The Quality Delivery Option in Treasury Bond Futures Contracts
Author(s): Michael L. Hemler
Source: *The Journal of Finance*, Vol. 45, No. 5 (Dec., 1990), pp. 1565-1586
Published by: Blackwell Publishing for the American Finance Association
Stable URL: http://www.jstor.org/stable/2328750
Accessed: 26/11/2009 07:22

THE JOURNAL OF FINANCE • VOL. XLV, NO. 5 • DECEMBER 1990

The Quality Delivery Option in Treasury Bond Futures Contracts

MICHAEL L. HEMLER*

ABSTRACT

This paper uses three methods to estimate quality option values for CBOT Treasury bond futures contracts. It presents evidence regarding: (1) payoffs from exercising this option at delivery, (2) estimates from a T-bond futures pricing model that incorporates this option, and (3) estimates obtained from an exchange option pricing formula. The results indicate that this option is worth considerably less than reported by Kane and Marcus (1986a). For example, payoffs obtained by switching from the bond cheapest to deliver three months prior to delivery to the one cheapest at time of delivery average less than 0.30 percentage points of par.

IN A FUTURES CONTRACT the short frequently has options as to what, when, where, and how much is delivered. These delivery options (which are known as the quality option, timing option, location option, and quantity option, respectively) have been the subject of considerable research.[1] In particular, recent studies have investigated whether the presence of delivery options reduces the corresponding futures price. The rationale is straightforward: if such options are valuable to the short, then he should be unable to acquire them costlessly. Therefore, because the short never explicitly pays for these options, the futures price should be bid down to compensate the long for the additional delivery risk.

This paper reports on the quality option in Chicago Board of Trade (CBOT) U.S. Treasury bond futures contracts, i.e., the short's right to deliver any T-bond issue that is at least 15 years from maturity or first call. There are good reasons for focusing on this particular quality option. First, the CBOT Treasury bond contract is the most actively traded and widely used futures contract in the

* The Fuqua School of Business, Duke University. This paper is based on my Ph.D. dissertation at the University of Chicago Graduate School of Business. I have benefited greatly from the comments of my dissertation committee: Kenneth French (chairman), George Constantinides and Robert Stambaugh (inside members); Wayne Ferson, Shmuel Kandel, and Arnold Zellner (outside members). I have received helpful comments from Douglas Breeden, William Christie, Charles Jacklin, Francis Longstaff, Merton Miller, Lance Smith, Eric Vezie, Bob Whaley, and members of the Department of Economic Analysis and Planning at the Chicago Board of Trade. I have also received valuable suggestions during workshops at the following universities: Chicago, Columbia, Duke, Indiana, Massachusetts Institute of Technology, Ohio State, Oregon, Pennsylvania, Stanford, State University of New York (Buffalo), and Washington University in St. Louis. The Chicago Board of Trade generously provided data and computer facilities used in this study. Finally, I wish to acknowledge the constructive comments of an anonymous referee and the Editor of this Journal.

[1] Recent work includes Arak and Goodman (1987), Benninga and Smirlock (1985), Boyle (1989), Broadie and Sundaresan (1988), Carr (1988), Cheng (1985, 1986), Gay and Manaster (1984, 1986, 1989), Hegde (1988, 1989), Kane and Marcus (1986a,b), and Livingston (1987).

United States. Second, there are numerous deliverable issues. More than thirty issues varying widely in coupon and time to maturity are currently eligible for delivery. Third, several issues are usually delivered during any given delivery month. The number of delivered issues typically ranges from three to seven; on occasion it has climbed as high as nineteen.

Most of the empirical evidence regarding the quality option suggests that its presence substantially affects futures prices. For example, Gay and Manaster (1984) value the quality option in CBOT wheat futures contracts using the Margrabe (1978) 2-asset exchange option pricing formula. Their most conservative estimate is that the option value equals roughly 2.2% of the corresponding futures price. Kane and Marcus (1986a) price the quality option in CBOT T-bond futures using Monte Carlo simulations of estimated term structures. They find option values ranging from 1.39 to 4.60 percentage points of par, which correspond to discounts of 1.9% to 6.2% from the equilibrium futures price that would result in the absence of this option. On the other hand, Hegde (1988) reports much smaller values for this option. He estimates the average value of delivery options over the last quarter of the nearby futures contract to be less than 0.5% of the mean futures price.

This paper presents estimates of the value of the quality option in the T-bond futures market during the sample period 1977–1986. A notable feature of this analysis is that it uses three distinct estimation techniques. It estimates historical payoffs obtained from exercising this option at delivery. It also estimates option values prior to delivery using a T-bond futures pricing model that explicitly incorporates the quality option. Finally, it estimates option values using an n-asset exchange option pricing formula where $n = 2, 3$. The advantage of using three techniques is that it yields a more complete picture of this option and allows one to cross-check the estimates more easily.[2]

The findings in this paper reinforce those of Hegde (1988) and, to a lesser extent, Gay and Manaster (1984). For example, the payoff obtained by switching from the bond cheapest to deliver three months prior to delivery to the one cheapest to deliver at time of delivery averages less than 0.3 percentage points of par. Similarly, option values obtained from the T-bond futures pricing model average less than 0.2 percentage points of par three months prior to delivery. The corresponding option values based on the exchange option pricing formula are somewhat higher, averaging approximately 0.7 and 1.2 percentage points of par when $n = 2$ and $n = 3$, respectively. Such values are substantially less than those reported by Kane and Marcus (1986a). One likely explanation for the discrepancy in results is that the estimates in Kane and Marcus, unlike those in this paper, rely on estimated term structures and associated volatilities over the period 1978–1982.[3]

Like most papers on the quality option, this paper assumes that all contracts are delivered on some predetermined day. Although this assumption is

[2] By way of constrast, Gay and Manaster (1984) use only one technique—the Margrabe (1978) 2-asset exchange option formula. Similarly, Kane and Marcus (1986a) use only one technique—average payoffs at time of delivery based on Monte Carlo simulations of estimated term structures.

[3] Section II discusses Kane and Marcus (1986a) in more depth and lists several potential explanations for why their results differ from ours.

undesirable, studies of the quality option typically have ignored timing options, and vice versa.[4] The quality and timing options are intertwined; valuing the two separately has proven difficult.

The paper proceeds as follows: Section I presents background material regarding the T-bond futures contract and delivery process. It also develops two models—a no-arbitrage T-bond futures pricing model and an exchange option pricing model—used to price the quality option. Section II presents empirical evidence regarding payoffs obtained by exercising the quality option at delivery along with option estimates based on the models in the preceding section. Section III summarizes and concludes the paper.

I. Pricing Models for the Quality Option

A. Preliminaries

The CBOT T-bond futures contract is an agreement that the short will deliver $100,000 face value U.S. Treasury bonds to the long during a specified delivery month in return for a specified price. Buyer and seller contract through the CBOT Clearing Corporation rather than with each other. This provides protection against default by either party and increases market liquidity. Unlike a forward contract, the agreement stipulates that both parties "mark-to-market" (i.e., pay any losses or receive any gains in cash on a daily basis) while the futures position remains open.

Despite its key role in the determination of T-bond futures prices, delivery need not occur. Most market participants liquidate their positions via offsetting trades before their contracts expire. When delivery does occur, it is the short who chooses the delivery date and delivery issue. The delivery date can be any business day of the delivery month (March, June, September, or December). The delivery issue can be any issue that is at least 15 years until maturity or first call. However, all bonds delivered against a given contract must be of the same issue.

