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Diagnosing Asset Pricing Models Using the Distribution of Asset Returns

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ABSTRACT

This paper develops a set of diagnostic tests which can shed light on why a particular model is failing and indicate what steps might be taken to make the model consistent with asset returns. Theoretical bounds on the moments of a stochastic discount factor are derived as a function of the moments of observed asset returns. Particular attention is paid to restrictions on moments other than the variance. These bounds can also be used to measure the information about the distribution of the discount factor contained in the moments of various asset returns. As an application of this methodology, bounds on the discount factor are estimated using size-based portfolios, and the results are used to analyze the small firm effect. Empirical results indicate, for the period 1926–1975, that moments of the returns of small firms contain information about the discount factor that is not contained in the moments of the returns of large firms and/or a proxy of the aggregate wealth portfolio. However, this difference disappears when more recent data is included.

ONE OF THE FUNDAMENTAL problems in finance is determining how future revenue streams are discounted to yield the value of an asset. In the absence of arbitrage opportunities, there is a single stochastic discount factor which can be used to value future revenue. This can be seen in the relationship between the price of a security, q_t, and the payoff of that security at some future period, $x_{t+\tau}$, which can be represented by $E[m_{t+\tau}x_{t+\tau}] = E[q_t]$ (or $E[m_{t+\tau}x_{t+\tau}] = 1$ if the x is viewed as a gross return) where the variable m represents some positive, stochastic discount factor. This paper extends a methodology proposed by Hansen and Jagannathan (1991a) to extract information about the stochastic discount factor from various moments of observed asset returns and to create bounds on corresponding moments of m. These bounds can be used as diagnostic tests to compare different models in the same framework. Specifically, I apply this methodology to show that the returns of small firms have different implications about the stochastic discount factor than do the returns of large firms.

Since different asset pricing models specify different measures of m, we can index models by the stochastic discount factor they imply (see Hansen and Richard (1987)). In the finance literature, m is often represented by the

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return on some portfolio. For example, the Capital Asset Pricing Model (CAPM) of Sharpe (1964) and Lintner (1965) implies that m is the linear combination of the returns on the aggregate wealth portfolio and some riskless return. For the Intertemporal Capital Asset Pricing Model (ICAPM), Brown and Gibbons (1985) and Rubinstein (1976) show that, under certain conditions, m can be represented by a nonlinear function of the return on aggregate wealth.

In consumption-based models, m can be interpreted as the intertemporal marginal rate of substitution (IMRS) of consumers. Given a representative consumer model with time and state-separable preferences, $m_{t+\tau} = U'(c_{t+\tau})/U'(c_t)$.[1] In this context, Breeden (1979), Grossman and Shiller (1981), and Hansen and Singleton (1982) define preferences such that m is a function of the growth rate of aggregate consumption. Epstein and Zin (1987) relax the state-separable nature of the consumer's utility function and express m as a function of both aggregate consumption and the return on the aggregate wealth portfolio.

By specifying a particular stochastic discount factor and making any necessary assumptions about the asset pricing environment, we can test the implied asset pricing model.[2] When a model is rejected, however, such tests provide little indication of the cause. If the model fails because of an incorrect specification of m, we would like to identify what features of the measure might be altered to make it consistent with asset returns.

Hansen and Jagannathan (1991a) examine the issue of what restrictions can be placed on the variance of the stochastic discount factor using information available from observed asset returns. Using the relationship $E[m_{t+\tau} x_{t+\tau}] = E[q_t]$, they derive a lower bound on the variance of any stochastic discount factor, a bound which can be estimated as a function of observed asset returns and used to form a mean-variance frontier for m. This technique creates a diagnostic test for a broad class of models since it allows the econometrician to determine whether the mean and variance of the discount factor implied by a given model lies inside the mean-variance frontier for m.

A mean-variance frontier for stochastic discount factors is only one dimension along which asset pricing models can be analyzed and compared. Other dimensions may be more interesting or may generate different implications when examining certain models or using certain types of asset returns. Also, such a mean-variance frontier only uses the information contained in the means, variances, and covariances of the assets under consideration. One purpose of this paper is to extend the methodology of Hansen and Jagannathan to produce bounds on moments of the stochastic discount factor other than variance. In particular, bounds on the δth norm of m ($E[m^\delta]^{1/\delta}$) are formed using the ρth moments of asset returns where $1/\delta + 1/\rho = 1$. From this we can construct a mean-δth norm frontier for the stochastic discount

[1] See Lucas (1978).

[2] For examples see Ferson (1983) or Hansen and Singleton (1983).

factor which will allow us to use the information contained in the moments of observed asset returns that is ignored in mean-variance analysis. Such diagnostics will have varying degrees of sensitivity to outlying return observations, since different values of ρ (different moments of asset returns) will change the weight given to outliers.

The other purpose of this paper is to use this methodology to examine the question of the small firm effect—an empirical irregularity well documented in the finance literature which states that small firms are not priced correctly by the CAPM. One advantage of using an extended version of the methodology of Hansen and Jagannathan is that it allows us to examine the small firm effect in a general, nonparametric fashion that avoids imposing a lot of structure. Also, the fact that the returns of small firms tend to be more skewed and leptokurtic than large firms can be exploited using this generalized technology. By constructing mean-δth norm frontiers using differing moments of size-based portfolios, one can test whether various moments of the returns of small firms impose restrictions on the distribution of the stochastic discount factor above and beyond those implied by the corresponding moments of the returns of large firms or proxies of the aggregate wealth portfolio.

When $\delta = 2$, this test can be shown, under certain circumstances, to be equivalent to tests of the efficiency of a given portfolio found in the CAPM literature (see Hansen and Jagannathan (1991a)). Estimation of the test statistics takes advantage of the General Method of Moments (GMM) technique proposed by Hansen (1982); hence, consistent estimates can be obtained even when the assumption of independent and identically distributed (IID) returns is relaxed to allow for serial correlation or conditional heteroskedasticity. In addition, this test does not rely upon the assumption of normally distributed asset returns.

Mean-δth norm frontiers for the stochastic discount factor ($\delta = 1.5$, 2, and 3) are generated and the empirical results imply that small firms contain information about the distribution of the stochastic discount factor that is not found in large firms or various proxies of the market portfolio for the period 1926:1 to 1975:12. When $\delta = 2$, this finding is similar to results that usual measures of the market portfolio are not mean-variance efficient and tend to misprice the stocks of small firms. When I allow for serial correlation and conditional heteroskedasticity in the asset returns, the returns of small firms still appear to impose greater restrictions on norms of the stochastic discount factor than do the returns of large firms or market portfolio proxies, although the results are less significant than when returns were assumed to be IID. However, for periods that include more recent data (up through 1987:12), I fail to reject the hypothesis that returns of small firms imply no additional restrictions beyond those implied by the returns of large firms and/or a market proxy. This finding holds whether or not returns were assumed to be IID.

I also find that the ability to reject the hypothesis that returns of small firms contain no information about the δth norm of m not already contained in the returns of a market proxy and/or large firms is an increasing function of ρ. This implies that the higher moments of small firms place, for certain time periods, more dramatic restrictions on the stochastic discount factor than do lower moments by creating more restrictive mean-norm frontiers.

Finally, higher moments of the returns of small firms are shown to imply greater restrictions on the stochastic discount factor implied by a representative consumer model with a time and state-separable utility function than do lower moments of those same returns.

My paper is organized as follows. In Section I, bounds on the δth norm of m are created exploiting the knowledge that m must be positive. Section II presents the estimation procedure used to calculate these bounds. These calculations reveal a method of testing whether or not a subset of a given universe of assets has information about the stochastic discount factor which is not contained in the remaining assets. Section III discusses the small firm effect, how the methodology is applied to analyze the problem, and the data sets used in the actual estimation. Empirical results are described in Section IV, and concluding remarks are contained in Section V.

I. Bounds on the Stochastic Discount Factor

Hansen and Jagannathan (1991a) develop bounds on the variance of m. In this section, their methodology is extended in order to derive bounds on moments or norms of the stochastic discount factor other than the variance. To facilitate this discussion, general features of the asset pricing environment as well as the notation used throughout the paper are presented.[3]

Assets are presumed to be purchased at time t and the payoffs from the assets received at time $t + \tau$, for some $\tau > 0$. The econometrician is presumed to observe a subset of the payoffs of all traded assets; denote this subset by the $k \times 1$ vector x. The vector x is not assumed to contain a perfectly riskless asset. Let P be the space of possible observed payoffs generated by the assets contained in x. Such a space can be represented by:

$$P = \{p : p = c \cdot x\} \quad \text{where} \quad c \in \mathbb{R}^k. \tag{1}$$

The only distributional assumptions made about asset prices and payoffs is that they are presumed to be generated by a stationary, ergodic, stochastic process. This implies that sample moments will, in the limit, converge to their population counterparts. This generality allows us to examine changes in the results of the diagnostic tests when the assumption of IID prices and returns (the working assumption in the CAPM literature) is relaxed.

