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# The Effect of Business Risk on Corporate Capital Structure: Theory and Evidence

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**ABSTRACT**

Under corporate and personal taxation, we demonstrate that the relation between optimal debt level and business risk is roughly U-shaped. This result follows from the fact that the tax liability is an option portfolio that is long in the corporate tax option and short in the personal tax option. Therefore, the net effect of a change in business risk on the optimal debt level depends upon the relative magnitudes of the resultant marginal changes in the values of these two options. Results of empirical tests offer support for the predicted U-shaped relationship.

DESPITE THE CONCENSUS THAT business risk is one of the primary determinants of a firm's capital structure, existing theoretical and empirical research does not provide an unambiguous answer to the question of whether an increase in a firm's business risk should lead it to lower the level of debt in its capital structure. Most finance textbooks casually assert an inverse relationship between optimal debt level (ODL) and business risk (BR).[1] The basis for this argument is that the existence of debt in the capital structure increases the probability of bankruptcy, and firms with more variable cash flows, that is, higher business risk, have a higher probability of bankruptcy for a given level of debt. Therefore, assuming that the marginal tax rate is the same for all firms, firms with higher BR should have less debt. However, it has been noted that "the existence of positive bankruptcy costs is not sufficient to ensure that the TS-BC (tax shields-bankruptcy costs) hypothesis will predict an inverse relationship (between business risk and ODL)."[2] Indeed, Jaffe and Westerfield (1987) prove that, given the appropriate choice of parameters, ODL will be an *increasing* function of business risk.[3] Empirical evidence can be found to support "increasing" as well as "decreasing"

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[1] See, for example, Brealey and Myers (1988, page 435).

[2] Castanias (1983).

[3] On the other hand, Castanias (1983) derives a decreasing ODL-BR relationship over a certain range of BR when cash flows are normally distributed.

relationships. Additionally, some empirical studies find no relationship between ODL and BR.[4] These studies, however, have generally proceeded by specifying a linear ODL-BR relationship.[5] Mikkelson (1984) points out that the lack of a properly specified functional form is an important weakness of empirical research in this area.

The main contribution of this paper is to derive the functional form of the ODL-BR relation within the classic personal and corporate taxation framework of optimal capital structure. We show that the ODL-BR relationship is roughly U-shaped; initially decreasing and ultimately increasing, in the DeAngelo and Masulis (1980) framework (henceforth referred to as D-M) in which bankruptcy need not be costly.[6] The basic intuition underlying this result is that most models, in which the value-maximizing firm trades off the costs and benefits of debt, can be conceptualized in terms of minimizing the value of the competing claims of non-owners, many of which can be described as options. In the D-M framework, through corporate taxation, the cash flow to the government either is equal to a fraction of the firm's cash flow in excess of its debt plus non-debt tax shields, or is equal to zero when the cash flows are insufficient to cover these obligations. Thus, the government's claim is a European call option on the firm's cash flows with an exercise price equal to the sum of debt and the non-debt tax shields. Similarly, as we later show in detail, personal tax payments of debtholders can be viewed as an option written by the government with an exercise price equal to the firm's debt level. The value-maximizing firm minimizes the value of the government's option portfolio which is long in the corporate tax option and short in the personal tax option. At the optimal debt level, the firm equates the tax-rate-weighted marginal effects of the debt level on the two options. The effect of a change in BR and ODL, therefore, depends upon the relative magnitudes of the marginal changes in the values of the two options induced by this change. For each given level of BR, we derive conditions under which each of the effects dominates, and show that the interplay of these two effects implies a U-shaped ODL-BR relationship. Our empirical tests provide qualified support for the hypothesis that the relationship is U-shaped.

The remainder of the paper is organized as follows: in Section I we develop a simple model of corporate capital structure which yields a U-shaped ODL-BR relationship; in Section II we outline the test methodology and discuss the selection of proxy variables to be used in the empirical tests. The results of

[4] Long and Malitz (1985) and Toy et al. (1974) find an increasing relationship, whereas Castanias (1983), Bradley, Jarrell, and Kim (1984), and Carleton and Silberman (1977) document a decreasing relationship. Ferri and Jones (1979), Flath and Knoeber (1980), and Titman and Wessels (1988) conclude that there is no significant relationship between corporate leverage and business risk.

[5] For example, the theoretical construct of Bradley, Jarrell, and Kim (1984) suggests an ambiguous, nonlinear relationship between ODL and business risk. However, on the basis of some numerical simulations, the authors conjecture an inverse ODL-BR relationship and use a linear form in their empirical tests.

[6] In fact, this result is consistent with one of the simulation results in Bradley, Jarrell, and Kim (1984).

the empirical tests are in Section III, and Section IV contains the concluding remarks.

## I. The Theory of Optimal Capital Structure

Consider a firm in a single period economy.[7] Let $X$ be the stochastic operating cash flows to the firm before debt service, taxes, and depreciation. We assume that $X(\mu, \sigma) = \mu + \sigma Z$, where both $\mu$ and $\sigma$ are greater than 0, and $Z$ is a random variable with zero mean, unit variance, a strictly increasing cumulative density function $F$, and density $f$. We assume that $F$ is twice continuously differentiable and supported by the interval $[L, U]$, where $L$ is negative and $-\infty < L < 0 < U < \infty$. The expected value and standard deviation of $X$ are given by $\mu$ and $\sigma$, respectively. Let $R$ represent the tax-deductible debt service charge, and $\Pi$ be the depreciation charge. Then, given that $X(\mu, \sigma) = \mu + \sigma Z$, the probability that a firm suffers a tax loss, $\text{Prob}\{X(\mu, \sigma) - R - \Pi \leq 0\}$, is equal to $F\{[R - \mu + \Pi]/\sigma\}$. Similarly, the probability of default is given by $\text{Prob}\{X(\mu, \sigma) - R \leq 0\} = F\{[R - \mu]/\sigma\}$.[8]

For simplicity, we assume that the marginal personal tax rate on equity is zero and let $\tau_p$ and $\tau_c$ be the marginal personal tax rate on debt income, and the corporate tax rate, respectively.[9] Personal taxes paid by debtholders are equal to $\tau_p \min\{X(\mu, \sigma), R\}$ which, in turn, are equal to $\tau_p[X(\mu, \sigma) - R - \Pi, 0\}$. The objective of the firm is to choose the capital structure that minimizes the expected value of the total tax liability which is given by

$$
\begin{aligned}
G &= \tau_c E\left[\max\{X(\mu, \sigma) - R - \Pi, 0\}\right] \\
&\quad + \tau_p E\left[X(\mu, \sigma) - \max\{X(\mu, \sigma) - R, 0\}\right] \\
&\equiv \tau_c C(\mu, R + \Pi, \sigma) + \tau_p\left[\mu - C(\mu, R, \sigma)\right], && (1) \\
&\equiv -\tau_c\left[C(\mu, R, \sigma) - C(\mu, R + \Pi, \sigma)\right] + (\tau_c - \tau_p)C(\mu, R, \sigma) + \tau_p \mu, && (1a)
\end{aligned}
$$

[7] The single-period assumption prevents us from explicitly considering tax-timing options equityholders have in a multi-period setting [Constantinides (1983, 1984)]. However, our assumption that the personal tax on equity is zero roughly captures the ability of equityholders to reduce their effective tax on capital gains income by using these options.

[8] Throughout the analysis, we assume that all debt service payments are deductible for the firm and taxable for the bondholders in order to be consistent with the assumptions of D-M, Dammon and Senbet (1988), and the classical bankruptcy cost-tax shields literature discussed in Masulis (1988). In actuality, interest payments determine the probability of tax loss whereas the total debt payment (interest and principle) is relevant for determining the probability of default [see Talmor, Haugen, and Barnea (1985) and Park and Williams (1985)].