The delivery process consists of a predetermined 3-day sequence of events initiated by the short. On the first day, Position Day, the short notifies the CBOT Clearing Corporation that he plans to make delivery. On the second day, Notice of Intention Day, the Clearing Corporation matches the oldest long to the delivering short and notifies them of the opposite party's intention to make or take delivery. The short then invoices the long, at which time the short specifies which issue he will deliver. On the third day, Delivery Day, the short delivers the T-bonds to the long in return for the invoice amount. Delivery and payment take place by book-entry via the Federal Reserve wire transfer system.

Because cash prices of the deliverable issues differ widely, the CBOT uses a system of conversion factors to adjust the delivery values of different issues according to their coupon rates and times to maturity or first call. The conversion factor for a given issue is the price (assuming a par value of $1) at which the issue will yield 8% at its current time to maturity or first call. The resulting

[4] For example, Gay and Manaster (1984) and Kane and Marcus (1986a) assume that delivery occurs on the first business day of the delivery month. Benninga and Smirlock (1985) assume delivery occurs on the last business day of the delivery month.

invoice price equals \$1,000 times the conversion factor multiplied by the futures settlement price from either Position Day or the last trading day for the contract (i.e., the eighth-to-last business day of the delivery month), whichever comes first. The total invoice amount paid to the short at delivery equals the invoice price plus interest which has accrued as of Delivery Day.[5]

B. A Pricing Model Based on a T-bond Futures Pricing Formula

Let the time period $[T_1, T_2]$ denote a given delivery month where T_1 and T_2 correspond to the first and last business days, respectively. Assume n T-bond issues are eligible for delivery; the corresponding T-bonds are called ith issue T-bonds where $i = 1, \cdots, n$. For each T_0 such that $T_0 \leq T_1$, define the following:

$F(T_0, [T_1, T_2])$ = Futures price at time T_0 for T-bond futures contract requiring the short to deliver at time $T \in [T_1, T_2]$,
$S_i(T_0)$ = Spot price quoted at time T_0 for an ith issue T-bond,
$B_i(T_0, T_1)$ = Value at time T_0 of the coupon payments made on an ith issue T-bond over the time period $[T_0, T_1]$,
$I_i(T_0)$ = Interest accrued at time T_0 for an ith issue T-bond,
$R(T_0, T_1)$ = Risk-free rate of return over the period $[T_0, T_1]$,
$C_i(T_1)$ = Conversion factor for invoicing an ith issue T-bond delivered at time T_1,
$q_i(T_0; [T_1, T_2], n)$ = Value at time T_0 of an (n, i)-quality option, which is the short's right to deliver jth issue T-bonds (where $j \in \{1, \cdots, n\}$ is arbitrary) rather than ith issue T-bonds at time of delivery $T \in [T_1, T_2]$.

When $T_1 = T_2$ we replace $[T_1, T_2]$ with T_1 in the preceding notation; e.g., $F(T_0, T_1)$ is equivalent to $F(T_0, [T_1, T_1])$. We adopt the standard convention of expressing $F(\cdot)$, $S_i(\cdot)$, $B_i(\cdot)$, and $I_i(\cdot)$ in percentage points of par. Because the contract calls for delivery of \$100,000 face value T-bonds, each percentage point of a price quote corresponds to \$1000 in terms of the futures contract. Note also that the market price paid for an ith issue T-bond at time T equals the quoted price $S_i(T)$ plus accrued interest $I_i(T)$ rather than the quoted price $S_i(T)$ alone.

To derive a no-arbitrage T-bond futures pricing model we assume:

(A1) Capital markets are perfect. In particular, there are no taxes, transaction costs, or short selling restrictions. Assets are perfectly divisible. Investors have equal and costless access to all relevant information concerning the securities traded. Furthermore, they take security prices as given, acting as if their market activities have no effect on the ruling prices.
(A2) Corresponding futures and forward prices are equal.
(A3) All futures contracts expire on the same business day of the delivery month, i.e., $T_1 = T_2$.

[5] See Chicago Board of Trade (1980, 1986, 1987) for additional information regarding contract specifications, conversion factors, the delivery process, and the invoicing procedure.

Consider two investment strategies at time T_0. Strategy 1 is to buy an ith issue T-bond in the cash market and hedge with the corresponding short position in the futures market. Strategy 2 has three components: buy an (n, i)-quality option, invest $[C_i(T_1)F(T_0, T_1) + I_i(T_1)]/[1 + R(T_0, T_1)]$ in risk-free bonds maturing at time T_1, and invest $B_i(T_0, T_1)$ in risk-free bonds (possibly of different maturities) so that the resulting payoffs identically match the coupon payments from the ith issue T-bond over the time interval $[T_0, T_1]$. Specifically, assume that the ith issue T-bond pays a coupon of b dollars on the coupon dates $t_1, \cdots, t_k$ where $T_0 < t_1 < \cdots < t_k < T_1$. Then

$$B_i(T_0, T_1) = \sum_{j=1}^{k} \frac{b}{1 + R(T_0, t_j)},$$

and the third part of Strategy 2 requires one to invest $b/[1 + R(T_0, t_j)]$ in risk-free bonds maturing at time t_j for each $j = 1, \cdots, k$.

What are the cash outflows associated with these strategies? By assumption we ignore daily resettlement, transaction costs, taxes, and timing delivery options available to the short. Because the futures contract requires no initial investment, the amounts paid to initiate these strategies at time T_0 are

$$S_i(T_0) + I_i(T_0) \tag{1}$$

for Strategy 1, and

$$q_i(T_0;\, T_1, n) + \frac{C_i(T_1)F(T_0, T_1) + I_i(T_1)}{1 + R(T_0, T_1)} + B_i(T_0, T_1) \tag{2}$$

for Strategy 2.

What are the cash inflows associated with these strategies? Both strategies yield the amount b on each of the coupon payment dates $t_1, \cdots, t_k$. Furthermore, both strategies yield the payoff

$$C_i(T_1)F(T_0, T_1) + I_i(T_1) + q_i(T_1;\, T_1, n) \tag{3}$$

at time T_1. This is obvious for Strategy 2; one obtains $q_i(T_1;\, T_1, n)$ from exercising the quality option and $C_i(T_1)F(T_0, T_1) + I_i(T_1)$ from the risk-free bonds maturing at time T_1. To prove it for Strategy 1, assume that jth issue T-bonds are cheapest to deliver at time T_1 where j is a fixed index satisfying $1 \leq j \leq n$. When $j = i$ the investor delivers ith issue T-bonds and receives $C_i(T_1)F(T_0, T_1) + I_i(T_1)$ in return. This implies that $q_i(T_1;\, T_1, n) = 0$, so that the payoff equals (3) as desired. On the other hand, when $j \neq i$ the investor delivers jth issue T-bonds rather than ith issue T-bonds. But the incremental amount received for switching from issue i to issue j must equal $q_i(T_1;\, T_1, n)$ by definition of the (n, i)-quality option. Therefore, the total amount received must equal (3) as claimed.