Define the price of p as $\pi(p) = c \cdot q$ where q is the vector of prices associated with the vector of payoffs x. As mentioned earlier, absence of arbitrage implies:

$$E[mp] = E[\pi(p)] \quad \text{for all} \quad p \in P \tag{2}$$

[3] The asset pricing environment discussed here is similar to that put forth by Hansen and Richard (1987).

where m is a positive random variable (see Hansen and Richard (1987)). Hence, we can view m as the stochastic discount factor or variable which prices assets contained in x. Note that equation (2) implies that if there is a riskless asset with a unit payoff, it has a price equal to Em.

The bounds derived in this paper take advantage of the fact that m must be positive. For example, if m is defined as the IMRS, free disposal ensures that marginal utility is greater than or equal to zero, and hence the measure of m is positive. Work by Rubinstein (1976), Harrison and Kreps (1979), and Hansen and Richard (1987) implies that, for any asset environment in which there are no arbitrage opportunities, the stochastic discount factor must always be greater than or equal to zero.

Explicit restrictions on the moments of the stochastic discount factor can be created by applying Hölder's inequality and exploiting the fact that the stochastic discount factor was assumed to be a positive random variable. This provides a direct link between the δth norm of the stochastic discount factor ($E[m^{\delta}]^{1/\delta}$) and the ρth moments of the asset returns, where $1/\delta + 1/\rho = 1$. These restrictions can be shown to be the greatest lower bounds on the noncentral moments of m (see Appendix A) and can be thought of as a set of statistics which can be used to assess various specifications of different asset pricing models. The bounds can then be estimated as a function of a given mean level to create a mean-δth norm frontier for the stochastic discount factor.

If moments of asset returns other than the variance are bounded, restrictions on m can be generated as a function of the returns.[4] These restrictions may provide different implications for different asset pricing models. For example, since a measure of m which lies in the mean-variance frontier may not lie in the mean-δth norm frontier, these regions could help differentiate among asset pricing models which satisfy the mean-variance restrictions. Furthermore, a new asset which does not cause a upward shift (i.e., generate a more restrictive region) in the mean-variance frontier for m may cause a significant upward shift in the mean-δth norm frontier for $\delta \neq 2$.

[4] There is a body of empirical evidence that the second moment of certain types of asset returns may not be finite. Fama (1965) tested for the normality of daily asset returns using the Dow Jones Industrial Stocks and found evidence that the sample was generated by a distribution which is leptokurtic or "fat-tailed" relative to the normal distribution. He concluded that stock returns conform to a stable Paretian distribution with an unbounded variance, an idea hypothesized earlier by Mandelbrot (1963). Blume (1970), Roll (1970), and Fama and Miller (1972) all use the stable Paretian distribution to describe the stochastic properties of asset returns. More recently, Wang and Lee (1988) examine the theory of arbitrage pricing when the second moments of asset returns are infinite.

The mean-variance frontier of Hansen and Jagannathan (1991a) depends upon the assumption that asset payoffs have finite variances. If a moment between the mean and variance is finite for all asset returns under consideration, we can still create a meaningful diagnostic test, even if the variance of the asset returns is infinite. If such a moment exists, then bounds on a higher (than the second) norm of the stochastic discount factor can be formed as a function of lower (than the second) moments of returns. This also will allow us to test how robust a specification of the m is to the assumption of bounded variances.

A. Restrictions on Noncentral Moments When a Riskless Asset Exists

In order to generate restrictions on the δth norm of m, the following restrictions are placed on the distributions of the random variables of interest:

Assumption (1): $E[m^{\delta}] < \infty$ and $E[p^{\rho}] < \infty$, where $1/\delta + 1/\rho = 1$ and the region of interest is the mean-δth norm frontier.

Define the truncated portfolio payoff p^{+} as:

$$p^{+} = \begin{cases} p \text{ if } p > 0 \\ 0 \text{ if } p \leq 0 \end{cases}. \tag{3}$$

We can use equation (2), definition (3), and Hölder's inequality to form the following relationship:

$$E\pi(p) = E[mp] = E[mp^{+}] \leq E[m^{\delta}]^{1/\delta} \cdot E[p^{+\rho}]^{1/\rho}. \tag{4}$$

When assumption (1) holds and $E[p^{+\rho}]^{1/\rho} > 0$, relationship (4) can be rewritten as:

$$E[m^{\delta}]^{1/\delta} \geq \frac{E\pi(p)}{E[p^{+\rho}]^{1/\rho}}. \tag{5}$$

Define $\lambda(\delta)$ as:

$$\lambda(\delta) = \sup_{p \in P} \frac{E\pi(p)}{E[p^{+\rho}]^{1/\rho}} \tag{6}$$

where $E[p^{+\rho}]^{1/\rho} > 0$ and $1/\delta + 1/\rho = 1$. Since equation (5) must hold for all δ such that $E[p^{+\rho}]^{1/\rho} > 0$, we can create the following bound on $E[m^{\delta}]^{1/\delta}$:

$$E[m^{\delta}]^{1/\delta} \geq \lambda(\delta). \tag{7}$$

This equation gives a bound on the δth moment of the stochastic discount factor as a function of the price of some portfolio and the ρth moment of the payoff of some option on that portfolio. Notice that $\lambda(\delta)$ describes a point and not a frontier, since if a riskless asset exists, Em is identified and equal to the price of the riskless asset. This bound does not require knowledge of the stochastic discount factor and can be formed with moments of random variables which can be observed.

Creating a bound on a norm of m using the knowledge that m must be positive can be shown to be equivalent to creating an option whose payoff prices assets and hence satisfies definition (2). This is consistent with the idea that options help complete the market (see Ross (1976)) and therefore should contain information about the stochastic discount factor. It is shown later in this paper that creating a bound on the δth moment of m can be accomplished by creating an option on a portfolio of x such that the option payoff z, when raised to the $\rho - 1$st power, correctly prices the asset payoffs contained in x:

$$E[z^{\rho-1}x] = E[q] \tag{8}$$

where $1/\delta + 1/\rho = 1$.

Equation (8) allows us to test whether a certain subset of the universe of assets contains information about m which is not contained in the remaining assets under consideration. This is equivalent to generating a mean-norm frontier with a subset of the universe of assets (say with some vector of risky asset payoffs $y \subset x$) and testing whether this frontier shifts up (becomes more restrictive) when the entire set (say the vector x) is used in generating the same mean-norm frontier. Such a test can be conducted by examining whether the underlying portfolio for the option payoff z can be formed with a strict subset of the set of risky assets x which, when raised to the $\rho - 1$st power will still correctly price all the asset payoffs in x, or alternatively:

$$E[\tilde{z}^{\rho-1}x] = E[q] \tag{8a}$$

where $\tilde{z}$ is an option payoff on a portfolio consisting only of assets associated with y. The last part of this section and Section II formally derive equations (8) and (8a), explain how to test whether (8a) holds for a given $y \subset x$ after accounting for sampling error, and examine some other empirical issues. Those readers who wish to skip the derivations should proceed to Section III.

B. Restrictions on Noncentral Moments in the Absence of a Riskless Asset

If a riskless asset o cannot be identified by the econometrician, Em is also not identified. However, restrictions analogous to bound (7) can be generated for all hypothesized values of Em. Consider the extended payoff space H which represents linear combinations of o and elements of P:

$$H = \{h : h = wo + p \text{ for some } w \in \mathbb{R} \text{ and for some } p \in P\} \tag{9}$$

and define the truncated portfolio payoff h^+ as:

$$h^+ = \begin{cases} h \text{ if } h > 0 \\ 0 \text{ if } h \leq 0 \end{cases}. \tag{10}$$

Note that a new pricing function, call it $\pi_v(h)$, can be extended to the larger payoff space H. The price of h is formally defined as:

$$\pi_v(h) = E[m(wo + p)] = wv + \pi(p) \tag{11}$$

where $v = Em$. Hence the price of any asset payoff in H is simply a linear combination of the price of the unobserved risk-free asset and a portfolio made up of risky assets. Since $\pi_v(h) = \pi(h)$ for all $h \in P$ (i.e., when w, the weight on the risk-free asset, is equal to 0), the logic that generated equation (7) can be repeated to yield:

$$E[m^\delta]^{1/\delta} \geq \lambda_v(\delta) = \sup_{h \in H} \frac{E\pi_v(h)}{E[h^{+\rho}]^{1/\rho}} \tag{12}$$

where $E[h^{+\rho}]^{1/\rho} > 0$. Therefore, given a hypothesized price of the risk-free asset, a lower bound on the δth moment of m can be calculated and a mean-δth norm frontier can be deduced.