[9] $\tau_p$ is the marginal tax rate of the investor who, at the margin, is indifferent between debt income and equity income. In the D-M framework, $\tau_p = (1 - P_D/P_E)$, where $P_D$ and $P_E$, determined endogenously in the equilibrium, are the market prices of before personal income tax expected cash flows to debt and equity. The D-M analysis assumes that the marginal tax rates are the same across all states. In Dammon (1988) and Ross (1985), the tax regime is progressive implying that the marginal tax rates are state dependent. As explained in footnote 12, the introduction of progressivity does not impact the U-shaped ODL-BR reletionship.

where $G$ is the value of the government's total tax claim, $C$ is the value of a European call option, and $\mu$ is the value of $X(\mu, \sigma)$ under risk-neutral valuation.[10] $\mu$ is analogous to the price of the underlying asset since the option is written on the cash flow of the firm. The second argument of $C(\cdot)$ represents the exercise price of the option. In the case of the option owned by the government through corporate taxation, the exercise price is $R + \Pi$, because, unless the cash flow is greater than this amount, the government does not collect any taxes.[11] Similarly, the debtholders have an option because, when the cash flow is greater than the debt level $R$, their tax payment is reduced from $\tau_p X(\mu, \sigma)$ to $\tau_p R$, that is, by $\tau_p[X(\mu, \sigma) - R]$. On the other hand, if the cash flow is less than $R$, the option expires worthless and the tax liability of the debtholders is $\tau_p X(\mu, \sigma)$. The option value is also a function of the risk-free rate. However, since all our analysis is done at a single point in time, there are no intertemporal cash flow tradeoffs made by the firm, and, therefore, the risk-free rate plays no role in our analysis. We have, therefore, simplified our notation by always working with the discounted values of all the model parameters. Note that the expression in squared brackets in equation (1a) is the difference of two call options and, thus, can be thought of as subordinated debt. The second term is the personal tax call option, and the third term is a constant. If the debt level is sufficiently low, the personal tax option is deep in the money and, hence, as shown in Galai and Masulis (1976), the effect of an increase in variance on the option value is small. As shown in Black and Cox (1976), an increase in variance will lower the value of the subordinated debt claim at low debt levels. Because the subordinated debt claim factors negatively into the equation, an increase in variance will increase the firm's tax liability at low debt levels. On the other hand, when the debt level is sufficiently high, Black and Cox have shown that subordinated debt behaves like a call option and thus increases in volatility increases the value of the subordinated debt claim. Because the subordinated debt effect factors negatively into the tax liability and is multiplied by a larger constant than the personal tax effect, increases in volatility lower the tax liability at sufficiently high levels of debt.

[10] Green and Talmor (1985) also utilize an option approach to analyze the impact of taxation on corporate decision making. The main difference between our analysis and Green and Talmor's is that they treat the capital structure variables as fixed, and investigate the effect of the government's corporate tax option on the firm's investment decisions. We, on the other hand, as in D-M, treat the firm's investment decision as fixed, and investigate the changes in the firm's endogenously determined optimal capital structure resulting from changes in the business risk of the cash flows from investment.

[11] As in D-M and Dammon and Senbet (1988), the single-period framework prevents us from incorporating the carryback and carryforward provisions of the tax code. The results in an overestimation of the cost of underutilization of tax shields in the current year. Because of the statutes regarding the number of years tax losses can be shifted, the utilization of all tax losses is not guaranteed. In addition, delayed utilization of tax losses is costly in terms of present value. The qualitative results in this paper, thus, are independent of the magnitude of the cost of lost tax shields.

In order to formalize this concept, let $C^p \equiv C(\mu, R, \sigma)$ and $C^c \equiv C(\mu, R + \Pi, \sigma)$ represent the personal tax and the corporate tax calls, respectively. Therefore, the firm's problem is to choose the $R$ which minimizes the total tax liability of the corporation's claimholders given by

$$\tau_c C^c + \tau_p[\mu - C^p].$$

The first-order condition of this problem is that, at the optimal $R$

$$\left.\frac{\partial G}{\partial R}\right|_{R=R^*} \equiv \tau_c \left.\frac{\partial C^c}{\partial R}\right|_{R=R^*} - \tau_p \left.\frac{\partial C^p}{\partial R}\right|_{R=R^*} = 0, \tag{2}$$

when $R^* > 0$. When $R^* = 0$, the first-order condition is

$$\left.\frac{\partial G}{\partial R}\right|_{R=0} \equiv \tau_c \frac{\partial C^c}{\partial R}_{R=0} - \tau_p \left.\frac{\partial C^p}{\partial R}\right|_{R=0} \geq 0. \tag{3}$$

Given risk-neutral valuation and the fact that we are working with discounted cash flows, the value of the government corporate tax call is given by

$$\tau_c C^c \equiv \tau_c E\{\max[X(\mu, \sigma) - R - \Pi, 0]\} = \tau_c E\{\max[\mu + \sigma Z - R - \Pi, 0]\}. \tag{4}$$

In equation (4) above, note that

$$X(\mu, \sigma) - R - \Pi = \mu + \sigma Z - R - \Pi > 0, \text{ if and only if, } Z > \frac{R - \mu + \Pi}{\sigma}.$$

In order to simplify notation, define $T(R, \sigma) \equiv \dfrac{R - \mu + \Pi}{\sigma}$. Then, the value of the corporate call is given by

$$\tau_c \int_T^{+\infty} (\mu + \sigma z - R - \Pi) f(z)\, dz.$$

Differentiating the above expression with respect to $R$, we obtain

$$\tau_C \left(\frac{\partial C^c}{\partial R}\right) = -\tau_c\{1 - F[T(R, \sigma)]\}, \tag{5}$$

where $F$ is the distribution function of $Z$. Similarly, defining $D(R, \sigma) \equiv \dfrac{R - \mu}{\sigma}$, we obtain

$$\tau_p \left(\frac{\partial C^p}{\partial R}\right) = -\tau_p\{1 - F[D(R, \sigma)]\}. \tag{6}$$

Define $D^*(\sigma) = D(R^*(\sigma), \sigma)$ and $T^*(\sigma) = D(R^*(\sigma), \sigma)$, where $R^*$ is the level of debt which maximizes firm value for a given $\sigma$. Hence, the first-order condition given in equation (2) can be expressed as[12]

$$\left.\frac{\partial G}{\partial R}\right|_{R=R^*} = \tau_p\{1 - F[D^*(\sigma)]\} - \tau_c\{1 - F[T^*(\sigma)]\} = 0, \tag{7}$$

and, the first-order condition given in equation (3), when $R^* = 0$, can be expressed as

$$\left.\frac{\partial G}{\partial R}\right|_{R=0} = \tau_p\{1 - F[D(0,\sigma)]\} - \tau_c\{1 - F[T(0,\sigma)]\} \geq 0. \tag{8}$$

In order to determine the relationship between $R^*$ and $\sigma$ from the first-order condition in equation (7), we use the Implicit Function Theorem to obtain

$$\frac{dR^*}{d\sigma} = -\frac{\partial^2 G/\partial R\,\partial\sigma}{\partial^2 G/\partial^2 R}. \tag{9}$$

Since $R^*$ minimizes $G$ (Lemma 1 in the appendix), the denominator $[\partial^2 G/\partial^2 R]_{R=R^*} > 0$. Hence, $dR^*/d\sigma$ is opposite in sign to the numerator. Because $\sigma > 0$, this implies that $dR^*/d\sigma$ has the same sign as $-\sigma[\partial^2 G/\partial R\,\partial\sigma]$. Differentiating equation (2) with respect to $\sigma$ and multiplying by $-\sigma$ yields

$$-\sigma\left(\frac{\partial^2 G}{\partial R\,\partial\sigma}\right) = \sigma\tau_p\left(\frac{\partial^2 C^p}{\partial R\,\partial\sigma}\right) - \sigma\tau_c\left(\frac{\partial^2 C^c}{\partial R\,\partial\sigma}\right). \tag{10}$$

$$= \tau_c f[T^*(\sigma)]T^*(\sigma) - \tau_p f[D^*(\sigma)]D^*(\sigma). \tag{11}$$