Since the strategies yield identical payoffs over the interval $(T_0, T_1]$, they must have the same price at time T_0. Otherwise, an arbitrage opportunity would result. Equating (1) and (2) gives

$$S_i(T_0) + I_i(T_0) = q_i(T_0;\, T_1, n) + \frac{C_i(T_1)F(T_0, T_1) + I_i(T_1)}{1 + R(T_0, T_1)} + B_i(T_0, T_1). \tag{4}$$

Solving for the futures price $F(T_0, T_1)$ in equation (4) yields

$$F(T_0, T_1) = \frac{[S_i(T_0) + I_i(T_0) - B_i(T_0, T_1) - q_i(T_0; T_1, n)][1 + R(T_0, T_1)] - I_i(T_1)}{C_i(T_1)}, \quad (5)$$

which is our fundamental T-bond futures pricing equation.

Equation (5) implies that the presence of the quality option lowers the equilibrium futures price. In effect, this price reduction compensates the long for the uncertainty regarding the issue delivered by the short. Equation (5) also yields

$$q_i(T_0; T_1, n) = S_i(T_0) + I_i(T_0) - B_i(T_0, T_1) - \frac{C_i(T_1)F(T_0, T_1) + I_i(T_1)}{1 + R(T_0, T_1)}$$

$$= \frac{f_1(T_0, T_1) - f_2(T_0, T_1)}{1 + R(T_0, T_1)}, \quad (6)$$

where

$$f_1(T_0, T_1) = [S_i(T_0) + I_i(T_0) - B_i(T_0, T_1)][1 + R(T_0, T_1)],$$

$$f_2(T_0, T_1) = C_i(T_1)F(T_0, T_1) + I_i(T_1).$$

This gives a convenient interpretation of the quality option as the present value of the forward price differential $f_1(T_0, T_1) - f_2(T_0, T_1)$. One can view $f_1(T_0, T_1)$ as the forward price established at time T_0 for delivery of ith issue (and only ith issue) T-bonds at time T_1.[6] On the other hand, $f_2(T_0, T_1)$ represents the price established at time T_0 for delivery of ith issue T-bonds at time T_1, provided that the short chooses to deliver that issue from among the n issues eligible for delivery.

C. A Pricing Model Based on an Exchange Option Pricing Formula

Another approach for valuing the quality option is to model it as an n-asset exchange option.[7] An n-asset exchange option is an option whereby the option

[6] This follows from a standard no-arbitrage argument. Observe that $f_1(T_0, T_1)$ can be rewritten as:

$$f_1(T_0, T_1) = [S_i(T_0) + I_i(T_0)][1 + R(T_0, T_1)] - B_i(T_0, T_1)[1 + R(T_0, T_1)],$$

which is clearly analogous to the pricing formula

$$F(t, T) = S(t)[1 + R(t, T)] - D(t, T)$$

commonly used (e.g., Cornell and French (1983)) for stock index futures. In the latter equation $S(t)$ represents the value of the stock index at time t, $F(t, T)$ equals the corresponding futures price, $R(t, T)$ denotes the risk-free rate observed at time t for the period $[t, T]$, and $D(t, T)$ is the value at time T of the dividend stream for the index over the time period $(t, T]$.

[7] We use the descriptive "n-asset" to differentiate this exchange option from the more commonly used 2-asset exchange option studied by Margrabe (1978). The n-asset exchange option pricing formula given in this section holds for any positive integer n greater than one. However, in our empirical work we restrict our attention to the cases $n = 2$ and $n = 3$. While computations for higher-dimensional cases are similar and could be done, it is unclear whether such calculations would give much additional insight. Usually there is little chance that the cheapest to deliver issue will not be one of two or three issues.

holder can exchange any asset j ($j = 1, \cdots, n$) for an asset i at some specified time. If x_i denotes the price of asset i, then exercising this option at time t^* yields the payoff

$$\max_{1 \le j \le n} \{x_i(t^*) - x_j(t^*)\}. \tag{7}$$

The quality option clearly qualifies as a special type of exchange option. (We shall prove this momentarily.) So the viability of this approach hinges on whether one can price an n-asset exchange option. We use the n-asset exchange option pricing formula obtained in Hemler (1988).[8]

To see that the quality option is a type of exchange option, assume (A1)–(A3) hold and consider the decision faced by the short at time of delivery. If the short delivers ith issue bonds at time T_1, he receives $C_i(T_1)F(T_0, T_1) + I_i(T_1)$ and gives up bonds worth $S_i(T_1) + I_i(T_1)$. His net profit equals $C_i(T_1)F(T_0, T_1) - S_i(T_1)$. By using the (n, i)-quality option, the short realizes a profit of

$$\max_{1 \le j \le n} \{C_j(T_1)F(T_0, T_1) - S_j(T_1)\} \tag{8}$$

rather than $C_i(T_1)F(T_0, T_1) - S_i(T_1)$. Let us temporarily assume that $C_j \equiv 1$ for $j = 1, \cdots, n$. Then one can rewrite (8) as follows:

$$\begin{aligned} \max_{1 \le j \le n} \{F(T_0, T_1) - S_j(T_1)\} &= F(T_0, T_1) - \min_{1 \le j \le n} \{S_j(T_1)\} \\ &= F(T_0, T_1) - S_i(T_1) + \max_{1 \le j \le n} \{S_i(T_1) - S_j(T_1)\}. \end{aligned} \tag{9}$$

The term $\max_{1 \le j \le n}\{S_i(T_1) - S_j(T_1)\}$ appearing in (9) equals the extra amount received for exercising the (n, i)-quality option. It reduces to (7) when $x_j(\cdot) = S_j(\cdot)$ for $j = 1, \cdots, n$.

Dropping the assumption $C_j(T_1) \equiv 1$ for $j = 1, \cdots, n$, one obtains the following analog of (9):

$$\begin{aligned} &\max_{1 \le j \le n} \{C_j(T_1)F(T_0, T_1) - S_j(T_1)\} \\ &= \max_{1 \le j \le n} \{F(T_0, T_1) - [S_j(T_1) + (1 - C_j(T_1))F(T_0, T_1)]\} \\ &= \max_{1 \le j \le n} \{F(T_0, T_1) - x_j(T_1; T_0, T_1)\} \\ &= F(T_0, T_1) - \min_{1 \le j \le n} \{x_j(T_1; T_0, T_1)\} \\ &= F(T_0, T_1) - x_i(T_1; T_0, T_1) + \max_{1 \le j \le n} \{x_i(T_1; T_0, T_1) - x_j(T_1; T_0, T_1)\} \end{aligned} \tag{10}$$

where

$$x_j(T; T_0, T_1) = (1 - C_j(T_1))F(T_0, T_1) + S_j(T), \qquad (j = 1, \cdots, n). \tag{11}$$

[8] Besides the result given in Hemler (1988), extensions and/or generalizations of the Margrabe (1978) formula have been obtained by Margrabe (1982), Stulz (1982), Cheng (1985, 1986), Chowdhry (1986), and Johnson (1987).

In this case the amount received by the short for exercising the (n, i)-quality option equals $\max_{1\le j\le n}\{x_i(T_1;\, T_0,\, T_1) - x_j(T_1;\, T_0,\, T_1)\}$. The main difference is that now the asset prices $x_j(\cdot;\, T_0,\, T_1)$ $(j = 1, \cdots, n)$ depend on the parameters T_0 and T_1.