The truncated payoff h^+ can be interpreted as the payoff of a European call option on a portfolio p with strike price equal to $-w$ if $w \leq 0$. If $w > 0$, then h^+ is the payoff of a European put option on $-p$ with strike price w. Notice that the bound implied by equation (12) depends upon the payoff of the option h^+ and the price of the underlying security p since $\pi_v(h) = wv + \pi(p)$. Creation of the option h^+ allows us to form sharper bounds than if we restricted ourselves to using just the underlying securities. Even sharper restrictions on the variance of m could be created by replacing $\pi(h)$ with $\pi(h^+)$ since $\pi(h)$ represents a lower bound on the price of the option h^+. Therefore, it would be ideal to use the price of this option in creating the bound. In practice, this becomes difficult since we do not observe prices on options with arbitrary strike prices. However, for those strike prices which are observed, options might be used to create more restrictive bounds on m for various levels of Em.[5]

The bounds described by equation (12) not only have the possibility to generate puzzles about a particular stochastic discount factor (i.e., does the δth norm of a specified m satisfy the restrictions implied by observable asset returns?), they also say something about arbitrage opportunities. Notice that if there exists an observable portfolio with a nonpositive payoff and a positive price, then the bounds implied by (12) can be made infinite.

C. Bounds on the Stochastic Discount Factor and Options

In the empirical estimation section of this paper, a pure riskless asset was presumed not to be observed by the econometrician; therefore, the remainder of this paper will focus on bounds of norms of m implied by equation (12). Recalling equations (1) and (9), the space H (which contains payoffs of the portfolio made up of the vector of risky assets and an unobserved risk-free asset) can also be represented as:

$$H = \{h = x^{*\prime}\beta\} \tag{13}$$

where x^* is x (the $k \times 1$ vector of asset payoffs presumed to be observed) concatenated with the riskless payoff o. The portfolio weight vector β is c (the vector of weights on the risky assets) concatenated with w (the weight on the riskfree asset) where $c \in \mathbb{R}^k$ and $w \in \mathbb{R}$. This characterization of β implies that there are $k + 1$ parameters of interest to be estimated.

Since the calculation of the bound on the δth norm of m, $\lambda_v(\delta)$, involves maximizing the ratio denoted in equation (12), there exists one free normalization which can be imposed before maximization. Hence, we can restrict the

[5] In future work, I plan to undertake an empirical analysis of the efficiency gains from using a collection of options and their associated prices to generate bounds on moments of m as opposed to generating the frontier using the underlying securities.

numerator to equal one and then minimize the denominator subject to this constraint. This normalization allows the maximization problem represented by equation (12) to be rewritten as:

$$\underset{\beta \in \mathbb{R}^{k+1}}{\text{Min}} \; E\left(\left(x^{*\prime}\beta\right)^{+}\right)^{\rho} \text{ s.t. } E\left(q^{*\prime}\beta\right) = 1 \tag{14}$$

where q^* is the $k \times 1$ vector of prices associated with payoffs x^*, the $+$ operator is as defined in equation (10), and β represents the portfolio weights. The ρth norm of $(x^{*\prime}\beta)^+$ is replaced by the ρth moment of $(x^{*\prime}\beta)^+$ since the norm is just an increasing, monotonic transformation of the corresponding moment and therefore will not affect the parameter estimates. The last element of q^* will contain the price of the risk free payoff (i.e., the mean of m).

The first order conditions of equation (14) can be written as:

$$\rho \cdot E\left[x^*\left(\left(x^{*\prime}\beta^*\right)^{+}\right)^{\rho-1}\right] - \phi^* \cdot E\left[q^*\right] = 0 \tag{15}$$

where β^* represents the argmin of (14),[6] and ϕ^* is the Lagrange multiplier. Define $\tilde{\beta}$ to be equal to $\rho \cdot \beta^*/\phi^*$. This allows equation (15) to be written as:

$$E\left[x^* z^*\right] = E\left[q^*\right] \tag{16}$$

where $z^* = ((x^{*\prime}\tilde{\beta})^+)^{\rho-1}$. Of particular interest is the scalar term z^*. Notice that z^* in (16) solves equation (2); therefore, z^* can be interpreted as a candidate for a stochastic discount factor. In Appendix A, it is shown that z^* attains the bound described in (12), and hence z^* is a greatest lower bound for the stochastic discount factor. Equation (16) implies that finding a lower bound on $E[m^\delta]^{1/\delta}$ is equivalent to finding the payoff of a European option on a portfolio of x^* which, when raised to the $\rho - 1$ power, prices the assets x^*.

II. Estimation, Tests of Information, and Tests of Degeneracy

To actually generate a region which bounds the δth norm of m, equation (14) can be estimated for different hypothesized mean levels of the stochastic discount factor to create a mean-δth norm frontier. In actual estimation, the population moments in (14) are replaced by their sample counterparts for sample size T:

$$\underset{b \in \mathbb{R}^{k+1}}{\text{Min}} \; \frac{1}{T}\sum_{t=1}^{T}\left(\left(x_{t+\tau}^{*\prime} b\right)^{+}\right)^{\rho} \text{ s.t. } \frac{1}{T}\sum_{t=1}^{T}\left(q_t^{*\prime} b\right) = 1 \tag{17}$$

which introduces sampling error. The method used to calculate the standard errors of objects which solve (17) is presented in Hansen and Jagannathan (1991b) for $\rho = \delta = 2$; for $\rho > 2$, the analysis in Hansen (1982) applies.

[6] The proof of the existence of β^* for an arbitrary ρ can be found in Snow (1991) and will be made available upon request. The proof of the existence of β^* for $\rho = 2$ can be found in Hansen and Jagannathan (1991a).

A. Tests of Information in a Subset of Assets

One advantage of writing the maximization problem in the form of equation (16) is that it yields a straightforward test of whether adding securities to the universe of assets provides any additional information about the stochastic discount factor. In other words, after accounting for sampling error, does increasing the number of assets result in a more restrictive frontier for the δth norm of m?

If an asset contains information about the bound which is not contained in other assets, its inclusion will increase the bound and hence will have a nonzero coefficient when problem (14) is solved.[7] Therefore, we can determine whether a subset of payoffs causes a statistically significant increase in the bound of a given norm of m beyond that implied by the remaining assets by testing whether the coefficients on the subset of payoffs are equal to zero. Due to the relationship between equations (14) and (16), this is equivalent to testing whether an option z^*, created as a function of the payoffs of a subset of the universe of assets, can satisfy equation (2) for the entire universe of assets under consideration. Note that this is similar to testing whether a function of a subset of assets can price other assets outside the subset.

To see how a test using the restriction implied by equation (16) can be conducted in practice, define $e_t(\beta^*) = x^*_{t+\tau} z^*_{t+\tau}(\beta^*) - q^*_t$, and note that equation (16) implies that $E[e_t(\beta^*)] = 0$. Therefore, there exist $k + 1$ moment conditions and $k + 2$ parameters to estimate if the mean of the stochastic discount factor is to be estimated as well.

If at least two elements of β^* are hypothesized to be equal to zero, there exist more moment conditions than parameters, and therefore equation (16) represents an overidentified system. Thus, the parameters can be estimated by a nonlinear General Method of Moments (GMM) procedure proposed by Hansen (1982) and the over-identifying restriction(s) tested. This can be done by replacing the population mean of e with its sample counterpart:

$$\hat{e}_T(b^*) = \left(\frac{1}{T} \sum_{t=1}^{T} x^*_{t+\tau} z^*_{t+\tau}(b^*) - \frac{1}{T} \sum_{t=1}^{T} q^*_t \right) \tag{18}$$

and estimating the parameters by choosing b^* to minimize the following quadratic form:

$$J_T = T \cdot \hat{e}_T(b^*)' W_T \hat{e}_T(b^*) \tag{19}$$

where W_T is a weighting matrix chosen to be the inverse of a consistent estimate of the covariance matrix of e. Hansen (1982) shows that by choosing the weighting matrix in this fashion, estimates of the parameters with the smallest covariance matrix among the class of GMM estimators can be calculated. A consistent estimate of the covariance matrix of e can be calculated by initially minimizing J_T with the weighting matrix equal to an

[7] This can be proven when an additional identification restriction is imposed.

identity matrix in order to get consistent estimates of β^*. The estimates of β^* can then be used in a spectral density estimator at frequency zero constructed with Bartlett weights as proposed by Newey and West (1987) in order to obtain a consistent, positive-definite estimate of the covariance matrix of e even if returns are assumed to be serially correlated (see also Andrew (1990) and Robinson (1991)).

Under the null hypothesis that $E[e_t(\beta^*)] = 0$, given that k_1 elements of β^* are assumed to be equal to zero ($2 \leq k_1 \leq k$), J_T is asymptotically distributed as a χ^2 with degrees of freedom equal to $k_1 - 1$ if E_m is not prespecified and degrees of freedom equal to k_1 if the econometrician sets the mean level (recall that the last element of q^* contains the price of the risk-free asset, equal to Em, which must be estimated if not prespecified).