When $R^* > 0$, the first-order condition in equation (7) implies

$$\tau_p = \tau_c\frac{1 - F[T^*(\sigma)]}{1 - F[D^*(\sigma)]}. \tag{12}$$

Substituting (12) into (11), yields

$$-\sigma\left(\frac{\partial^2 G}{\partial R\,\partial\sigma}\right) = \tau_c\{1 - F[T^*(\sigma)]\}\{h[T^*(\sigma)]T^*(\sigma) - h[D^*(\sigma)]D^*(\sigma)\}, \tag{13}$$

where $h(\cdot)$ is the hazard rate of the distribution of $Z$ and is given by $f/1 - F$. The key to determining the relationship between the optimal debt level $R^*$ and $\sigma$ is to determine the sign of equation (13). In order to establish the

[12] It is easy to see that the introduction of progressive personal taxes would modify the first-order condition as follows. The personal tax term, $\tau_p(1 - F[\cdot])$, would be replaced by $\tau_p(1 - F_d[\cdot])$, where $F_d$ is the cumulative distribution function induced by the state-price density function for debt income. The corporate tax term would be replaced by $\tau_c(1 - F_e[\cdot])$, where $F_e$ is the cumulative distribution function induced by the state-price density function for equity income. As long as both $F_d$ and $F_e$ have the increasing hazard rate property and the hazard rate of $F_e$ is greater than that of $F_d$, the results are unchanged.

U-shaped relationship between $R^*$ and $\sigma$, we need to show that $dR^*/d\sigma$ is negative when $\sigma$ is sufficiently small, and positive when $\sigma$ is sufficiently large. The following theorem utilizes equation (13) to establish the U-shaped relationship between $R^*$ and $\sigma$. The proof of this theorem utilizes a series of lemmas (2 through 7) that are proved in the appendix. Lemma 2 establishes some properties of the ratio of distribution functions that are used in the subsequent proofs. The other lemmas sign equation (13) for various ranges of $\sigma$.

**Theorem 1:** *If, (a) there exists at least one level of $\sigma$ at which the equilibrium expected taxable corporate income and the debt level are positive, that is, if there exists a $\sigma^1$ such that $0 < R^*(\sigma^1) < \mu - \Pi$; (b) the cash flow distribution of the firm satisfies the increasing hazard rate property, that is, $f(z)/(1 - F(z))$ is strictly increasing in $z$; then, the DeAngelo-Masulis model implies a roughly U-shaped relationship between $R^*$ and $\sigma$.*

*Proof:* To show that $dR^*/d\sigma$ is positive when $\sigma$ is sufficiently large, we first show that there exists a unique $\sigma$, $\sigma_0$, at which $R^*(\sigma_0) = \mu - \Pi$ (Lemma 3 in the appendix). This implies that $T^*(\sigma_0) = 0$ and $D^*(\sigma_0) < 0$ and, therefore, equation (13) will have a positive sign. As shown in Lemmas 4 and 5 in the appendix, for all $\sigma > \sigma_0$, $U > T(\sigma) > 0$. We also know that $h[T(\sigma)] > h[D(\sigma)]$ because $T(\sigma) > D(\sigma)$ and the hazard rate is increasing. Thus, when $D(\sigma) \leq 0$, equation (13) will have a positive sign. Even when $D(\sigma) > 0$, which is possible for sufficiently high $\sigma$, because $h[T(\sigma)] > h[D(\sigma)]$, $T(\sigma) > 0$, $D(\sigma) > 0$, and $T(\sigma) > D(\sigma)$, equation (13) will have a positive sign.

Now consider the case when $\sigma$ is small. We show in the appendix that when $\sigma$ is sufficiently small, $R^*(\sigma) > 0$ (Lemma 5 in the appendix), $L < T^*(\sigma) < 0$ (Lemmas 4 and 5 in the appendix), and $D^*(\sigma) \leq L$ (Lemma 7 in the appendix). $D^*(\sigma) \leq L$ implies that $h(D^*(\sigma)) = 0$ which implies that $h[D^*(\sigma)]D^*(\sigma) = 0$. $L < T^*(\sigma) < 0$ implies that $1 - F[T^*(\sigma)] > 0$ and $h(T^*(\sigma)) > 0$. This implies that $h[T^*(\sigma)]T^*(\sigma) < 0$ and $\tau_c\{1 - F[T^*(\sigma)]\} > 0$. Hence, $-\sigma(\partial^2 G/\partial R\,\partial\sigma) < 0$ for $\sigma$ sufficiently small. Thus, $R^*(\sigma)$ will be roughly U-shaped, initially decreasing and ultimately increasing.

Conditions (a) and (b) of the theorem are quite reasonable. Condition (c) is a distributional condition that is satisfied by almost all of the unimodal distributions used in finance. For example, uniform, normal, truncated normal, gamma, and Weibull distributions satisfy this property. The increasing hazard rate assumption is a natural distributional assumption to impose on the problem because, otherwise, we cannot be assured that a unique optimal debt level exists in the D-M framework.[13]

In view of the earlier discussion following equations (1) and (1a), the underlying intuition of the above result is as follows. Recall that the subordinated debt component of the government's tax claim on the firm factors in

[13] This is a common assumption, see, for example Grossman and Hart (1982). In fact, as shown in the appendix (Lemma 1), if $F$ is a "decreasing hazard rate" distribution, an interior optimal capital structure cannot exist.

negatively; that is, it reduces the firm's tax liability. At low levels of debt, an increase in business risk lowers the value of this component and, therefore, the firm's optimal response is to lower its debt level. On the other hand, when the level of debt is high, the subordinated debt component behaves like a call option and, thus, an increase in business risk increases its value. In this case, the optimal response for the firm is to increase its debt level. Taken together, the above imply a roughly U-shaped relation between business risk and optimal debt level.

To illustrate the U-shaped relationship, we graph the total expected tax liabilities of three firms that are identical in every respect except for the variance of their cash flows. For each firm, suppose the non-debt tax shield is 200, the corporate tax rate 0.35, the personal tax rate 0.25, and the mean cash flow for each of the three firms is 1,000. The cash flows of the three firms are normally distributed with a common mean of 1,000. The high risk firm ($H$) has a cash flow standard deviation of 450, the medium ($M$) 200, and the low risk firm ($L$) has a standard deviation of 50. Figure 1 is a graph of the government's expected tax claim $G$ for the three firms as a function of $R$, the debt payment. The functions $G_i(R)$, $i = H, M, L$ for the firms attain a unique interior minimum at $R_i^*$; $R_H^* = 869.47$, $R_M^* = 719.10$, and $R_L^* = 771.70$. Clearly, the medium risk firm has a lower optimal debt level than either the low risk or the high risk firm.