To summarize, assume that (A1)–(A3) hold. Given a fixed delivery date T_1, let $c(x_1, \cdots, x_n, T)$ denote the value at time T of an option to exchange any asset j $(j = 1, \cdots, n)$ for a fixed asset i at time T_1. Then for $T \le T_1$,

$$q_i(T;\, T_1,\, n) = c(x_1(\cdot;\, T,\, T_1),\, \cdots,\, x_n(\cdot;\, T,\, T_1),\, T) \tag{12}$$

where the asset prices $x_j = x_j(\cdot;\, T,\, T_1)$ $(j = 1, \cdots, n)$ are as defined in (11). Suppose that each asset price x_j follows geometric Brownian motion, i.e.,

$$\frac{dx_j}{x_j} = \mu_j dt + \sigma_j dz_j, \qquad (j = 1, \cdots, n)$$

where z_j is a Wiener process, μ_j, σ_j are constants, and $\rho_{ij} = \operatorname{corr}(dz_i, dz_j)$ for $j = 1, \cdots, n$. For convenience and without loss of generality, assume $i = 1$. Then

$$c(x_1, \cdots, x_n, T) = x_1 \int_{R^{n-1}} \max\left\{0,\, 1 - \frac{x_2}{x_1}\exp(f_2(z)),\, \cdots,\, 1 - \frac{x_n}{x_1}\exp(f_n(z))\right\} k^{n-1}_{(T_1-T)/2}(z)\, dz \tag{13}$$

where

$$R^{n-1} = (n-1)\text{-dimensional Euclidean space}$$

$$k^{n-1}_\tau(z) = (4\pi\tau)^{-(n-1)/2}\exp\left(-\frac{|z|^2}{4\tau}\right) \qquad (\tau > 0,\, z \in R^{n-1})$$

$$f_j(z) = -\nu_j^2\,\frac{(t^* - t)}{2} - \sum_{i=1}^{n} r_{ji}\sqrt{\lambda_i}z_i \qquad (i = 2, \cdots, n;\, z \in R^{n-1}).$$

and the ν_j, r_{ij}, and λ_i $(2 \le i, j \le n)$ are parameters depending on $x_1, \cdots, x_n$. (See Hemler (1988) for details.) Combining these last two equations yields

$$q_i(T;\, T_1,\, n) = x_1 \int_{R^{n-1}} \max\left\{0,\, 1 - \frac{x_2}{x_1}\exp(f_2(z)),\, \cdots,\, 1 - \frac{x_n}{x_1}\exp(f_n(z))\right\} k^{n-1}_{(T_1-T)/2}(z)\, dz \tag{14}$$

where the asset prices $x_j = x_j(\cdot;\, T,\, T_1)$ $(j = 1, \cdots, n)$ are as defined in (11).

Admittedly, the assumption that each asset price x_j follows geometric Brownian motion is made for tractability rather than for theoretical reasons. The assumption has obvious drawbacks. It is inconsistent with the bond price equaling one at maturity and bond price variance approaching zero at maturity. But it yields tractable closed-form solutions that are relatively easy to evaluate numerically. Furthermore, the bond issues of interest here are at least 15 years to time of maturity/first call, so the constraint that bond prices equal one at maturity should be unimportant empirically. Given the scarcity of empirical work regarding n-asset exchange options, the computational advantages associated with this assumption would seem to outweigh the theoretical drawbacks.

D. Closing Remarks

Equations (6) and (14) play key roles in our analysis. The value of the quality option is unobservable and must be estimated. To do so one can solve for the quality option value $q_i(T_0; T_1, n)$ in terms of the other variables in equation (5); this yields equation (6). Alternatively, one can use an exchange option formula such as the one developed by Margrabe (1978). This leads to equation (14). Both methods have merit. Therefore, we use both in our empirical work.

Although we use both equations (6) and (14) for estimation purposes, it is important to note that the two types of estimates have different properties. In particular, estimates from equation (14) are always nonnegative while estimates from equation (6) are often negative. Estimates from equation (14) are integrals of nonnegative functions and must be nonnegative. On the other hand, estimates from equation (6) consist of ratios whose various terms are based on actual market prices. Such prices are "noisy," leading to estimation error. Whenever that estimation error is large relative to the quality option value, negative estimates are likely.

It is critical to keep this difference in mind when interpreting the empirical results. A significant proportion of the option estimates from equation (6) are negative. Such behavior is reasonable provided that the option value is small relative to the estimation error, i.e., there is a small signal-to-noise ratio where the signal is the true value of the option and the noise is the estimation error. To gauge the relative size of this measurement error, we estimate option values at time of delivery and 3, 6, 9, and 12 months prior to delivery. (See Table III.) Because the option values are being measured relative to the cheapest to deliver issue, the true value of the option at time of delivery should be zero. (That is, given that one has the cheapest to deliver issue, there is no value in switching to another issue.) Hence, the corresponding estimates in Table III should not statistically differ from zero. That is exactly what one finds. The average estimated value is 0.062 percentage points of par; the associated standard error of the mean is 0.065. The median estimate is 0.015, which implies that nearly 50% of the estimates are negative. In a sense, such results tell us that the estimation technique is calibrated correctly. When the option value is known to be zero, the estimates are close to zero and fairly evenly distributed between positive and negative values.

The assumptions (A1)–(A3) invoked to derive these pricing equations are clearly unrealistic. Capital markets are imperfect. Forward and futures prices differ. Traders typically deliver on several days during a given delivery month. But it does not follow that replacing (A1)–(A3) with more realistic assumptions is desirable or necessary.

Consider the assumption that futures and forward prices are equal. In view of Cox, Ingersoll, and Ross (1981), one might expect T-bond futures prices to be less than the corresponding forward prices. This would imply that estimates based on equation (6) are biased upward. However, empirical evidence given in Rendleman and Carabini (1979), Cornell and Reinganum (1981), and Elton, Gruber, and Rentzler (1984) indicates that the futures-forward price differential is insignificant.[9] So adjusting equation (6) for daily resettlement could increase, rather than decrease, estimation error.

Another potential source of estimation error is transaction costs. For example, introducing bid-ask spreads forces one to replace equation (6) with an analogous inequality. However, estimation error due to bid-ask spreads should be relatively small in magnitude. At most it is roughly the size of a typical spread in the T-bond spot market, i.e., four to six 32nds of a percentage point.

Taxes are a third factor that could affect our results. It is difficult to adjust for taxes—tax rules regarding futures contracts change continually. Moreover, in certain cases the inclusion of taxes in the analysis would not affect equation (6). For qualifying hedge transactions the taxpayer records ordinary income or loss when the contract is closed.[10] So taxpayers who invest in a risk-free hedge or in risk-free bonds are taxed at the same rate. Hence, adjusting for taxes in the derivation of (6) would effectively amount to multiplying both sides of (4) by the factor $1 - \tau$ where τ is the tax rate on ordinary income. This factor cancels from both sides of the equality leaving (6) unchanged.

Finally, consider various timing options available to the short. One option is the short's right to deliver on any business day of the delivery month. A "wild-card" option arises because the futures market closes at 2:00 p.m. (Central Standard Time) while the short has until 8:00 p.m. to announce his intention to deliver. Since the T-bond cash market is a dealer market which is open from 2:00 to 8:00 p.m., the short has a 6-hour period during which he can decide to deliver at that day's settlement price. An "end-of-month" option exists because trading in the expiring futures contracts ends on the eighth-to-last business day of the delivery month. The futures settlement price on the last trading day determines the invoice price for bonds delivered during the last 7 business days of the delivery month. Thus, the short has a 7-day period for deciding when to deliver at this fixed invoice price.