It should be noted that conducting these tests for various levels of ρ allows the econometrician to examine how sensitive a particular asset pricing model is to outlying payoff events. As δ is decreased, the Hölder inequality implies that ρ must be increased ($1/\delta + 1/\rho = 1$). In other words, frontiers for the δth norm of m, where δ is close to one, will be a function of higher moments of returns and therefore will be more sensitive to outliers. Likewise, if δ is large, the frontiers will be a function of smaller moments of returns and will be less sensitive to outliers.

B. Degeneracy of Frontiers

There is an important issue as to whether the mean-norm frontiers implied by equation (12) place restrictions on the stochastic discount factor beyond the trivial restriction that $E[m^\delta]^{1/\delta} \geq Em$. This restriction follows directly from Jensen's Inequality which implies that $E[m^\delta] \geq E[m]^\delta$ for $\delta \geq 1$. Therefore, we are interested in situations where $E[m^\delta]^{1/\delta} = Em$, which also implies that $\lambda_v(\delta) = Em$ since we can always choose h to be the payoff of a portfolio consisting of only the riskless asset o. This corresponds to the case when investors are risk neutral, and therefore m is a degenerate random variable. Let v represent a specified mean level of the stochastic discount factor. If m is a degenerate random variable, then equation (2) can be rewritten as:

$$vEx = E\pi(x). \tag{20}$$

Let n equal the total number of assets used in generating the region. Normalizing the prices of the payoffs to equal one we can write:

$$g_t = x_t m_t - i \tag{21}$$

where i is a vector of ones with length equal to the number of assets used in creating the region. Note that $E[g_t] = 0$ under the null hypothesis that m is a degenerate random variable; therefore, we can again apply the GMM methodology. By replacing $E[g_t]$ by its sample counterpart $\hat{g}_T = 1/T\sum_{t=1}^{T} g_t = v\hat{\mu} - i$ (where $\hat{\mu}$ is the vector of sample payoff means) and prespecifying a particular value of υ, the following test statistic can be formed:

$$D_T = T \cdot \hat{g}_T' W_T \hat{g}_T \tag{22}$$

where W_T is a positive-definite weighting matrix chosen to be equal to the inverse of a consistent estimate of the covariance matrix of g. Test statistic D_T can be shown to be asymptotically distributed as a χ^2 under the null hypothesis with n degrees of freedom. The weighting matrix can be estimated using the spectral density estimator at frequency zero with Bartlett weights proposed by Newey and West (1987).

III. Analyzing the Small Firm Effect

A. Size Effects

The idea that the size of the firm affects the return on the associated stock has been inspired by a number of papers in the finance literature. Banz (1981) showed that in the CAPM framework stock returns associated with small firms had, on average, a higher risk adjusted return than the stock returns of large firms, implying that the static CAPM is misspecified. Due to Roll's (1977) critique, Banz uses several differing measures of a market portfolio (a measure of invested market wealth), including the Value-Weighted and Equal-Weighted NYSE portfolios. He found that although the results were robust to the choice of the market portfolio, they were not necessarily robust to the choice of time period examined. This fact is reinforced by the findings of Brown, Kleidon, and Marsh (1983) which showed evidence of a reversal of the size anomaly for certain years. Keim (1983) and Reinganum (1981) also present evidence that the size of the firm has some effect on stock returns.

Even though the small firm effect is generally couched in the framework of the static CAPM, the effects of size may be an important issue in other asset pricing models as well. Some recent research indicates that the statistical properties of stock returns may be a function of size. Huberman, Kandel, and Karolyi (1987) show that the degree of correlation between different stocks may be stronger for stocks of similarly sized firms than for stocks of firms of different size. Work by Fama and French (1988) shows that the degree of mean reversion for a given stock may also be a function of the size of the firm.

In terms of the APT, there are a number of papers which indicate that factors other than typical proxies of the aggregate wealth portfolio are needed to explain the size effect. Chen (1983) noted that the null hypothesis of no size effect could not be rejected after accounting for factor loadings, a finding which seems to imply that there are other factors which account for the CAPM mispricing. In an attempt to identify these factors, Chan, Chen, and Hsieh (1985) tested a pricing equation which was a function of observable economic factors and found evidence that other factors related to risk (namely a measure of the spread of the returns of risky assets) appear to help explain the small firm effect. More recently, Connor and Korajczyk (1988), using an asymptotic principle components technique, noted that a five-factor version of the APT seems to perform better in explaining the size effect than the CAPM using the NYSE Value-Weighted portfolio and just as well as the CAPM using the NYSE Equal-Weighted portfolio.

Whether or not size is a direct factor in the pricing of assets or is a proxy for some other factor, it is of interest to see if various specifications of the stochastic discount factor are able to price the stocks of different firm sizes correctly since size represents a significant cross-sectional difference.

B. Methodology

In order to examine whether the restrictions on the stochastic discount factor implied by small firm returns differ significantly from the restrictions generated by large firm returns, mean-δth norm frontiers of the stochastic discount factor are calculated for the different data sets. Creation of these regions permits us to determine whether an estimate of the stochastic discount factor could lie within the region generated by one class of size-based returns and outside the region generated by another class. If so, then the possibility exists that a particular model may correctly price certain assets and not others. In order to take sampling error into account, tests of equation (16) were conducted to determine whether small firm payoffs imply restrictions on the stochastic discount factor beyond those restrictions implied by the payoffs of large firms. These regions and test statistics provide a way of analyzing whether the returns of small firms provide different implications about the stochastic discount factor than do the returns of large firms.

In order to examine the same issue in terms of the intertemporal CAPM, measures of a market portfolio (namely, the NYSE Equal-Weighted and Value-Weighted portfolios) were added to the firm size data, and the regions and tests were regenerated. This was done in order to see if the returns of either large or small firms contain any significant information about the stochastic discount factor once the information contained in the market portfolio and an unobserved risk-free asset are taken into account.

C. Data

The data sets used in estimation consisted of monthly observations over three different time periods. The first time period, 1926:1 to 1975:12, was selected in order to match the time period used by Banz (1981), since his is one of the first and most frequently quoted papers in the small firm literature. The second period, which extends from 1926:1 to 1987:12, was used in order to take advantage of a longer time series and thereby possibly increase the power of the tests. The third time period stretches from 1959:4 to 1987:12 and was chosen to match available aggregate consumption data in order to examine the restrictions size-based portfolio returns placed on a specific stochastic discount factor.

The assets used in generating the regions were the returns on the NYSE firm-size deciles and the measure of the aggregate wealth portfolio (either the NYSE Value-Weighted or Equal-Weighted portfolio). All data were taken from the CRSP data base. Data sets which included the returns of the top three deciles and/or the bottom three deciles were used in creating the different mean-norm frontiers because of the variations in the moments of the corresponding returns and the sorting by firm size. Table I summarizes the mean, standard deviation, and the 3/2 and third noncentral moments for all the deciles (DEC) for the period 1926:1 to 1987:12. Decile 10 contains the largest firms, and DEC1 the smallest.

Several different collections of returns were constructed and are summarized below with the abbreviations used in the tables of this paper:

3 Sm	The three bottom deciles of the NYSE
3 Sm + VWI	The three bottom deciles and the NYSE Value-Weighted Index
3 Sm + EWI	The three bottom deciles and the NYSE Equal-Weighted Index
3 Lg	The three top deciles of the NYSE
3 Lg + VWI	The three top deciles and the NYSE Value-Weighted Index
3 Lg + EWI	The three top deciles and the NYSE Equal-Weighted Index
3 Sm + 3 Lg	The three top and three bottom deciles of the NYSE
3 Lg + 3 Sm + VWI	The three top deciles, the three bottom deciles, and the NYSE Value-Weighted Index
3 Lg + 3 Sm + EWI	The three top deciles, the three bottom deciles, and the NYSE Equal-Weighted Index.

Table I

Descriptive Moments of Decile Returns on the NYSE

These estimates were calculated using real monthly returns, 1926:1–1987:12. All of the data were taken from the CRSP files.

	Asset	Mean	3/2 Noncentral	Std. Dev.	3rd Noncentral
TOP	DEC10	1.0038	1.0064	0.0427	1.0170
	DEC9	1.0053	1.0087	0.0476	1.0226
	DEC8	1.0063	1.0104	0.0505	1.0267
	DEC7	1.0068	1.0013	0.0525	1.0288
	DEC6	1.0071	1.0018	0.0537	1.0301
	DEC5	1.0072	1.0019	0.0556	1.0309
	DEC4	1.0084	1.0138	0.0577	1.0353
	DEC3	1.0087	1.0145	0.0617	1.0379
	DEC2	1.0094	1.0157	0.0641	1.0412
BOTTOM	DEC1	1.0102	1.0173	0.0724	1.0474

These collections were created for all three time periods. The nominal returns in the data collections for the first two time periods were deflated by a CPI,[8] and the returns for the third time period were deflated by an implicit consumption deflator in order to create real returns.