Figure 1 also provides insight into another aspect of corporate policy, the ability of firms to change their cash flow variance by hedging. As shown in Smith and Stulz (1985), under corporate taxation and bankruptcy costs, firms

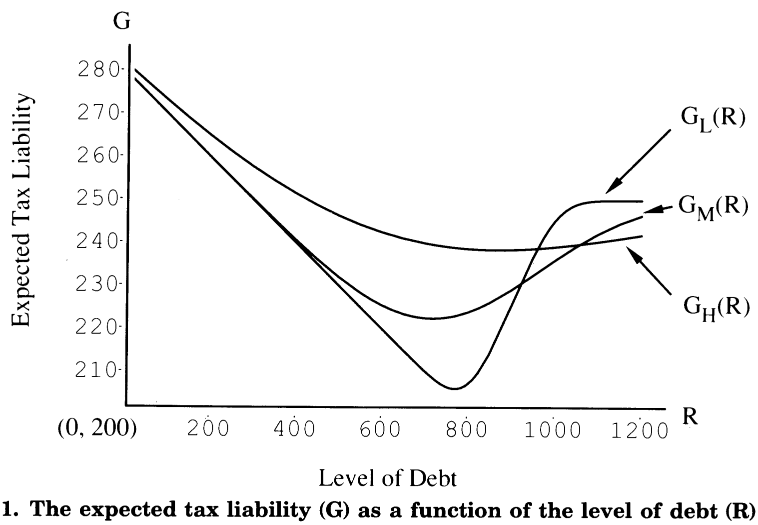


**Figure 1. The expected tax liability (G) as a function of the level of debt (R) for three firms with different levels of business risk.** For all three firms, the mean cash flow is 1,000, non-debt tax shields are 200, corporate and personal tax rates on debt are 0.35 and 0.25, respectively. The cash flow standard deviations for the low (L), medium (M), and high (H) risk firms are 50, 200, and 450, respectively. All cash flows are normally distributed.

have the incentive to hedge their cash flows. In this regard, it is also interesting to note that at the optimum level of debt, the tax liability is lowest for the low risk firm and highest for the high risk firm. Therefore, at the optimum $R$, the value of the tax liability is decreasing in the variance of cash flows. This shows that firms have an incentive to reduce their cash flow variance by hedging in the DeAngelo-Masulis framework which explicitly incorporates corporate as well as personal taxes.[14]

## II. Selection of Empirical Proxies

The formulation of the capital structure decision in an options framework permits us to generate a regression equation that can be empirically estimated. Note that the derivative of the option valuation formula with respect to the exercise price is homogeneous of degree one, implying that multiplying each of its independent variables by a constant $k$ alters the value of the function by the proportion $k$. Consider a firm with mean cash flow equal to 1, depreciation charges equal to $\pi$, and standard deviation of cash flows equal to $v$. At the optimal capital structure, denoted by the debt level $r$, this firm satisfies the first-order condition

$$\tau_c\left(\frac{\partial C(1, r+\pi, v)}{\partial R}\right) - \tau_p\left(\frac{\partial C(1, r, v)}{\partial R}\right) = 0. \tag{14}$$

Note that, if we multiply $r$, $\pi$, and $v$ by $\mu$, then the linear homogeneity of the derivative of the option price with respect to the exercise price implies that

$$\mu\left(\tau_c\left(\frac{\partial C(1, r+\pi, v)}{\partial R}\right) - \tau_p\left(\frac{\partial C(1, r, v)}{\partial R}\right)\right)$$
$$= \tau_c\left(\frac{\partial C(\mu, r\mu+\pi\mu, v\mu)}{\partial R}\right) - \tau_p\left(\frac{\partial C(\mu, r\mu, v\mu)}{\partial R}\right) = 0.$$

Hence, the optimal debt level of a firm with a mean cash flow equal to $\mu$, a variance of cash flows equal to $\sigma = v\mu$, and non-debt tax shields equal to $\Pi = \mu\pi$, will equal $R^* = \mu r^*$. Where $r^*$, which is the optimal debt level of a firm with mean cash flow of \$1, is a function only of $v = \sigma/\mu$ and $\pi = \Pi/\mu$. Which, of course, implies the following;

$$R^*(\mu, \sigma, \Pi) = \mu r^*(v, \pi) \Rightarrow R^*/\mu = r^*(v, \pi). \tag{15}$$

From equation (15), we know that $R^*/\mu = r^*(v, \pi)$. For the purpose of empirical tests, in view of the U-shaped ODL-BR relationship, we approximate this functional form by

$$R^*/\mu = b_0 + b_1 v + b_2 v^2 + b_3 \pi$$

[14] For a formal demonstration of the optimality of hedging in this framework, see Kale and Noe (1990).

In the above, the quadratic form, $b_1 v + b_2 v^2$, captures the predicted U-shape of the ODL-BR relationship. The predicted signs of the coefficients are $b_1 < 0$, $b_2 > 0$, and $b_3 < 0$. Also, note that, in a D-M framework, $r^*(0,0) = 1$ because a firm with no risk and no NDTS will lever up to its mean cash flow and, thus, the model predicts that $b_0 = 1$. Therefore, subtracting one from both sides we obtain

$$(R^*/\mu) - 1 = b_1 v + b_2 v^2 + b_3 \pi. \tag{16}$$

Equation (16) forms the basis of the empirical tests which follow.

Even though our theory makes predictions regarding only the ODL-BR relationship, in the empirical tests which follow, we include additional variables to minimize possible misspecification errors owing to omitted variables. Our theory specifies the empirical proxies for business risk and leverage. Earlier research, notably that of Titman and Wessels (1988), suggests the empirical proxies for the remaining variables.

According to equation (16) above, leverage ($Y$) is represented by $(R^*/\mu) - 1$, where $R^*$ is the taxable debt service and $\mu$ is the expected cash flow. We measure $R^*$ by a firm's interest expense. However, since a firm's expected cash flow is unobservable, we use the current year's cash flow to the firm as the proxy. If the conditional mean is dependent on variables not in the econometrician's information set, then using the current year cash flow (the "errors-in-variable" estimator) may be preferable.[15] The advantage of using the errors-in-variable estimator is that it does not require the assumption that the conditional mean of the cash flow series be a constant or some other deterministic function of time. However, while this is an unbiased estimator of expected future cash flow, it introduces measurement error into the dependent variable.

As suggested by our theoretical construct, business risk is represented by the coefficient of variation ($CV$), $\sigma/\mu$, where $\sigma$ is the standard deviation of the firm's cash flows. Unlike the case for proxying expected cash flows, the estimate of $\sigma$ must be made from historical data. We construct two proxies for $\sigma$: (i) the standard error of an ordinary least squares regression on time (OLS), and (ii) the maximum likelihood estimate of the standard deviation of a linear regression of cash flows on time assuming a first-order autoregressive process in the error term (AR).[16] Both the OLS and AR estimation procedures assume that the conditional variance of the firm's cash flows is constant over time. Thus, our estimation procedures of the conditional moments of the cash flow distribution, $\mu$ and $\sigma$, allows the conditional mean to vary over time, while the conditional variance is restricted to be constant. The assumption of constant conditional variance is consistent with an econ-

[15] For a discussion, see McCallum (1976), Wickens (1982), and Gottfries and Persson (1988).

[16] For some firms, in the AR procedure, the estimation algorithm did not converge and standard deviation estimates from OLS were substituted in order to equate sample sizes.

omy in which shocks affect estimated mean cash flows but leave variances unchanged.[17]

As many authors have pointed out, capital structure decisions could be affected by several other considerations including non-debt tax shields, systematic risk, firm size, agency costs, and asymmetric information. To control for these effects, we design a number of proxies as follows. The observable indicators of non-debt tax shields that we employ are investment tax credits (ITC), depreciation expense (DEP), and research and development tax credits (RDTC). The computation of RDTC is in accordance with the then existing Internal Revenue Service guidelines which allowed firms a credit equalling 25% of their "increased" expenditures for research and development. The "increase" was computed by subtracting the average amount spent during the past three years from current R&D expenditures.[18] With these three variables, we construct the measure of non-debt tax shields, NDTS = $(0.46^*\text{DEP} + \text{ITC} + \text{RDTC})/\mu$, where $\mu$ denotes the expected cash flow. Some empirical studies [Boquist and Moore (1984) and Bowen, Daley, and Huber (1982)] include operating tax loss carryforwards in their proxy for NDTS. This variable, however, is determined jointly by the firm's past levels of NDTS and its equilibrium probability of tax loss. As shown in the appendix (Lemma 8), this probability is monotonically increasing in the variance of the firm's cash flows. However, these carry forwards do affect the firm's taxable income. Therefore, in order to control for this effect without biasing our results, we include tax loss carryforwards (CARRY) and the multiplicative term (CARRY $\times$ CV) as a control variable.