A shortcoming of this paper is that it ignores these timing options by invoking assumption (A3), i.e., by assuming all contracts are delivered on some predetermined day. Although assumption (A3) might be unpalatable, studies of the quality option typically have ignored timing options, and vice versa. To gauge the impact of this assumption on our results, we have performed sensitivity analysis by using alternate delivery date assumptions. One assumption is that delivery occurs on the first date eligible for delivery, i.e., the first business day of the delivery month. The other is that the short announces his intention to deliver on the last

[9] French (1983) finds a statistically significant difference between futures and forward prices for silver and copper. But the theoretical models developed in Cox, Ingersoll, and Ross (1981), Richard and Sundaresan (1981), and French (1982) do not help to explain these price differentials.

[10] Taxpayers generally must mark-to-market all regulated futures contracts held at the end of the year and include unrealized gains and losses in income for that year. The net gain or loss (including both realized and unrealized gains and losses) is considered to be a 40% short-term and 60% long-term capital gain or loss. However, transactions that qualify as a hedge for tax purposes are exempt from the mark-to-market rules.

possible trading day for the contract, i.e., the eighth-to-last business day of the delivery month. Because the results are very similar, we do not always report both sets of results in this paper. However, they are available on request from the author.

From our perspective, it is important to realize how ignoring the timing options affects our results. The presence of timing options means that our estimates of the payoffs from exercising the quality option at delivery are biased downward. On the other hand, our estimates based on equation (6) are biased upward to the degree that they represent the combined value of quality and timing options rather than the quality option value alone.

II. Empirical Results

A. The Data

The data consist of T-bond futures and spot prices together with T-bill discount yields. The T-bond futures and spot prices are from Interactive Data Corporation. The futures prices are settlement prices; the spot prices are over-the-counter bid quotations representing mid-afternoon market conditions. Interactive Data Corporation receives its Treasury spot price data from Associated Press, which in turn receives them from the Federal Reserve Bank of New York. The T-bill data are bid and asked discount yield quotations from the Federal Reserve Bank of New York as published in *The Wall Street Journal.* The forty-seven T-bond issues used constitute all issues eligible for delivery at least once prior to or during December 1986.[11]

The conversion factors used for invoicing are calculated as follows:[12]

$$\mathrm{CF} = 1.04^{-X/6}\left[\frac{C}{2} + \left(\frac{C}{0.08}\,[1 - 1.04^{-2N}] + 1.04^{-2N}\right)\right] - \frac{C(6 - X)}{12}$$

where

CF = Conversion factor,
C = Coupon rate of bond in decimal form,
N = Whole number of years to call if callable or to maturity if not callable,
X = Number of months that time to maturity or first call exceeds N, rounded down to the nearest quarter. (If $X = 0$, 3, or 6, use the formula as shown. If $X = 9$, set $2N = 2N + 1$ and $X = 3$.)

They are rounded to four decimal places and equal those given in Chicago Board of Trade (1985a).

Two delivery date assumptions are used. One is that delivery occurs on the first date eligible for delivery, i.e., the first business day of the delivery month.

[11] Two issues merit special attention: the 3% of 1995 and the 3½% of 1998. Both are "flower bonds" acceptable at face value in payment of Federal Estate Tax when owned by the decedent at time of death. They possess very low coupons, trade at relatively high prices, and are unlikely to be delivered. Indeed, no flower bond has ever been delivered, although no rule precludes such delivery.

[12] See Chicago Board of Trade (1985b).

The other is that the short announces his intention to deliver on the last possible trading day for the contract, i.e., the eighth-to-last business day of the delivery month. Each assumption has merit. The first is more common in the literature and is reasonable considering the relative lack of trading volume during the delivery month. The second more accurately describes the actual empirical distribution of the delivery date. One encounters the same problem with either assumption—one is ignoring the timing option available to the short, which could bias estimates.

With these delivery date assumptions in mind, we collected data at 3-month intervals corresponding to the 38 delivery months that occurred prior to 1987. Because both Position and Delivery Days are necessary for the calculations, we use data corresponding to these days for each delivery month given the respective delivery date assumption.[13] For example, the invoicing procedure at delivery uses the Position Day futures settlement price along with interest accrued as of Delivery Day. When there is no risk of confusion, we refer to the 3-day delivery period as one delivery date. When more precision is necessary, we add superscripts to the delivery date. For instance, T_1^P and T_1^D denote the Position and Delivery Days corresponding to the delivery date T_1.

To obtain risk-free rates of return, let T_0 be a fixed delivery date and let T_1, T_2, T_3, and T_4 denote the next four delivery dates. For $i = 0, \cdots, 4$ the holding periods $[T_0^P, T_i^D]$ are of length 2 business days, 3 months, 6 months, 9 months, and 12 months, respectively. Using the five T-bills maturing closest to $T_0^D, \cdots, T_4^D$, we calculate rates of return $R(T_0, T_i)$ from the bid and asked quotations $B(T_0^P, T_i^D)$ and $A(T_0^P, T_i^D)$ via:

$$R(T_0, T_i) = \frac{100}{P(T_0^P, T_i^D)} - 1,$$

where the price $P(T_0^P, T_i^D)$ is given by

$$100 - \left(\frac{\text{Number of Days in } [T_0^P, T_i^D]}{360}\right)\left(\frac{B(T_0^P, T_i^D) + A(T_0^P, T_i^D)}{2}\right).$$

When no T-bill matures exactly at time T_i^D the above formula is adjusted to correct for the different number of days. Risk-free rates for discounting coupon payments are calculated similarly. The four potential coupon payment dates $t_1, \cdots, t_4$ immediately following T_0 satisfy

$$T_0^D < t_1 < T_1^D < \cdots < T_3^D < t_4 < T_4^D.$$

The rate $R(T_0, t_i)$ is computed using the same T-bill used to compute $R(T_0, T_i)$ for $i = 1, \cdots, 4$.

B. Historical Payoffs From Exercising the Quality Option

Our first empirical results concern payoffs obtained from exercising the quality option at delivery. We present realized values $q_i(T_1; T_1, n)$ of the quality option

[13] Recall that delivery takes place over 3 business days. The short announces his intention to deliver on Position Day but only makes delivery 2 days later on Delivery Day.

for the period 1977–1986. Specifically, consider the quality option relative to issue i at the delivery date T_1. Let issue j be the cheapest to deliver issue at time T_1. Then the payoff from exercising this option at delivery equals

$$\begin{Bmatrix} 0, & \text{if } i = j; \\ [C_j(T_1)F(T_1, T_1) - S_j(T_1)] - [C_i(T_1)F(T_1, T_1) - S_i(T_1)], & \text{if } i \neq j. \end{Bmatrix}$$

We use prices from the Position Day T_1^P where issue i is the cheapest to deliver issue 3, 6, 9, or 12 months prior to delivery.[14]

Our decision to study the quality option associated with cheapest to deliver issues impacts heavily on our empirical results. The option value obviously depends on the choice of issue i. The more economical it is to deliver an issue, the smaller the value of the corresponding quality option. For example, consider the quality option associated with "flower bonds." Flower bonds have never been delivered by the short; they must be relatively costly to deliver. Hence, quality options associated with flower bonds must be relatively large in value compared to the quality options examined here.

We measure the quality option relative to the cheapest to deliver issue because that issue provides a natural benchmark. One wants to know whether the quality option plays a significant role in the pricing of T-bond futures. If the market knows issue i is cheapest to deliver prior to delivery and believes it will stay cheapest to deliver, the option is worthless. T-bond futures and spot prices are related as in equation (6) where $q_i(T_0; T_1, n) = 0$. One does not need quality options to explain the determination of futures prices. So although $q_j(T_0; T_1, n)$ might be nonzero for all $j \neq i$, it is not particularly interesting.