IV. Empirical Results

Restrictions on the δth norm of m ($\delta = 1.5, 2, 3$) as a function of Em were estimated by calculating the bound on the δth norm of m, $\lambda_v(\delta)$, for the various return collections and the different time periods. The estimation was done by normalizing, without loss of generality, the asset prices to one and conducting a nonlinear search to find portfolio weights to optimize problem (17). Tests were conducted to analyze two specific questions. First, are the frontiers generated by data sets including the returns of small firms more restrictive than the frontiers generated with the same data sets excluding the small firm returns? Secondly, do different δth-norm frontiers provide different implications for the stochastic discount factor implied by a representative consumer model with time and state-separable utility?

A. Restrictions Implied by Size-Based Portfolios

The estimated regions for the time period 1926:1 to 1987:12 are found in Figures 1a through 3c.[9] Figures 1a, 1b, and 1c contain the frontiers for the 3/2 norm, standard deviation and 3rd norm of the stochastic discount factor when the asset collection x consisted of payoffs of just size-based portfolios. The solid lines represent the restrictions implied by the returns of 3 Sm, the dashed lines the restrictions implied by the returns of 3 Lg and the dotted lines the restrictions generated by a collection of returns of the top three and bottom three deciles (3 Sm + 3 Lg).

Figures 2a through 3c contain the regions generated using the returns of size-based portfolios and a proxy of the aggregate wealth portfolio. The solid, dashed, and dotted lines in Figures 2a, 2b, and 2c were generated by the 3 Sm + VWI, 3 Lg + VWI, and 3 Sm + 3 Lg + VWI data sets, respectively, while the same lines in Figures 3a, 3b, and 3c were created using the 3 Sm + EWI, 3 Lg + EWI, and 3 Sm + 3 Lg + EWI data sets. The solid 45° line represents the points where $Em = E[m^\delta]^{1/\delta}$. If the mean-$\delta$th norm frontier touches this line, the region has no empirical content for those values of Em (i.e., restrictions no greater than those implied by Jensen's Inequality).

Some definite patterns emerge in the three sets of graphs. In Figures 1a, 1b, and 1c, the region generated by the small firms stays below the region generated by the large firm deciles except for values of Em between approximately 0.9935 and 0.9995 (the band is slightly smaller for Fig. 1a and just a little larger for Fig. 1c). The band is significant since this translates into an

[8] I would like to thank George Tauchen for making this index available.

[9] Similar figures were created for the period 1926–1975 but are not reproduced here due to their visual similarity to Figures 1a through 3c.

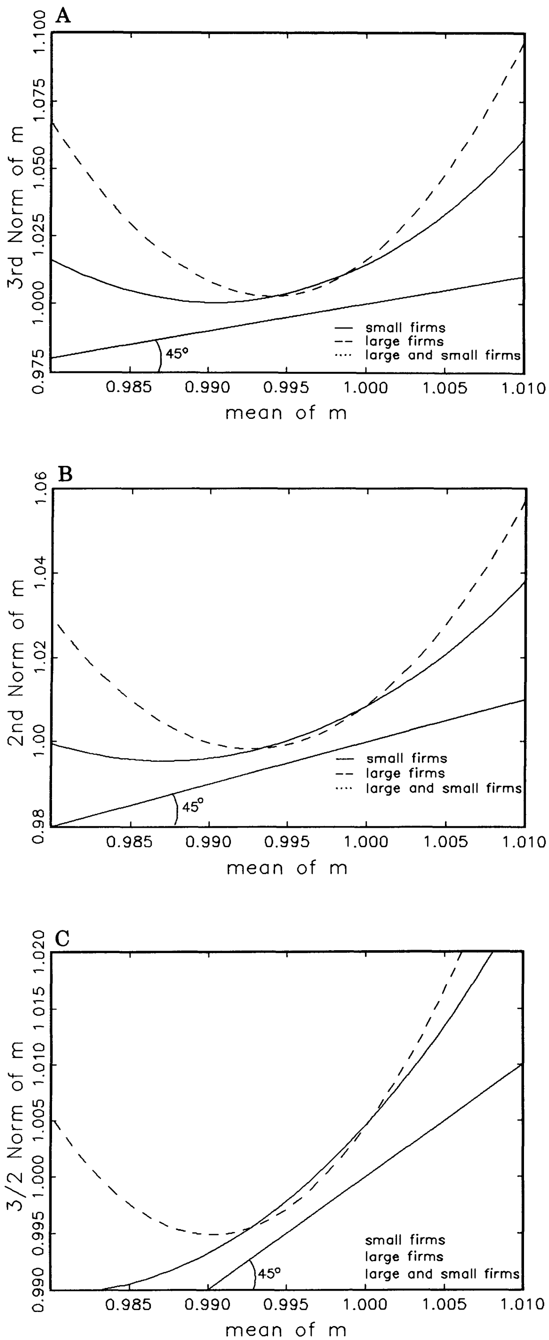

Figure 1. Bounds on the 3rd, 2nd, and 3/2 norm of the stochastic discount factor implied by the returns of size-based portfolios for the period January 1926 through December 1987.

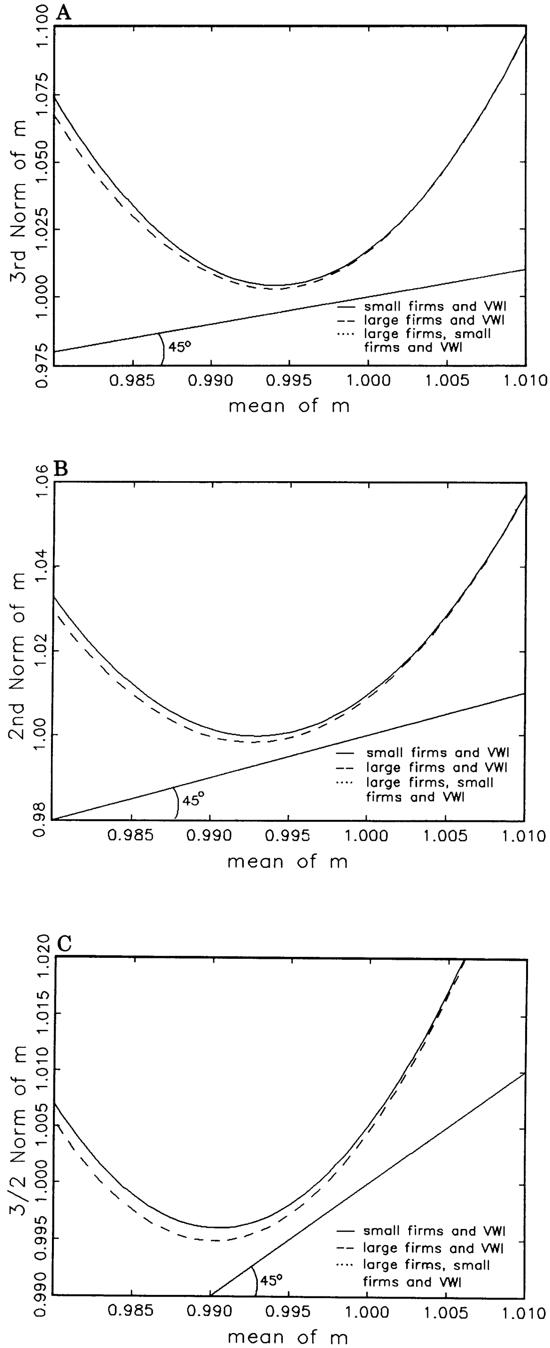

Figure 2. Bounds on the 3rd, 2nd, and 3/2 norm of the stochastic discount factor implied by the returns of size-based portfolios and the NYSE Value-Weighted Index for the period January 1926 through December 1987.

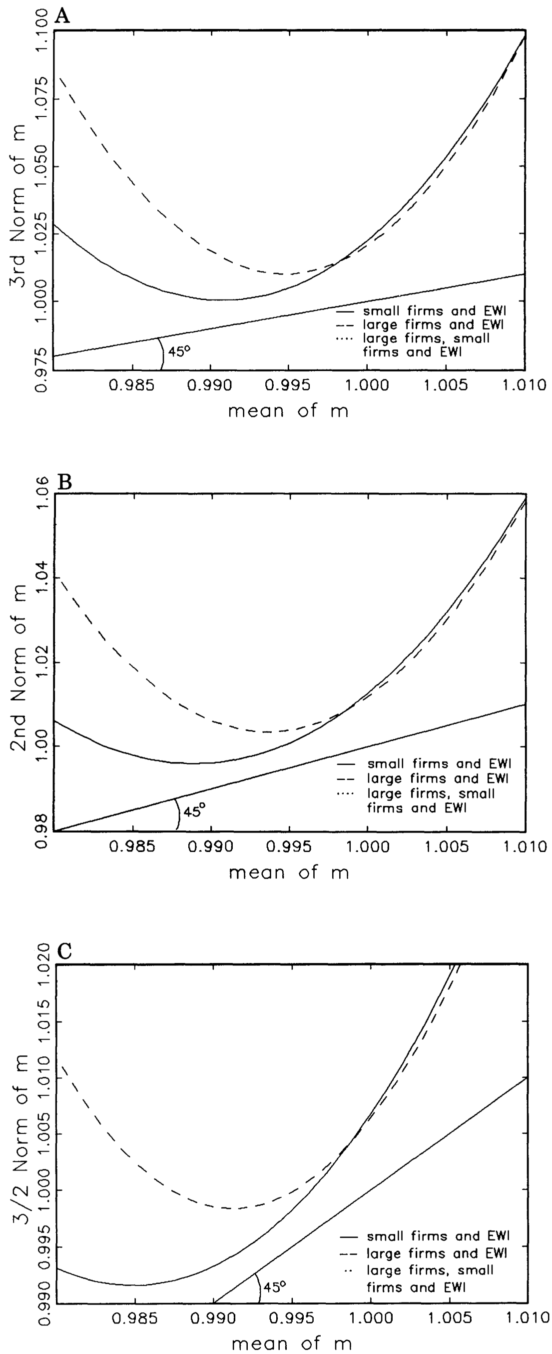

Figure 3. Bounds on the 3rd, 2nd, and 3/2 norm of the stochastic discount factor implied by the returns of size-based portfolios and the NYSE Equal-Weighted Index for the period January 1926 through December 1987.

