution is

$$u(z, T) = \frac{1}{\sqrt{2\pi T}} \int_0^{+\infty} u(\xi, 0) \left[e^{\frac{-(\xi - z)^2}{2T}} - e^{\frac{-(\xi + z)^2}{2T}} \right] d\xi \tag{A10}$$

Because $u(\xi, 0) = 0$ for $\xi < \ln K$, we have the special solution with respect to condition (A7)

$$u(z, T) = \frac{1}{\sqrt{2\pi T}} \int_{\ln K}^{+\infty} \left(e^{\frac{1}{2}\xi} - Ke^{-\frac{1}{2}\xi} \right) \left[e^{\frac{-(\xi - z)^2}{2T}} - e^{\frac{-(\xi + z)^2}{2T}} \right] d\xi \tag{A11}$$

This may be rearranged to obtain

$$u(z,T) = -Ke^{-\frac{1}{2}z}e^{\frac{1}{8}T}N(-d_1) + e^{\frac{1}{2}z}e^{\frac{1}{8}T}N(-d_2) + Ke^{\frac{1}{2}z}e^{\frac{1}{8}T}N(d_3) - e^{-\frac{1}{2}z}e^{\frac{1}{8}T}N(d_4) \qquad \text{(A12)}$$

where

$$d_1 = \left(\ln K - z + \frac{1}{2}T\right)/\sqrt{T}$$

$$d_2 = \left(\ln K - z - \frac{1}{2}T\right)/\sqrt{T}$$

$$d_3 = \left(\ln K - z - \frac{1}{2}T\right)/\sqrt{T}$$

$$d_4 = \left(\ln K - z + \frac{1}{2}T\right)/\sqrt{T}$$

Putting this in terms of the original variables we obtain formula (15) in the text.

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