Tables I and II summarize the empirical distribution of these payoffs for each delivery date assumption. The results are qualitatively the same. The cheapest to deliver issue prior to delivery generally differs from the cheapest to deliver issue at time of delivery. In each case considered the issues differed at least 75% of the time. The empirical distributions of the payoffs are highly skewed to the right and have a positive probability of equaling zero. Such behavior is consistent with the payoffs coming from a truncated lognormal distribution that one might expect. Most important, the payoffs are typically quite small, both in an absolute sense and relative to the futures price at time of delivery. For example, absolute payoffs in Table I average 0.245, 0.462, 0.663, and 0.780 percentage points of par as the time to delivery increases. Relative payoffs in Table I average 0.332%, 0.619%, 0.886%, and 1.035%, respectively.[15]

[14] As our cheapest to deliver issue prior to delivery, we use the issue which maximizes

$$\left[\frac{C_i(T_1)F(T_0, T_1) + I_i(T_1)}{1 + R(T_0, T_1)} + B(T_0, T_1)\right] - [S_i(T_0) + I_i(T_0)]$$

at time T_0 prior to delivery. Issue i is often called cheapest to deliver if, and only if, it maximizes the implied repo rate RP_i associated with issue i. We maximize the preceding expression rather than RP_i, choosing to maximize net present value rather than internal rate of return. We maximize over all issues eligible for delivery at time T_1 that have been issued prior to time T_0. Terms are calculated on the respective Position Days except for $I_i(T_1)$, which is evaluated at T_1^D.

[15] Standard errors reported in Tables I and II are intended for descriptive purposes, not for hypothesis testing. Suppose the payoffs X_i for a fixed time to delivery are significantly autocorrelated,

Table I

Summary Statistics for Payoffs from Exercising the Quality Option Assuming That the Delivery Day Equals the First Business Day of the Delivery Month

Payoffs are reported in percentage points of par and as a percentage of the futures price used for invoicing at delivery. The payoffs represent the amount gained by delivering the issue that is cheapest to deliver at time of delivery rather than the issue that was cheapest to deliver 3, 6, 9, or 12 months prior to delivery. Given a random variable X, the qth quantile equals the smallest number x such that $P[X \leq x] \geq q$. The quantiles reported here are the corresponding sample analogs.

	Based on the issue that is cheapest to deliver			
Summary statistic	3 months prior to delivery	6 months prior to delivery	9 months prior to delivery	12 months prior to delivery
	Payoffs expressed in percentage points of par			
Minimum	0.000	0.000	0.000	0.000
0.10th Quantile	0.000	0.000	0.000	0.000
0.25th Quantile	0.007	0.015	0.090	0.108
0.50th Quantile	0.139	0.268	0.372	0.399
0.75th Quantile	0.381	0.785	1.089	1.110
0.90th Quantile	0.696	1.272	1.481	1.949
Maximum	1.168	1.300	2.980	3.458
Mean	0.245	0.462	0.663	0.780
(Std. Error)	(0.049)	(0.076)	(0.119)	(0.149)
1st Order Auto.	−0.058	0.070	0.253	0.520
(Std. Error)	(0.164)	(0.167)	(0.169)	(0.172)
	Payoffs expressed as percentage of futures price			
Minimum	0.000	0.000	0.000	0.000
0.10th Quantile	0.000	0.000	0.000	0.000
0.25th Quantile	0.007	0.023	0.136	0.152
0.50th Quantile	0.171	0.298	0.635	0.556
0.75th Quantile	0.574	1.051	1.451	1.529
0.90th Quantile	0.992	1.639	2.234	2.289
Maximum	1.239	2.041	3.285	4.747
Mean	0.332	0.619	0.886	1.035
(Std. Error)	(0.065)	(0.105)	(0.153)	(0.200)
1st Order Auto.	0.025	0.075	0.253	0.476
(Std. Error)	(0.164)	(0.167)	(0.169)	(0.172)

These results contrast sharply with those of Kane and Marcus (1986a). Using a 3-month horizon, they find that the value of the quality option ranges from 1.39 to 4.60 percentage points of par for the September 1981, March 1982, September 1982, and March 1983 contracts. They obtain their values via Monte Carlo analysis based on historically estimated yield curve dynamics. They use estimated term structures over the period 1978–1982, estimates of the volatility of those term structures, and the actual sets of delivery-grade bonds available during this period. They calculate values relative to a hypothetical contract

as would appear likely in the 9 and 12 month cases. Even if the X_i are stationary with $\text{var}(X_i) = \sigma^2$ for $i = 1, \cdots, n$, the equality $\text{var}(\bar{X}) = \sigma^2$ does not necessarily hold. This weakens the rationale for using the standard error of the mean $S/\sqrt{n}$ to estimate the standard deviation of $\bar{X}$. Similar remarks apply for Tables III and IV.

Table II

Summary Statistics for Payoffs from Exercising the Quality Option Assuming That the Position Day Equals the Last Trading Day of the Delivery Month

Payoffs are reported in percentage points of par and as a percentage of the futures price used for invoicing at delivery. The payoffs represent the amount gained by delivering the issue that is cheapest to deliver at time of delivery rather than the issue that was cheapest to deliver 3, 6, 9, or 12 months prior to delivery. Given a random variable X, the qth quantile equals the smallest number x such that $P[X \leq x] \geq q$. The quantiles reported here are the corresponding sample analogs.

	Based on the issue that is cheapest to deliver			
Summary statistic	3 months prior to delivery	6 months prior to delivery	9 months prior to delivery	12 months prior to delivery
	Payoffs expressed in percentage points of par			
Minimum	0.000	0.000	0.000	0.000
0.10th Quantile	0.000	0.051	0.000	0.000
0.25th Quantile	0.035	0.097	0.099	0.052
0.50th Quantile	0.151	0.281	0.421	0.504
0.75th Quantile	0.333	0.618	1.070	1.237
0.90th Quantile	0.663	1.467	1.863	1.863
Maximum	1.476	1.689	2.266	2.972
Mean	0.262	0.468	0.668	0.752
(Std. Error)	(0.058)	(0.081)	(0.116)	(0.144)
1st Order Auto.	−0.073	−0.054	0.306	0.389
(Std. Error)	(0.164)	(0.167)	(0.169)	(0.172)
	Payoffs expressed as percentage of futures price			
Minimum	0.000	0.000	0.000	0.000
0.10th Quantile	0.000	0.071	0.000	0.000
0.25th Quantile	0.048	0.142	0.160	0.072
0.50th Quantile	0.181	0.383	0.583	0.569
0.75th Quantile	0.418	0.755	1.177	1.305
0.90th Quantile	0.869	1.722	1.925	2.404
Maximum	1.952	2.404	3.383	4.142
Mean	0.346	0.605	0.868	0.941
(Std. Error)	(0.078)	(0.109)	(0.153)	(0.180)
1st Order Auto.	−0.031	−0.078	0.279	0.337
(Std. Error)	(0.164)	(0.167)	(0.169)	(0.172)

which would require that the bond most likely to be delivered ex ante (i.e., the most frequently delivered bond according to the simulation results) actually be delivered. Their value equals the average increment to the short's profits over 10,000 simulated outcomes.