annual return on a riskfree security of about 0.6% to 7.8%—a very plausible range of values. This indicates that the payoffs of small firms may have information about the stochastic discount factor (thereby increasing the lower bound on the various moments of m) not contained in the returns of large firms.

When the returns on the NYSE Value-Weighted and Equal-Weighted Indices (VWI and EWI, respectively) were added to the small and large firm data sets and the mean-norm frontiers recalculated, the results appear to change slightly. In Figures 2a through 2c, the region created by using 3 Sm + VWI lies above the region created by 3 Lg + VWI for almost all levels of the expected value of m. However, in Figures 3a through 3c, the frontiers generated by 3 Sm + EWI tended to lie above the frontiers generated by 3 Lg + EWI when Em was greater than 0.998 and tended to lie below when Em was less than 0.998.

Comparing the regions given by the solid and dashed lines in Figures 1a through 3c yields some mixed results; the small firm regions lie above the large firm regions often for only specific values of Em. However, this type of comparison does not tell us whether, in a statistically meaningful way, various moments of the returns of small firms contain information about the distribution of the stochastic discount factor which is not contained in the corresponding moments of the returns of large firms and/or a market proxy. A more meaningful analysis is to determine whether, after generating the frontiers with the returns of large firms and/or a market proxy, adding in the returns of small firms causes the frontier to shift upward and hence become more restrictive. This is equivalent to testing equation (16) with x containing some group of assets including the small firm returns and imposing the restriction that the coefficients on the small firm payoffs in z^* are equal to zero.[10]

In Table II, the first two rows under each subperiod heading represent tests of whether there is any additional information about the stochastic discount factor in the returns of size-based portfolios that is not already contained in the return of a market proxy and an unobserved riskfree return as described in Section II.C. (e.g., is the region created with 3 Sm + VWI strictly above the region created with just VWI after accounting for sampling error?).

The test statistics were calculated under the restriction that the coefficients on the returns of the three size-based deciles equal zero. This implies

[10] Recall that calculation of the bound described in equation (12) takes advantage of the positivity of m. Hansen and Jagannathan (1991a) show that the mean-variance frontier for m can be created without requiring that m be a positive random variable. They also show that there is a duality between mean-variance frontiers for m (where the positivity of m is not imposed) and the mean-variance frontier for asset returns. Therefore, this duality can be used to show that an upward shift in this mean-variance frontier for m implies a leftward shift in the mean-variance frontier for asset returns. Bounds were created with and without requiring m to be positive, using the size-based data sets, and they were identical for many values of Em. For those values of Em where the positivity of m does impose any additional restriction on the variance of m, we can view tests of equation (16) for $\delta = 2$ as being equivalent to tests of the efficiency of a given portfolio.

Table II

Tests of Whether the Returns of Large or Small Firms Contain Information about the δth Norm of the Stochastic Discount Factor Which Is Not Contained in the Return of a Proxy of the Aggregate Wealth Portfolio

The test statistic J_T is distributed as a $\chi^2(2)$ under the null hypothesis that the coefficients on the returns of either the three top deciles or three bottom deciles are zero in equation (16). P-values are reported in parentheses. J_T was estimated using monthly data. IID columns list estimates calculated under the assumption of IID returns while NW columns list estimates calculated using the procedure proposed by Newly and West with lag = 13 to allow for serial correlation in the asset returns.

	$\delta = 3$		$\delta = 2$		$\delta = 3/2$	
Data Set	IID	NW	IID	NW	IID	NW
			1926–1975			
3 Sm + EWI	10.31	6.01	10.63	6.89	10.60	7.14
	(.006)	(.049)	(.005)	(.032)	(.005)	(.028)
3 Sm + VWI	6.02	6.12	6.02	5.97	6.27	6.40
	(.049)	(.047)	(.049)	(.051)	(.043)	(.041)
3 Lg + EWI	5.52	3.71	5.85	3.99	6.35	4.47
	(.063)	(.157)	(.054)	(.136)	(.042)	(.107)
3 Lg + VWI	3.10	2.26	3.33	2.42	3.41	2.54
	(.212)	(.323)	(.189)	(.298)	(.182)	(.281)
			1926–1987			
3 Sm + EWI	5.96	4.09	7.13	4.77	7.39	5.20
	(.051)	(.129)	(.028)	(.092)	(.025)	(.074)
3 Sm + VWI	4.29	3.51	4.61	4.72	4.78	4.92
	(.117)	(.173)	(.099)	(.094)	(.097)	(.085)
3 Lg + EWI	4.37	3.14	5.12	3.74	5.47	4.13
	(.112)	(.209)	(.077)	(.154)	(.065)	(.127)
3 Lg + VWI	2.88	2.14	2.81	2.06	2.82	2.10
	(.236)	(.343)	(.245)	(3.56)	(.244)	(.350)
			1959–1987			
3 Sm + EWI	0.44	0.44	0.37	0.40	0.40	0.45
	(.803)	(.801)	(.834)	(.820)	(.820)	(.797)
3 Sm + VWI	1.35	1.58	1.25	1.61	1.29	1.71
	(.508)	(.453)	(.534)	(.447)	(.524)	(.425)
3 Lg + EWI	0.39	0.35	0.39	0.35	0.36	0.34
	(.822)	(.838)	(.823)	(.838)	(.826)	(.842)
3 Lg + VWI	0.47	0.35	0.75	0.59	0.99	0.80
	(.792)	(.838)	(.687)	(.745)	(.611)	(.842)

that, under the null hypothesis, the test statistic J_T should be distributed as a χ^2 with two degrees of freedom since Em was not prespecified. Test statistics in the columns labeled IID were calculated under the assumption of independent and identically distributed asset returns, since this is the standard working assumption in many tests of asset pricing models. Test statistics listed in columns labeled NW were computed using the procedure proposed by Newey and West (1987) to allow for serial correlation and conditional heteroskedasticity in the asset returns. Lag length in the creation of the Bartlett weights used in constructing the estimate of the variance-covariance matrix was set to 13 in these tests.[11]

The results in Table II indicate that, although information about the stochastic discount factor contained in the moments of the returns of large firms tends to be captured by the corresponding moments of the return of a market proxy, small firm returns can provide restrictions beyond those implied by the market proxy. This appears to be especially true for the period 1926–1975. Notice that the p-values for J_T when testing for the significance of the coefficients on the small firm deciles all fall below 0.051, while the p-values of J_T when testing for the significance of the large firm deciles fall between 0.042 and 0.323.

The results for the period 1926–1987 are much less significant as a whole. For the period 1959–1987, the hypothesis that the information in the small firm returns is accounted for by the returns of large firms and/or a market proxy cannot be rejected at the usual levels of significance and for all the values of δ used in the calculations. Like the 1926–1975 and 1926–1987 periods, these test statistics also tend to increase as ρ increases. The results presented in Table II indicate that over this period small firm and large firm deciles appear to have no information about the stochastic discount factor once we account for the information in a market proxy since none of the p-values from these tests are less than 0.425, and hence there seems to be no size effect during this period.

There are a number of possible explanations for this result. Brown, Kleidon, and Marsh (1983) note that there are shifts in the direction and size of the firm-size effect from 1967 to 1979. Hence, the choice of a certain subperiod of time may have affected the results. However, it should be noted that their approach used conditioning information, while the approach used in this paper averages over the objects on which Brown, Kleidon, and Marsh conditioned. The second explanation is that the difference in results could be due to a loss in the power of the tests because the 1959–1987 period has a significantly smaller number of return observations (744 observations for 1926:1–1989:12, 600 observations for 1926:1–1975:12 and 345 observations for 1959:4–1987:12). Lastly, the marginal information content of small firm returns may have diminished with time, or the results may simply be a statistical artifact of the first period.