Dammon (1988) and Ross (1985) demonstrate that, when tax rates are progressive, the correlation between firm value and the state-dependent tax rate is an important determinant of the value of the debt tax shield. The net tax subsidy depends positively on the level of state prices which are negatively correlated with aggregate consumption. Therefore, the more positive the correlation between firm cash flow and aggregate consumption, the less the value of the net tax subsidy of debt. This theory predicts a negative relationship between firm asset beta and leverage. In our empirically test, we control for this effect by including the asset beta from the capital asset pricing model (BETA).[19]

To proxy firm size, we use the natural log of the firm's sales in the current year (ln SA). For the variables which capture the agency [Smith and Warner (1979)] and asymmetric information [Myers and Majluf (1984)] determinants

[17] As an alternative to using current flow as the proxy for $\mu$, we repeated the empirical tests using averages over the longest possible period of data availability, 19 years on the COMPUSTAT, as proxies for $\mu$. The results on the ODL-BR relationship for one of the years were similar to those reported in the paper, while for another, they were generally insignificant.

[18] *The Price Waterhouse* "Guide to the New Tax Law," 1986, page 156. Also, in case the current R&D expense is less than the average, RDTC = 0.

[19] The equity beta was first obtained from the monthly stock returns on the CRSP tapes and the asset beta was then obtained by multiplying the equity beta by one minus the debt-equity ratio.

of capital structure, we generally use proxies suggested by Titman and Wessels (1988). We proxy for intangible assets with two indicators, the reported intangible assets expressed as a fraction of total assets (INT) and the ratio of advertising expense to sales (ADV). Growth opportunities, and to a smaller extent firm profitability, are captured by the ratio of capital expenditure to total assets (CPX). The informational asymmetry regarding assets in place is proxied by ending inventory scaled by the firm's total assets (INV).[20]

## III. Data, Hypotheses, and Results

### A. Data

All our data, with the exception of BETA, are obtained from the standard versions of the Annual 1986 COMPUSTAT tapes.[21] BETA is estimated by unlevering the equity beta estimated from monthly stock returns for the last five years obtained from the Center for Research in Security Prices (CRSP) tapes using the CRSP value-weighted market index. We conduct our cross-sectional tests for two years, 1985 and 1984. For both years, we use a 19-year period in estimating the standard deviation of cash flows used in calculating CV (the 19-year restriction is placed by data availability on the COMPUSTAT tapes). We select only those firms that have a complete history of cash flows for this period. All the other variables used in the study are the realized values for the current year. The sample is further restricted to only those firms that do not have missing values in the current year for INT, R&D, DEP, ITC, ADV, and CPX. In addition, since the estimated cash flow, sales and total assets are used to scale variables, we require that the values for these be strictly positive. Also, we exclude all utilities, financial, and real estate firms from the sample.[22] The cash flow, used in calculating CV and $\mu$, is the operating income before depreciation, interest, and taxes for the current year as reported in the COMPUSTAT tapes.

Descriptive statistics for the 1985 and the 1984 samples are presented in table I. The estimates of CV are higher with the OLS procedure (0.332 in 1985 and 0.297 in 1984) than with the AR procedure (0.296 in 1985 and 0.271 in 1984) in both samples. A possible explanation for this is that OLS estimates may be biased because of positive serial correlation in the cash flow

[20] Titman and Wessels (1988) include the R&D expenditure of the firm to capture technological sophistication and detect a negative relationship between it and leverage. We have, however, used R&D increases in computing the NDTS of the firm. Titman and Wessels concede that their approach may bias the significance of the NDTS downward and, thus, it is also true that the significance level of NDTS in our results may be biased upwards.

[21] Our empirical results, therefore, suffer from the well known survivorship bias induced by using only those firms appearing in the COMPUSTAT tapes. However, as we argue later, in our study this misspecification would tend to bias the results toward *rejecting* the theoretically derived U-shaped relationship between ODL and CV.

[22] Using 3-digit SIC codes, utilities are in SIC codes 480–499; financial and real estate firms are in SIC codes 600–679.

Table I

## Descriptive Statistics of Leverage and its Determinants

Mean, standard deviation, and the range for the 1985 and 1984 samples with 233 firms in 1985 and 243 firms in 1984 with complete data on the Annual COMPUSTAT tapes. All variables except the coefficient of variation (CV) are values for the current year, either 1984 or 1985. The mean used to calculate CV is the current year cash flow. The standard deviation is estimated (from cash flow series of past 19 years) two ways, ordinary least squares (OLS) and first order autoregressive process (AR). In the last method, where the process did not converge, values from OLS are substituted to equate sample sizes. CAPM asset beta is estimated from monthly returns in the last five years from CRSP tapes using the CRSP value-weighted market index. Except for CV, Leverage, and Asset Beta, all values are in millions of dollars. Leverage is defined as the ratio of interest expense to cash flow.

| | 1985 | | | 1984 | | |
|---|---|---|---|---|---|---|
| Variable | Mean | Minimum | Maximum | Mean | Minimum | Maximum |
| Cash flow | 332.999 | 0.774 | 10,422.941 | 314.709 | 0.190 | 9665.699 |
| CV (OLS) | 0.332 | 0.042 | 6.030 | 0.297 | 0.043 | 8.745 |
| CV (AR) | 0.296 | 0.025 | 4.672 | 0.271 | 0.025 | 8.745 |
| Interest | 46.510 | 0.000 | 944.900 | 42.478 | 0.000 | 932.500 |
| Leverage | 0.310 | 0.000 | 17.941 | 0.275 | 0.000 | 8.974 |
| Sales | 2646.630 | 14.980 | 96,371.625 | 2435.560 | 12.547 | 83,889.875 |
| Depreciation | 121.070 | 0.251 | 6208.496 | 106.904 | 0.186 | 4965.652 |
| Inv. Tax. Cr. | 10.244 | 0.000 | 508.500 | 9.063 | 0.000 | 404.100 |
| Tax Carry for. | 16.466 | 0.000 | 1172.000 | 20.124 | 0.000 | 1414.000 |
| R&D Exp. | 82.626 | 0.000 | 3625.200 | 69.784 | 0.000 | 3075.800 |
| Intang. Asset | 83.755 | 0.000 | 5945.246 | 45.426 | 0.000 | 1938.499 |
| Advtg. Exp. | 52.443 | 0.000 | 1080.000 | 38.154 | 0.000 | 892.300 |
| Capital Exp. | 218.222 | 0.445 | 11,711.602 | 216.983 | 0.271 | 11,122.902 |
| Inventory | 342.517 | 0.508 | 8269.695 | 328.301 | 0.332 | 7359.652 |
| Asset Beta | 0.677 | 0.006 | 2.143 | 0.610 | −0.009 | 2.193 |

series. The 1985 mean values for all the variables are higher than the corresponding mean values in 1984. The values for leverage, defined as the ratio of interest expense to the cash flow, are 0.310 in 1985 and 0.275 in 1984 which compare well with the leverage values reported in earlier empirical studies.

### B. Hypotheses

In order to test the relationship between leverage and its determinants we estimate the following equation for both samples.

$$\begin{aligned} Y = \beta_0 &+ \beta_1 \mathrm{CV} + \beta_2 \mathrm{CV}^2 + \beta_3 \ln \mathrm{SA} + \beta_4 \mathrm{NDTS} \\ &+ \beta_5 \mathrm{INT} + \beta_6 \mathrm{ADV} + \beta_7 \mathrm{CPX} + \beta_8 \mathrm{INV} + \beta_9 \mathrm{CARRY} \\ &+ \beta_{10} \mathrm{CARRY} \times \mathrm{CV} + \beta_{11} \mathrm{BETA} + \underline{D} \cdot \underline{B} + e_i \end{aligned}$$

where $\underline{D}$ is the vector of three industry dummy variables representing extractive (SIC codes 100–149), manufacturing (SIC codes 200–399), and transportation (SIC codes 400–479). Since our theoretical model predicts a U-shaped relationship between business risk and ODL, we specify a quadratic functional form for the CV-ODL relationship in our tests. Thus, the misspecification bias alluded to by Mikkelson (1984) may be substantially lower in our tests. A U-shaped ODL-CV relationship implies that $\beta_1 < 0$ and $\beta_2 > 0$.