Several possible explanations exist for the different results. Likely candidates include:

1) The quality options could be associated with different issues. We use the issue which is cheapest to deliver prior to delivery; they use the issue which their simulations indicate is most likely to be delivered ex ante.
2) We use historical values of the option; they use simulated values.
3) We have one realization available per delivery month; they average 10,000 outcomes.

4) The studies use different bond data. Our data are from Interactive Data Corporation; their data are from The Center for Research in Security Prices (CRSP). Hence, the dates used in the studies differ because the CRSP government bond tapes provide end-of-month prices.
5) Kane and Marcus estimate their yield curve dynamics over the period 1978–1982. Because interest rates were extremely volatile during this period, their estimated term structures might be quite different from the ones which subsequently occurred.
6) Kane and Marcus estimate values for September 1981, March 1982, September 1982, and March 1983. If the values are positively related to interest rate volatility, one expects their estimates to exceed average values of our estimates for the period 1977–1986.

We can only speculate as to which of the above best explains the discrepancy between our results and those in Kane and Marcus (1986a). Given the behavior of interest rates during the period 1978–1982, however, it would seem that the term structures estimated by Kane and Marcus are a prime candidate. Explanations based on the choice of issue and slightly different data (i.e., the first and fourth items above) are unlikely. Nor does the choice of delivery months by Kane and Marcus (i.e., the sixth item listed above) explain the difference because our estimates for those months do not differ significantly from those for other months. Moreover, timing options do not explain the difference because both analyses make similar assumptions regarding the delivery date.

C. Option Estimates Based on the First Pricing Model

Tables I and II give values of the quality option at delivery but provide no information regarding its value prior to delivery. We now obtain such information via equation (6). That is, we estimate the value $q_i(T_0;\, T_1,\, n)$ of the option via the expression:

$$[S_i(T_0) + I_i(T_0)] - \left[\frac{C_i(T_1)F(T_0,\, T_1) + I_i(T_1)}{1 + R(T_0,\, T_1)} + B(T_0,\, T_1)\right].$$

For each delivery date T_1, we use five different time horizons corresponding to the time of delivery and 3, 6, 9, or 12 months prior to delivery. Because we want the quality option associated with the cheapest to deliver issue at time T_0, we calculate the minimum value of this expression over all issues eligible for delivery at time T_1 which also had been issued prior to time T_0. We calculate all terms on the respective Position Days except for $I_i(T_1)$, which is evaluated at T_1^D.

Table III summarizes results assuming delivery occurs on the first business day of the delivery month. (Results for the other delivery date assumption are similar and hence suppressed.) As expected, the average value of the estimates increases with the time to delivery. Average values are small, both in an absolute sense and relative to the futures price at time of delivery. They are always less than either 0.6 percentage points of par or 0.8% of the corresponding futures price, even when the time to delivery equals 12 months. For a time horizon of

Table III

Summary Statistics for Quality Option Estimates Based on a T-bond Futures Pricing Model Assuming That the Delivery Day Equals the First Business Day of the Delivery Month

Estimates are reported in percentage points of par and as a percentage of the futures price used for invoicing at delivery. Estimates are obtained using equation (6). Given a random variable X, the qth quantile equals the smallest number x such that $P[X \leq x] \geq q$. The quantiles reported here are the corresponding sample analogs.

	Based on the issue that is cheapest to deliver				
Summary statistic	At time of delivery	3 months prior to delivery	6 months prior to delivery	9 months prior to delivery	12 months prior to delivery
	Estimates expressed in percentage points of par				
Minimum	−0.812	−1.197	−1.429	−1.515	−1.340
0.10th Quantile	−0.409	−0.735	−0.969	−0.947	−0.904
0.25th Quantile	−0.149	−0.092	−0.211	−0.298	−0.414
0.50th Quantile	0.015	0.241	0.393	0.291	0.384
0.75th Quantile	0.247	0.444	0.806	1.136	1.216
0.90th Quantile	0.600	0.824	1.093	1.387	1.603
Maximum	1.157	1.176	1.854	2.524	2.987
Mean	0.062	0.126	0.267	0.400	0.509
(Std. Error)	(0.065)	(0.095)	(0.137)	(0.167)	(0.187)
1st Order Auto.	0.140	0.551	0.682	0.738	0.701
(Std. Error)	(0.164)	(0.164)	(0.167)	(0.169)	(0.172)
	Estimates expressed as percentage of futures price				
Minimum	−0.861	−1.815	−2.147	−2.254	−1.996
0.10th Quantile	−0.594	−1.237	−1.612	−1.464	−1.408
0.25th Quantile	−0.236	−0.091	−0.310	−0.449	−0.622
0.50th Quantile	0.016	0.295	0.529	0.372	0.425
0.75th Quantile	0.259	0.577	1.112	1.551	1.792
0.90th Quantile	0.893	1.095	1.460	1.800	2.210
Maximum	1.541	1.523	2.343	3.228	3.907
Mean	0.086	0.143	0.321	0.500	0.658
(Std. Error)	(0.088)	(0.131)	(0.188)	(0.229)	(0.259)
1st Order Auto.	0.178	0.592	0.716	0.769	0.746
(Std. Error)	(0.164)	(0.164)	(0.167)	(0.169)	(0.172)

3 months, estimates never exceed a maximum value of either 1.2 percentage points of par or 1.6% of the corresponding futures price.

Although numerous estimates in Table III are negative, that does not indicate per se that the estimation technique is flawed. As discussed earlier in the closing remarks for Section I, such behavior is reasonable provided the option value is small relative to the estimation error. The option value estimates at time of delivery indicate that the estimation technique is calibrated correctly; the true option value is zero while the average option estimate is only about one standard deviation from zero. Hence, the fact that nearly 50% of those estimates are negative does not indicate a problem with the estimation technique.

D. Option Estimates Based on the Second Pricing Model

Estimates based on equation (14) have desirable properties—they are zero at delivery and nonnegative prior to delivery. However, they also pose unique problems. The underlying assumption that each asset price follows geometric Brownian motion with constant drift and infinitesimal variance parameters has distinct drawbacks. It is inconsistent with the bond price equaling one at maturity and bond price variance approaching zero at maturity. But it yields tractable closed-form solutions that are relatively easy to evaluate numerically. Because the bond issues studied are at least 15 years to time of maturity/first call, the computational advantages associated with the assumption of geometric Brownian motion would seem to outweigh the potential liabilities. In addition, when implementing equation (14) one encounters difficulties similar to those associated with the Black–Scholes formula, except that here they take on added dimensions. One must estimate a covariance matrix rather than a variance and calculate a multi-dimensional integral rather than a cumulative (one-dimensional) normal distribution function.

The estimation process is straightforward. We rank the issues in terms of cheapness of delivery; assets 1, 2, and 3 correspond to the cheapest, second cheapest, and third cheapest to deliver issues, respectively. We calculate values of the asset $y_i = x_i/x_1$ for $i = 1, 2, 3$. We estimate the covariance matrix for the corresponding returns via maximum likelihood using the sixty observations immediately preceding the estimation date, provided sixty observations are available. When sixty observations are unavailable, we use the maximum number available.[16] We use the IMSL Fortran subroutine EIGRS to obtain the necessary eigenvalues and eigenvectors of the estimated covariance matrix; the values obtained correspond to the parameters λ_i and r_{ij} appearing in equation (14). In the two-asset case we use the IMSL Fortran subroutine MDNOR to obtain values of the standard normal cumulative distribution function. In the three-asset case we use the NAG (Numerical Algorithms Group Inc.) Fortran subroutine D01FCF to evaluate the two-dimensional integrals.[17]

Table IV summarizes results assuming two or three assets, times to delivery of 3 or 6 months, and delivery on the first business day of the delivery month. (Results corresponding to the other delivery date assumption are very similar and hence suppressed.) Once again the values are smaller in magnitude than

[16] If the most recent issue is one of the cheapest to deliver issues, there might only be 10–12 observations available.