Table III represents tests of equation (16) using data collections which include both the large firm and small firm deciles (3 Lg + 3 Sm) and these same deciles with some measure of the market (3 Lg + 3 Sm + VWI and 3 Lg + 3 Sm + EWI).[12] These tests are equivalent to asking (for the 1926–1987 period) whether, in a statistically meaningful way, the dotted

[11] Other lag lengths were used in estimating J_T in Table II. Decreasing the lag length almost uniformly increased the significance of the test statistic, while increasing the lag length tended to decrease the significance of the test.

[12] Other lag lengths were used in estimating J_T in Table III. When decile data with no market portfolio or with the NYSE value portfolio were used in calculating J_T, changing the lag length had little effect. However, when decile data with the NYSE Equal-Weighted return were used, the values of J_T tended to decrease as the lag length was increased.

Table III

Tests of Whether the Returns of Small Firms Contain Information about the δth Norm of the Stochastic Discount Factor Which Is Not Contained in the Returns of Large Firms and a Proxy of the Aggregate Wealth Portfolio

The test statistic J_T is distributed as a $\chi^2(2)$ under the null hypothesis that the coefficients on the returns of the three bottom deciles are zero in equation (16). P-values are reported in J_T was estimated using monthly data. IID columns list estimates calculated under the assumption of IID returns while NW columns list estimates calculated using the procedure proposed by Newey and West with lag = 13 to allow for serial correlation in the asset returns.

	$\delta = 3$		$\delta = 2$		$\delta = 3/2$	
Data Set	IID	NW	IID	NW	IID	NW
			1926–1975			
3 Lg + 3 Sm	5.78	5.78	6.01	6.18	6.02	6.34
	(.055)	(.056)	(.049)	(.046)	(.049)	(.042)
3 Lg + 3 Sm	[a]	4.29	9.52	4.45	10.68	4.82
+EWI		(.117)	(.009)	(.108)	(.005)	(.089)
3 Lg + 3 Sm	[a]	[a]	5.50	5.01	5.22	5.25
+VWI			(.063)	(.082)	(.057)	(.073)
			1926–1987			
3 Lg + 3 Sm	3.77	4.32	4.53	4.69	4.61	5.09
	(.152)	(.115)	(.103)	(.096)	(.100)	(.079)
3 Lg + 3 Sm	[a]	3.21	8.35	5.14	10.77	4.53
+EWI		(.202)	(.012)	(.023)	(.005)	(.104)
3 Lg + 3 Sm	3.21	3.73	3.84	4.09	4.31	4.51
+VWI	(.201)	(.155)	(.147)	(.129)	(.116)	(.105)
			1959–1987			
3 Lg + 3 Sm	1.31	1.04	1.50	1.09	1.21	0.89
	(.518)	(.593)	(.471)	(.578)	(.546)	(.641)
3 Lg + 3 Sm	1.61	2.02	1.56	1.79	1.40	1.56
+EWI	(.447)	(.365)	(.458)	(.409)	(.497)	(.460)
3 Lg + 3 Sm	[a]	[a]	2.04	2.58	1.98	2.42
+VWI			(.360)	(.276)	(.372)	(.299)

[a] The relative gradients at convergence were significantly different from zero, and therefore the estimates were deemed to be unreliable.

frontiers in Figures 1a through 3c are strictly higher than the dashed regions for all values of Em. As in part of Table II, tests of equation (16) were conducted under the null hypothesis that the coefficients on the payoffs of small firm deciles are equal to zero and hence do not place any additional restrictions on the stochastic discount factor above and beyond the returns of the large firms and/or a proxy of the market portfolio. For the period 1926–1975, nearly all the test statistics have p-values less than 0.1. Again, when the longer data set (1926–1987) was used, the significance of the test statistics dropped. None of the p-values are less than 0.276 for the last subperiod (1959–1987), indicating that the returns of small firms for this time period seem to have no marginal impact in generating restrictions on the discount factor.

It should be noted that, for higher levels of ρ (recall that ρ corresponds to the moments of the returns used in generating the bound on the δth norm of m), the significance of the test statistics tended to be higher. This implies that as ρ is increased, or, as δ is decreased due to Hölder's inequality, the region created by the returns of small firms, large firms, and a market index become significantly more restrictive than the region created by just the returns of large firms and/or a market index. Therefore, the opportunity for the δth norm of a specified m to lie inside the region created by returns of the large firm deciles and/or a market index, but outside the region created by those same returns plus the returns of small firms, increases as ρ increases.

B. Implications of Size-Based Portfolios for a Specific Model

In order to examine how size-based portfolios might deliver differing implications about a particular asset pricing model, the sample moments of the stochastic discount factor implied by a utility-based model were calculated. In this case, the stochastic discount factor can be interpreted as the IMRS. The representative consumer is assumed to maximize expected utility which is time and state-separable and has the following form:

$$U(c(t)) = \beta^t \frac{c(t)^{\gamma+1}}{\gamma + 1} \tag{23}$$

where U is assumed to be concave and $0 < \beta < 1$. This implies that:

$$m = \beta(c(t+1)/c(t))^{\gamma}. \tag{24}$$

This measure of m has been proposed by Lucas (1978), Grossman and Shiller (1981), and Hansen and Singleton (1982).

In estimating the population moments of m, $c(t)$ was replaced with estimates of per capita aggregate consumption. Aggregate consumption was defined as nondurables plus services and was taken from the National Income and Product Accounts as well as the Survey of Current Business for the period 1959:3–1987:12. Per capita consumption was formed by dividing the real consumption series by resident population data taken from the CITIBASE data base and the Current Population Reports.

The estimated regions for the period are found in Figures 4a through 4c. These regions were generated using the 3 Sm, 3 Lg, and 3 Lg + 3 Sm data sets and are represented by solid, dashed, and dotted lines, respectively.[13] The sample counterparts of the ordered pairs $(Em, E[m^{\delta}]^{1/\delta})$ are plotted as a function of γ and are represented as small triangles. Beta was set equal to 0.99. Gamma was varied between 0 and -200 by increments of -2. Gamma equal to zero is represented by the right-most triangle near the bottom of the

[13] The same regions were constructed using the other data collections listed in Section V, although they are not presented here since the results were very similar to those implied by the 3 Sm, 3 Lg, and 3 Sm + 3 Lg data sets. Tables III and IV make this fact apparent.

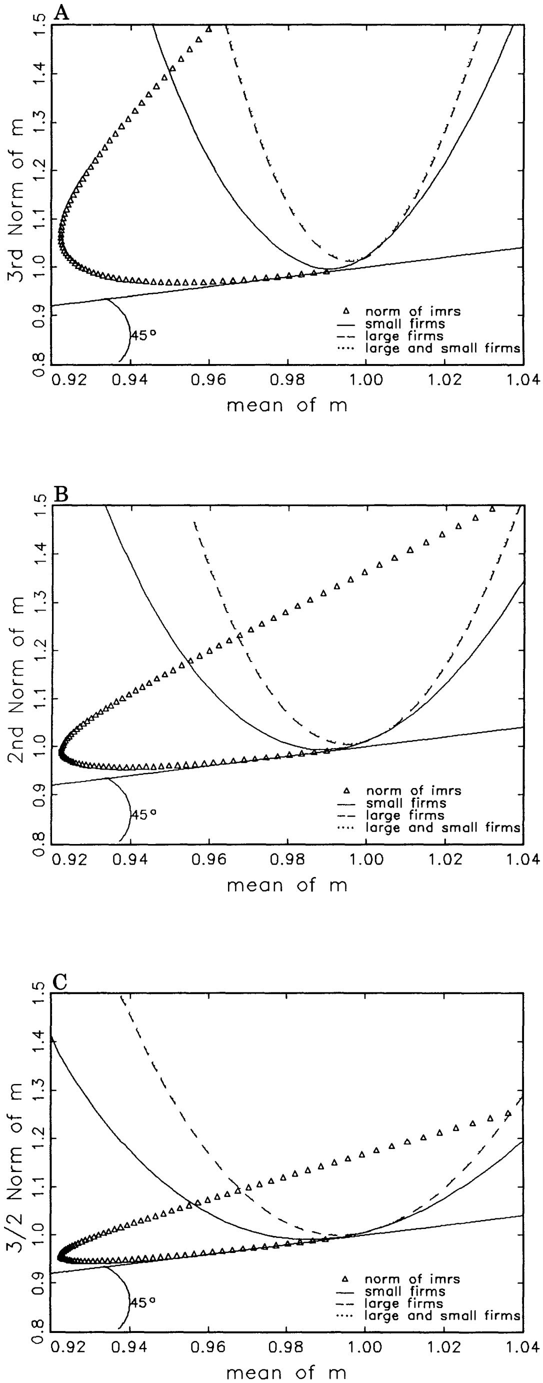

Figure 4. Bounds on the 3rd, 2nd, and 3/2 norm of the stochastic discount factor implied by the returns of size-based portfolios for the period April 1959 through December 1987.

region. As γ is decreased, the point estimate of (Em, $E[m^{\delta}]^{1/\delta}$) moves left, away from the regions, before eventually turning around and entering the various regions at very low levels of γ. As β is decreased (increased) the path of the ordered pairs is shifted in a parallel fashion to the left (right).