For reasons mentioned earlier, other variables are included in the cross-sectional analysis. If bankruptcy costs contain a fixed component, larger firms will have higher optimal debt levels implying that the coefficient of the size proxy, $\beta_3$, is positive. With regard to non-debt tax shields, the D-M framework suggests that $\beta_4$ and $\beta_9$ should both be negative. Proxies for agency and asymmetric information effects are, unavoidably, difficult to construct and, hence, noisy. The costly contracting hypothesis of Smith and Warner (1979) predicts that high-growth firms, with large investments in intangible assets, would face higher covenant enforcement and monitoring costs and, hence, should have lower debt levels. Therefore, the costly contracting hypothesis would predict that INT, ADV, and CPX should be negatively correlated with debt levels, implying negative values for $\beta_5$, $\beta_6$, and $\beta_7$. A firm with a high proportion of inventory in its asset base should have less informational asymmetry regarding its assets in place, and, according to Myers and Majluf (1984), such a firm will suffer lower dilution costs from issuing equity and, thus, should have less debt in its capital structure, implying a negative $\beta_8$. Finally, the theoretical constructs of Dammon (1988) and Ross (1985) predict that the coefficient of BETA, $\beta_{11}$, will be negative.

### C. Results

The above equation is estimated for each of the different $CV$ proxies on the 1985 and 1984 samples. Initial OLS estimation resulted in highly nonspherical disturbance terms indicating heteroskedasticity of an unknown form. We correct for heteroskedasticity by using White (1980) asymptotically consistent standard error estimates. The regression results for the years 1985 and 1984 (that is four estimated regressions) are reported in Table II.

The signs of $CV$ and $CV^2$ are negative and positive, respectively, in each of the four regressions. While the coefficient of $CV^2$ is always significantly positive, that of $CV$ is significant only in 1985. Thus, the results indicate the existence of considerable nonmonotonicity in the ODL-BR relation and offer qualified support to this relation being U-shaped. Given the survivorship bias induced by the use of COMPUSTAT tapes, one would expect that the firms which were deleted from the tapes had lower mean cash flows and higher variance of earnings and, therefore, higher $CV$. Omitting these firms, under our hypothesis, would exclude primarily observations lying on the upward sloping portion of the ODL-BR curve. Thus, the omission of these firms should diminish the observed coefficient of $CV^2$.

The regression results also provide weak evidence in support of a size effect although it is opposite to the conventional belief that larger firms have a disproportionately larger debt capacity. The Dammon-Ross hypothesis of a negative relation between asset beta and leverage is supported with significantly negative coefficients in two of the four regressions. While the results for NDTS are mixed, they are generally consistent with the earlier empirical findings of a positive relation with leverage. The estimated coefficients on the other variables are not consistent across the regressions and, hence, interpretation is difficult. The coefficient for intangible assets (INT) is generally zero, except in two of the regressions, but the signs are opposite. The same is true in case of the coefficient for advertising (ADV). The coefficient of capital expenditure (CPX) is significantly negative in two regressions and zero otherwise and, thus, is weakly supportive of the costly contracting hypothesis. The coefficient of inventory (INV) is significantly positive in two of the four regressions and zero otherwise which is inconsistent with the asymmetric information theory. The reason for the mixed results in the case of agency and asymmetric information-related variables could be due to the inability of the proxies to capture these subtle effects. Therefore, our empirical results should be interpreted with caution. The coefficient for tax carryforwards (CARRY) is significant in only one of the regressions. The coefficient of the cross-product term (CAR $\times$ CV) is always negative and significant in all but one of the regressions. This indicates that the marginal effect of carryforwards diminishes as the business risk of the firm increases. The industry dummies are not significant in any of the four regressions.

## IV. Concluding Remarks

In this paper, we derive the functional relationship between business risk and optimal debt level in the DeAngelo and Masulis (1980) framework. We show that the relationship is roughly U-shaped, decreasing for low levels of business risk, and increasing for high levels. We utilize this functional form to test the relationship between business risk and optimal debt levels on a cross-section of firms in 1985 and 1984. The findings from these tests

**Table II**

**Regression Results for the Cross-Sectional Determinants of Capital Structure**

The 1985 sample includes 233 firms with complete data on the Annual COMPUSTAT tapes for the 19-year period 1967–1985 and the 1984 sample has 243 firms for the period 1966–1984. The general model is

$$Y = \beta_0 + \beta_1\mathrm{CV} + \beta_2\mathrm{CV}^2 + \beta_3 \ln\mathrm{SA} + \beta_4\mathrm{NDTS} + \beta_5\mathrm{INT} + \beta_6\mathrm{ADV} + \beta_7\mathrm{CPX} + \beta_8\mathrm{INV} + \beta_9\mathrm{CARRY} + \beta_{10}\mathrm{CAR} \times \mathrm{CV} + \beta_{11}\mathrm{BETA} + \underline{D} \cdot \underline{B} + e_i$$

All variables except the coefficient of variation (CV) are current year values, i.e., 1985 or 1984. The mean ($\mu$) used to calculate CV is the current year cash flow. The measure of leverage (Y) = (interest expense/$\mu$) − 1. Business risk (CV) measured by the coefficient of variation of cash flows; firm size measured by natural log of sales (lnSA); and non-debt tax shields (NDTS) = (0.46* depreciation expense + investment tax credits + R&D tax credits)/($\mu$). Intangible assets (INT), capital expenditures (CPX), and ending inventories (INV) are scaled by total assets, advertising expense (ADV), and tax carryforward (CARRY) are scaled by $\mu$. BETA is the CAPM asset beta. $\underline{D}$ is a vector of three industry dummies for extractive, manufacturing, and transportation sectors. The standard deviation is estimated two different ways, ordinary least squares (OLS) and first-order autoregressive process (AR). In the AR method, where the process did not converge, values from OLS are substituted to equate sample sizes. ($t$-values, using the heteroskedasticity-consistent standard errors [White (1980)] are in parentheses)

| | 1985 | | 1984 | |
|---|---|---|---|---|
| Variables | OLS | AR | OLS | AR |
| Constant | −0.445 | −0.076 | −0.955 | −0.944 |
| | (−4.45)** | (−0.27) | (−7.06)** | (−7.00)** |
| CV | −0.749 | −1.440 | −0.376 | −0.143 |
| | (−2.65)** | (−2.28)** | (−1.23) | (−0.54) |
| $\mathrm{CV}^2$ | 0.678 | 1.007 | 0.092 | 0.080 |
| | (15.55)** | (3.14)** | (5.46)** | (3.30)** |
| lnSA | −0.004 | −0.017 | −0.026 | −0.019 |
| | (−0.42) | (−1.10) | (−1.73)* | (−1.58) |
| NDTS | −0.240 | −0.239 | 1.568 | 1.240 |
| | (−1.93)* | (−0.35) | (2.15)** | (2.33)** |

**Table II**—*Continued*

| | 1985 | | 1984 | |
|---|---|---|---|---|
| Variables | OLS | AR | OLS | AR |
| INT | −0.474 | −0.645 | 0.621 | 0.336 |
| | (−1.78)* | (−1.35) | (1.49) | (1.17) |
| ADV | −0.654 | −0.699 | 0.170 | 0.294 |
| | (−3.03)** | (−1.55) | (0.51) | (0.76) |
| CPX | −0.368 | −0.930 | 0.344 | 0.341 |
| | (−2.06)** | (−2.21)** | (1.26) | (1.40) |
| INV | −0.201 | −0.359 | 0.573 | 0.489 |
| | (−1.34) | (−1.26) | (2.11)** | (2.22)** |
| CARRY | 0.122 | 0.077 | 0.106 | −0.051 |
| | (2.91)** | (1.22) | (1.30) | (−0.49) |
| CAR × CV | −0.810 | −0.227 | −0.000 | −0.226 |
| | (−8.36)** | (−3.11)** | (−0.00) | (−4.43)** |
| BETA | −0.137 | −0.054 | −0.210 | 0.060 |
| | (−3.43)** | (−1.46) | (−4.09)** | (0.94) |
| No. sig. dummies | 0 | 0 | 0 | 0 |
| Adj. $R^2$ | 0.950 | 0.815 | 0.802 | 0.821 |

**, significant at the 0.05 level; *, significant at the 0.10 level.

generally support the predicted U-shape for the relation between business risk and optimal debt level. However, it must be noted that while we have derived the theoretically correct functional form for the relation between the optimal debt level and one of its important determinants, business risk, we have certainly not eliminated all potential sources for misspecification. Misspecification biases could arise from any or all of the following reasons: (i) errors in variables caused by omission of relevant indicators or the noisy measurement of the variables included, (ii) nonlinearities in the functional relations of optimal debt level with other determinants besides business risk, and (iii) quadratic approximation of the U-shape of the relation with business risk. These problems are important and deserving of attention by researchers. Even if these econometric issues are resolved, the broader issue of optimal capital structure and its determinants remains unresolved. This paper makes a modest contribution to this area by deriving and operationalizing the functional relation and optimal debt level and business risk.