Since estimates of the moments of m were created for this period, tests of degeneracy were conducted for these regions by estimating equation (20) for various prespecified levels of Em. The χ^2 statistics and probability values are listed in Table IV. For the small firm data, values of Em for which the null hypothesis of risk neutrality cannot be rejected at the 0.05 significance level range between 0.985 and 0.999. For the large firms, the range of Em is 0.991 to 1.0. These tests imply that these data sets are somewhat limited during this period in their ability to generate testable restrictions and that diagnostic tests used with this data would be highly dependent upon the choice of β.

Note that the estimated mean-δth moment pairs all start outside of their corresponding regions. However, it is the point where they enter the region which is of particular interest. In Figure 4a (the mean-3rd norm frontier for the stochastic discount factor), the duplets enter the small firm region at γ approximately equal to -138 and enter the large and full decile regions around γ equal to -150. In Figure 4b, the levels of γ must be decreased to -142 and -152 for the estimated pairs to lie inside their respective regions.

Table IV

Test for Degeneracy of the Regions

The test statistic D_T is distributed as a $\chi^2(3)$ under the null hypothesis that $EmEx = E\pi(x)$. D_T was estimated using monthly data for the period 1926–1959 ($T = 345$) for different prespecified values of Em. The weighting matrix was estimated using the procedure proposed by Newey and West with lag = 13.

3 Lg			3 Sm		
Em[a]	$X^2(3)$	P-value	Em[a]	$X^2(3)$	P-value
0.985	7.32	0.06	0.991	7.76	0.05
0.986	6.08	0.11	0.992	6.20	0.10
0.987	5.04	0.17	0.993	5.03	0.17
0.988	4.18	0.24	0.994	4.22	0.24
0.989	3.51	0.32	0.995	3.78	0.29
0.990	3.02	0.39	0.996	3.71	0.29
0.991	2.71	0.44	0.997	4.01	0.26
0.992	2.59	0.46	0.998	4.67	0.19
0.993	2.65	0.45	0.999	5.69	0.12
0.994	2.88	0.41	1.000	7.07	0.07
0.995	3.30	0.35			
0.996	3.89	0.27			
0.997	4.66	0.20			
0.998	5.60	0.13			
0.999	6.71	0.08			

[a] All values of Em outside the range of 0.985 to 0.999 for 3 Sm and outside the range of 0.991 to 1 for 3 Lg implied p-values less than 0.05.

The levels of γ drop further (around -146 and -156) in Figure 4c.[14] Although this indicates that examining lower moments of m (or higher moments of returns) results in increasingly restrictive regions, the extremely negative levels of γ give some cause for concern since this dramatically increases the importance of outliers in the consumption series. Therefore, this result may not exist when different functional forms of the utility functions are used.

The notion that the mean-δth norm frontier for this particular stochastic discount factor becomes more restrictive as δ decreases is important. Mehra and Prescott (1985) and Hansen and Jagannathan (1991a) both note that in a representative consumer model with time and state-separable utility, extremely low levels of γ must be specified in order to make the volatility of the implied stochastic discount factor consistent with the volatility of asset returns. Constantinides (1990) and Heaton (1991) both attempt to solve this volatility problem by relaxing the time-separable assumption on utility to allow for intertemporal complementarity. Although this may solve the problem in the mean-variance dimension for the stochastic discount factor, the greater puzzle would appear to be in satisfying mean-δth moment restrictions for $1 < \delta < 2$—in other words, choosing a specification of m such that the lower moments of the stochastic discount factor are consistent with the higher moments of asset returns.

V. Conclusion

Since higher moments may contain interesting information about the stochastic discount factor, this paper extends the work of Hansen and Jagannathan to include a method of generating restrictions on general noncentral moments of the stochastic discount factor. These bounds can be created for any norm of m as long as the relevant moments of the distribution of asset returns are finite.

Mean-δth norm frontiers ($\delta = 1.5$, 2, 3) for m were created using the size-based data sets for three different time periods. Results for the 1926–1975 time period provide strong evidence that small firms contain information about the stochastic discount factor which is not contained in large firms or measures of a market portfolio. For $\delta = 2$, this is consistent with results which indicate that usual proxies of the market portfolio are not mean-variance efficient and that a specification of m that is a function of only a market proxy will on average misprice the stocks of small firms. When a longer period of returns was used (1926–1987), the results were much less dramatic and often not statistically significant.

[14] The evidence is consistent with the results of Hansen and Jagannathan which indicate that when a time separable power utility function is used γ must be very negative in order for the mean-standard deviation pair to lie inside the region. The level of γ which causes the pair to lie inside the region is not reported in the paper; however, these authors note that around γ equal to -100, the pairs stop moving away and turn back toward the region.

Examination of returns from 1959–1987 yielded no evidence of a small firm effect. There are several possible explanations for this finding. Some studies, such as Brown, Kleidon, and Marsh (1983) indicate that the magnitude and direction of the small firm effect were not constant during certain periods of time. The smaller sample period also provides a less powerful test; therefore, these results may simply reflect the loss of certain information. Lastly, the marginal information content of small firm returns may have decreased over time, or the results may simply be a statistical artifact of the first period.

Finally, I find that different mean-norm frontiers can deliver different implications about m. The significance of tests of whether certain moments of the returns of small firms have information about the stochastic discount factor which is not contained in the corresponding moments of the returns of large firms or a market proxy was almost uniformly an increasing function of the moment of the asset returns used in the test. In other words, as ρ (δ) increased (decreased), the level of most of the test statistics rose. Also, when a consumer with a time and state-separable utility function is assumed, the level of the risk aversion coefficient required for the norm to lie inside the region rose as δ decreased. This indicates that mean-δth norm frontiers may present different or more startling puzzles than the mean-variance frontier when examining or modeling a given measure of the stochastic discount factor.

APPENDIX A

In this appendix, z^* is shown to be a random variable which attains the bound specified by equation (12) and hence is a greater lower bound for the stochastic discount factor. Recall equations (12) and (16):

$$E[m^{\delta}]^{1/\delta} \geq \lambda_v(\delta) = \sup_{h \in H} \frac{E\pi_v(h)}{E[h^{+\rho}]^{1/\rho}} \tag{12}$$

where $E[h^{+\rho}]^{1/\pi} > 0$.

$$E[x^* z^*] = E[q^*] \tag{16}$$

where $z^* = ((x^{*\prime}\tilde{\beta})^+)^{\rho-1}$.

Let β^* be the vector of portfolio weights which solves (12). After substituting β^* into (12) and noting that $E\pi[x^{*\prime}\beta^*] = E[q^{*\prime}\beta^*]$ since π is a linear functional, we can write

$$\lambda_v = \frac{E[q^{*\prime}\beta^*]}{E[(x^{*\prime}\beta^*)^{+\rho}]^{1/\rho}} \tag{A.1}$$

where $E[(x^{*\prime}\beta^*)^{+\rho}]^{1/\rho} > 0$.

Dividing both numerator and denominator by ρ/ϕ and recalling $\tilde{\beta} = \rho\beta^*/\phi$, equation (A.1) becomes:

$$\lambda_v = \frac{E[q^{*\prime}\tilde{\beta}]}{E\left[\left(x^{*\prime}\tilde{\beta}\right)^{+\rho}\right]^{1/\rho}} \tag{A.2}$$

where $E[(x^{*\prime}\tilde{\beta})^{+\rho}]^{1/\rho} > 0$.

Equation (16) and the requirement that $1/\delta + 1/\rho = 1$ implies that:

$$\begin{aligned} E[q^{*\prime}\tilde{\beta}] &= E\left[\left(x^{*\prime}\tilde{\beta}\right)z^*\right] = E\left[\left(x^{*\prime}\tilde{\beta}\right)^{+}z^*\right] \\ &= E\left[z^{*\rho/(\rho-1)}\right] = E\left[z^{*\delta}\right]. \end{aligned} \tag{A.3}$$

Note that z^* has a finite δth moment and hence is an element of L^δ. Substituting the relationships implied by equation (A.3) into equation (A.2) allows us to write:

$$\lambda_v(\delta) = \frac{E[z^{*\delta}]}{E\left[z^{*\rho/(\rho-1)}\right]^{1/\rho}} = \frac{E[z^{*\delta}]}{E\left[z^{*\delta}\right]^{1/\rho}} = E\left[z^{*\delta}\right]^{(\rho-1)/\rho} = E\left[z^{*\delta}\right]^{1/\delta}. \tag{A.4}$$

Hence, there exists a nonnegative random variable in L^δ which has a δth norm that attains the bound described in equation (12) and satisfies equation (2); therefore, $\lambda_v(\delta)$ is a greatest lower bound for m given that $Em = v$.

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