## Appendix

Lemma 1: *If $R^*$ satisfies equation* (7) *and $L < T(R^*, \sigma) < U$, it satisfies the second-order condition for a minimum.*

*Proof:* The first-order condition in equation (7) implies that

$$\frac{\tau_c}{\tau_p} = \frac{1 - F[D^*]}{1 - F[T^*]}. \tag{A1}$$

Also note that $\partial T^*/\partial R = \partial D^*/\partial R = 1/\sigma$, and that $F' = f$. Hence, if we differentiate again with respect to $R$, we obtain

$$\left.\frac{\partial^2 G}{\partial^2 R}\right|_{R=R^*} = \frac{\tau_c f[T^*] - \tau_p f[D^*]}{\sigma}. \tag{A2}$$

Hence,

$$\operatorname{sign}\left(\left.\frac{\partial^2 G}{\partial^2 R}\right|_{R=R^*}\right) = \operatorname{sign}\left(1 - \frac{\tau_p f[D^*]}{\tau_c f[T^*]}\right). \tag{A3}$$

If we substitute equation (A1) into equation (A3), and using the definition of the hazard rate, $h(\cdot) = f/(1 - F)$, we obtain

$$\operatorname{sign}\left(\left.\frac{\partial^2 G}{\partial^2 R}\right|_{R=R^*}\right) = \operatorname{sign}\left(1 - \frac{h[D^*]}{h[T^*]}\right). \tag{A4}$$

The fact that $T^* > D^*$ and the assumption that $Z$ has an increasing hazard rate, implies that the term in brackets on the right hand side of equation (A4) is positive and, hence, the second order condition for an interior maximum is satisfied.

Lemma 2: *Let $Q(x, y) = (1 - F[x - y])/1 - F[x]$. Then if the hazard rate $h$ is strictly increasing on $[L, U]$, $Q$ is always nondecreasing in $x$ and is strictly increasing in $x$ for $U > x > L$; and nondecreasing in $y$ and strictly increasing in $y$ when $U > x - y > L$.*

*Proof:* Let $q(x, y) = \log Q(x, y) = \log(1 - F[x - y]) - \log(1 - F[x])$. Note that the derivative with respect to $s$ of $(\log(1 - F[s])$ is the hazard rate $h[s]$. By the Fundamental Theorem of Calculus,

$$\log(1 - F(x)) - \log(1 - F(x - y)) = \int_{x-y}^{x} \log(1 - F(s))'\,ds$$
$$= -\int_{x-y}^{x} h(s)\,ds.$$

Therefore,

$$q(x, y) = \int_{x-y}^{x} h(s)\,ds,\ y > 0.$$

Note that, by the assumption of increasing hazard rate, if $U > x > L$, then $\partial q/\partial x = h(x) - h(x - y) > 0$. This implies that $q$ is always nondecreasing in $x$ and is strictly increasing in $x$ for $U > x > L$. Because $Q$ is a strictly monotone function of $q$, $Q$ satisfies the same conditions. Next note that, because $h$ is always non-negative and is positive on the support of $Z$, $\partial q/\partial y = h(x - y) > 0$ if $U > x - y > L$. This implies that $q$ is always nondecreasing in $y$ and is increasing in $y$ when $U > x - y > L$.

Lemma 3: *There exists a unique $\sigma, \sigma_0$, at which $T^*(\sigma_0) = 0$.*

*Proof:* First, we show that there exists a level of $\sigma, \sigma_0$, at which the first-order condition is satisfied with $T^*(\sigma_0) = 0$. To prove this, let

$$H(\sigma) = \frac{\partial G}{\partial R}\bigg|_{R=\mu-\Pi} = \tau_p\{1 - F[-\Pi/\sigma]\} - \tau_c\{1 - F[0]\}.$$

Therefore,

$$\sup_{\sigma>0} H(\sigma) = \sup_{\sigma>0}\left[\tau_p\{1 - F[-\Pi/\sigma]\} - \tau_c\{1 - F[0]\}\right.$$
$$\left. = \tau_p - \tau_c\{1 - F[0]\}\right].$$

By assumption (a) of Theorem 1, there exists $\sigma^1$ such that $R^*(\sigma^1) < \mu - \Pi$ which satisfies the first-order condition (7). That is,

$$\frac{\tau_c}{\tau_p} = \frac{1 - F[D^*(\sigma^1)]}{1 - F[T^*(\sigma^1)]}. \tag{A5}$$

Because $D = T - \Pi/\sigma$, equation (A5) can be expressed as

$$Q(T^*(\sigma^1), \Pi/\sigma^1) = \frac{\tau_c}{\tau_p}$$

Because $T^*(\sigma^1) < 0$ and $Q(\cdot)$ is increasing in its first argument by Lemma 2, this implies that

$$Q(T^*(\sigma^1), \Pi/\sigma^1) < Q(0, \Pi/\sigma^1) = \frac{1 - F[-\Pi/\sigma^1]}{1 - F[0]} \leq \frac{1}{1 - F[0]}.$$

Hence $\tau_c/\tau_p < 1/(1 - F[0])$, which implies that $\tau_p - \tau_c\{1 - F[0]\} > 0$. Hence, $\sup_{\sigma > 0} H(\sigma) > 0$. On the other hand, $\lim_{\sigma \to \infty} H(\sigma) = \tau_p(1 - F[0]) - \tau_c(1 - F[0]) < 0$. Because $H$ is continuous, by the Intermediate Value Theorem, there exists a $\sigma_0$ such that $H(\sigma_0) = 0$. Therefore, $T^*(\sigma_0) = 0$. Because $H(\sigma)$ is strictly decreasing at $\sigma_0$, and is nonincreasing everywhere, $\sigma_0$ is the unique solution to $H(\sigma) = 0$.

Lemma 4: (*a*) $L < T^*(R^*(\sigma), \sigma) < U$; (*b*) *if a minimizing value of* $G(R, \sigma)$ *exists for a given* $\sigma$, *it is unique*; *and* (*c*) *for all* $\sigma > 0$, *a minimizer* $R^*$ *exists and* $R^*(\sigma)$ *is a continuous function of* $\sigma$.

*Proof:* (a) If $T(R, \sigma) \leq L$. This implies that $D(R, \sigma) < L$. In this case, the LHS of equation (7) reduces to $\tau_p - \tau_c$, which is negative, implying that $R$ cannot be a minimizer. This establishes that $T(R^*(\sigma), \sigma) \geq L$. To show that $T(R^*(\sigma), \sigma) < U$, we employ the following argument by contradiction. Suppose $R'$ is the minimizing value of $G(R, \sigma)$, and that $T(R', \sigma) \geq U$. Define $R^T(\sigma)$ as the solution to $T(R, \sigma) = U$, and define $R^D(\sigma)$ as the solution to $D(R, \sigma) = U$. It follows that $R^D(\sigma) > R^T(\sigma)$. If $R' \geq R^D(\sigma)$, the first-order condition obviously holds because $\{1 - F[D(R', \sigma)]\}$ and $\{1 - F[T(R', \sigma)]\}$ are both equal to zero. However, for all $R \in (R^T(\sigma), R^D(\sigma))$, the first-order condition in (7) is positive. Hence a reduction in debt from $R'$ to $R^T(\sigma)$ will lower the tax liability. Thus $R'$ cannot be a minimizer of $G(R, \sigma)$. Similarly if $R' \in [R^T(\sigma), R^D(\sigma)]$, then equation (7) is positive, and $R'$ cannot be a minimizer of $G(R, \sigma)$.

(b) When $R^*$ such that $T(R^*, \sigma) \in (L, U)$ solves the first-order condition, uniqueness follows from the fact that the first-order condition in equation (7) can be expressed as

$$\left.\frac{\partial G}{\partial R}\right|_{R=R^*} = \tau_p\{1 - F[T(R^*, \sigma)]\}\left\{Q(T(R^*, \sigma), \Pi/\sigma) - \frac{\tau_c}{\tau_p}\right\} = 0,$$

where $Q(\cdot, \cdot)$ is defined in Lemma 2. If $R^*$ is a minimizer, then as shown in part (a), $T(R^*, \sigma) \in (L, U)$. Hence $\{1 - F[T(R^*, \sigma)]\} \in (0, 1)$. Since, by Lemma 2, $Q$ is strictly increasing in its first argument in this range of $T(\cdot)$, and $T(\cdot)$ is strictly increasing in $R$, if $R > R^*$, $dG/dR > 0$ and if $R < R^*$, $dG/dR < 0$. A similar argument works when $R^* = 0$.

(c) Let $\overline{R}(\sigma) = \mu - \Pi + \sigma U$, $\underline{R}(\sigma) = \max\{0, \mu - \Pi + \sigma L\}$, and $\Gamma(\sigma) = [\underline{R}(\sigma), \overline{R}(\sigma)]$. Because $G(R, \sigma)$ is a continuous function on the positive quadrant, and $\Gamma(\sigma)$ is a continuous compact-valued correspondence, by the Theorem of the Maximum (see, for example, Stokey, Lucas, and Prescott (1989, Theorem 3.6, page 62) argmax$\{R \in \Gamma(\sigma)$: $G(R, \sigma)]$ is nonempty, compact-valued, and upper hemicontinuous. Because for each $\sigma$, the argmax$\{R \in \Gamma(\sigma)$: $G(R, \sigma)\}$ is a singleton as shown in (b) above, in fact, argmax$\{R \in \Gamma(\sigma)$: $G(R, \sigma)\}$ is a continuous function of $\sigma$. Part (a) implies that the $R$ minimizing $G(R, \sigma)$ always lies in $[\underline{R}(\sigma), \underline{R}(\sigma)]$, therefore argmax$\{R \in \Gamma(\sigma)$: $G(R, \sigma)\}$ = argmax$\{R \geq 0$: $G(R, \sigma)\}$. Therefore, argmax$\{R \geq 0$: $G(R, \sigma)\}$ is a continuous function of $\sigma$ which we denote by $R^*(\sigma)$.

Lemma 5: *If* $\sigma < \sigma_0$, *then* $T^*(\sigma) < 0$.

*Proof:* Note that for the first-order condition in (7) to be satisfied at $\sigma_0, \tau_p\{1 - F[D^*(\sigma_0)]\} - \tau_c\{1 - F[0]\} = 0$. We know, from the proof of Lemma 3, that $\tau_p - \tau_c\{1 - F[0]\} > 0$. Therefore, for the above equation to hold, $\{1 - F[D^*(\sigma_0)]\}$ must be less than one. Hence $D^*(\sigma_0)$ must be greater than $L$ implying that $h[D^*(\sigma_0)] > 0$. This implies that, by equation (13), $dR^*/d\sigma$, evaluated at $\sigma_0$, is greater than zero because $T^*(\sigma_0) = 0$ and $D^*(\sigma_0) < 0$. Thus, there exists $\sigma > (<)\sigma_0$ such that $T^*(\sigma) > (<)0$. Because $R^*$ is a continuous function of $\sigma$, and $T^*$ is a continuous function of $R$, and the fact that $\sigma_0$ is the only level of $\sigma$ at which $T^*(\sigma) = 0$, then for all $\sigma > (<)\sigma_0$. $T^*(\sigma) > (<)0$.

Lemma 6: *There exists* $\varepsilon_0 > 0$ *such that if* $\sigma < \varepsilon_0$, *then* $R^*(\sigma) > 0$.

*Proof:* Consider $\partial G/\partial R$ evaluated at $R = 0$.

$$\left.\frac{\partial G}{\partial R}\right|_{R=0} = \tau_p\{1 - F[-\mu/\sigma]\} - \tau_c\{1 - F[(\Pi - \mu)/\sigma]\}. \tag{A6}$$

Because $F$ is continuous and $\Pi < \mu$,

$$\begin{aligned}\lim_{\sigma\to\infty}\left.\frac{\partial G}{\partial R}\right|_{R=0} &= \lim_{\sigma\to 0}[\tau_p\{1 - F[-\mu/\sigma]\} - \tau_c\{1 - F[(\Pi - \mu)/\sigma]\}] \\ &= \tau_p\{1 - F[-\infty]\} - \tau_c\{1 - F[-\infty]\} = \tau_p - \tau_c < 0.\end{aligned}$$

Hence, there exists an $\varepsilon > 0$ such that, if $\sigma < \varepsilon$, then $\left.\frac{\partial G}{\partial R}\right|_{R=0} < 0$. This implies that, at $R = 0$, the value of $G$ is decreasing in $R$. The firm's objective is to minimize $G$, and hence $R^*(\sigma) > 0$.

Lemma 7: *If* $\sigma$ *is sufficiently small,* $h[D^*(\sigma)] = 0$.

*Proof:* Recalling that $L < 0$, let $\underline{\sigma} = \min\{\sigma_0, (\Pi/-\mathrm{L})\}$ and $\sigma < \underline{\sigma}$. Then

$$D^*(\sigma) = T^*(\sigma) - \frac{\Pi}{\sigma} < T^*(\sigma) - \frac{\Pi}{\underline{\sigma}} < L.$$

The equality follows by the definition of $D^*$ and $T^*$, the first inequality follows from the fact that $\sigma < \underline{\sigma}$, and the second inequality follows by Lemma 5. Therefore, because $D^*(\sigma) < L$, $h[D^*(\sigma)] = 0$.

Lemma 8: *For* $\sigma > 0$, $T^*(\sigma)$ *is nondecreasing in* $\sigma$.

*Proof:* Consider the two cases: (i) $R^*(\sigma) > 0$ and (ii) $R^*(\sigma) = 0$. If (i) holds, the first-order condition is satisfied. The first-order condition, as shown in the proof of Lemma 5, can be expressed as equation (A6). By Lemma 2, $Q(\cdot, \cdot)$ is nondecreasing in both its arguments. An increase in $\sigma$ lowers the second argument of $Q$ in equation (A6). Hence, to maintain equality, the first argument, $T^*(\sigma)$, must not decrease. If (ii) holds, because $R^*(\sigma)$ is continuous (Lemma 4) and $\mu - \Pi > 0$, for all $\varepsilon$ sufficiently small, $R^*(\sigma + \varepsilon) + \Pi - \mu < 0$. Therefore,

$$T^*(\sigma + \varepsilon) = \frac{R^*(\sigma + \varepsilon) + \Pi - \mu}{\sigma + \varepsilon} > \frac{R^*(\sigma + \varepsilon) + \Pi - \mu}{\sigma}$$
$$\geq \frac{R^*(\sigma) + \Pi - \mu}{\sigma} = T^*(\sigma).$$

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