[17] We integrate over the hyper-rectangle $\{(z_1, z_2) \mid -4.0 \le z_1 \le 4.0, -4.0 \le z_2 \le 4.0\}$. (Because the integrand is bounded by the standard bivariate normal density function, the integral over the region outside this rectangle is bounded above by the integral of the standard bivariate normal density function over the same region, which equals 0.0002.) The routine operates by repeated subdivision of this hyper-rectangle into smaller hyper-rectangles. The routine estimates the integral over each subregion and its associated estimated error. If the total estimated error over all subregions exceeds the requested relative error, further subdivision is necessary. The subregion having the largest estimated error is halved, and the process continues until an estimate satisfying the requested relative error is obtained. We use a relative error of 0.001.

Table IV

Summary Statistics for Quality Option Estimates Based on an Exchange Option Pricing Formula Assuming That the Delivery Day Equals the First Business Day of the Delivery Month

Estimates are reported in percentage points of par and as a percentage of the futures price used for invoicing at delivery. Estimates are obtained using equation (14). Given a random variable X, the qth quantile equals the smallest number x such that $P[X \leq x] \geq q$. The quantiles reported here are the corresponding sample analogs.

	Based on the issue that is cheapest to deliver			
Summary statistic	3 months prior to delivery (2-asset case)	6 months prior to delivery (2-asset case)	3 months prior to delivery (3-asset case)	6 months prior to delivery (3-asset case)
	Estimates expressed in percentage points of par			
Minimum	0.134	0.204	0.251	0.362
0.10th Quantile	0.179	0.287	0.307	0.441
0.25th Quantile	0.269	0.383	0.521	0.660
0.50th Quantile	0.666	0.844	1.223	1.805
0.75th Quantile	1.098	1.457	1.935	2.676
0.90th Quantile	1.262	1.731	2.204	3.105
Maximum	2.142	1.983	3.947	4.839
Mean	0.713	0.966	1.243	1.735
(Std. Error)	(0.079)	(0.098)	(0.137)	(0.186)
1st Order Auto.	0.726	0.722	0.693	0.717
(Std. Error)	(0.164)	(0.167)	(0.164)	(0.167)
	Estimates expressed as percentage of futures price			
Minimum	0.174	0.234	0.276	0.404
0.10th Quantile	0.204	0.322	0.384	0.586
0.25th Quantile	0.349	0.413	0.593	0.885
0.50th Quantile	0.803	0.942	1.371	2.104
0.75th Quantile	1.488	2.017	2.468	3.504
0.90th Quantile	1.966	2.780	3.382	4.915
Maximum	3.224	2.894	5.941	7.284
Mean	0.971	1.324	1.690	2.382
(Std. Error)	(0.119)	(0.150)	(0.208)	(0.284)
1st Order Auto.	0.779	0.772	0.752	0.775
(Std. Error)	(0.164)	(0.167)	(0.164)	(0.167)

reported by Kane and Marcus (1986a).[18] Even when one uses three assets, the estimates 3 months prior to delivery only average 1.243 or 1.250 percentage points of par (alternatively, 1.690% or 1.695% of the corresponding futures price) depending on the delivery date assumption used. The maximum values are considerably higher than the average values. For example, the maximum values for a 3-month time horizon correspond to March 1982 and equal 3.947 percentage points of par and 5.941% of the futures price. Such figures are comparable to

[18] Kane and Marcus estimate the payoff from exercising this option at delivery, as opposed to an option value prior to delivery. So their estimates are not exactly comparable to those in Table IV. The estimates in Table IV are closer in spirit to those given by Gay and Manaster (1984). However, even those estimates are not directly comparable because Gay and Manaster study wheat futures rather than T-bond futures.

those obtained by Kane and Marcus (1986a); in our case, however, they probably reflect a lack of data. Because one of the issues used (i.e., the 14% of 2011) was issued on November 16, 1981, we must estimate the corresponding covariance matrix over the period from November 10, 1981 to November 25, 1981. Only twelve observations are available for estimation purposes. Furthermore, since four of these observation dates occur prior to the issue date of November 16, one must use "When Issued" prices for those dates rather than true spot prices. Thus, the covariance matrix estimates (and hence the option estimates) are questionable. Indeed, when one estimates the corresponding option value using the alternative delivery date assumption, the values drop to 2.887 percentage points of par and 4.638%, respectively.

III. Conclusion

Our paper uses three estimation techniques to obtain quality option values in the T-bond futures market during the sample period 1977–1986. We find that the payoff obtained by switching from the bond cheapest to deliver 3 months prior to delivery to the one cheapest to deliver at time of delivery averages less than 0.3 percentage points of par. Similarly, option values obtained from the T-bond futures pricing model average less than 0.2 percentage points of par 3 months prior to delivery. The corresponding option values based on the exchange option pricing formula are somewhat higher, averaging approximately 0.7 to 1.2 percentage points of par when $n = 2$ and $n = 3$, respectively. The results reinforce those of Hegde (1988) and, to a lesser extent, Gay and Manaster (1984), but differ substantially from those of Kane and Marcus (1986a); they apply only to quality options associated with cheapest-to-deliver issues. Other quality options can, and will, attain larger values.

One can view these results from the perspective of market efficiency. Recall equation (6) which gave the value of the option based on a T-bond futures pricing model. Negative option estimates based on equation (6) when $T_0 = T_1$ correspond to positive payoffs for an investor from the following hedging strategy: buy the ith bond on Position Day, take the corresponding short futures position, and announce the intention to deliver 2 business days later. The larger the payoffs from this strategy, the greater the evidence of inefficiency. Kolb, Gay, and Jordan (1982) evaluate similar payoffs on fifteen dates, one for each delivery month from December 1977 through June 1981. In general, their results are consistent with the efficiency of this market. Resnick and Hennigar (1983) evaluate similar payoffs using daily data from January 1979 through May 1981 for contracts that expire in the next 2 delivery months. Interestingly, they conclude that the T-bond futures market was less efficient when T-bill yields, or the risk-free borrowing rate, were high. They find a positive relation between the T-bill rate and the net hedge return, with a majority of the hedge returns positive when the T-bill bid yields exceeded 10%. Such results are consistent with evidence presented here. The corresponding option estimates in Table III are often negative, especially in 1980–1982 when T-bill yields were high.

Though our paper answers certain questions concerning the quality option, it also raises new ones. Perhaps the most interesting questions concern the option estimates obtained via equation (6). The results in Kolb, Gay, and Jordan (1982), Resnick and Hennigar (1983), and Hemler (1988) suggest that deviations from equation (6) are related to the level or volatility of interest rates. An interesting question is: What is driving the time series behavior of deviations from the T-bond futures pricing model? Is it the level or volatility of interest rates, and if so, why? In short, how can one improve equation (6)? We hope further research will answer these and other related questions.

REFERENCES

Arak, M. and L. Goodman, 1987, Treasury bond futures: Valuing the delivery options, *Journal of Futures Markets* 7, 269–286.

Benninga, S. and M. Smirlock, 1985, An empirical analysis of the delivery option, marking to market, and the pricing of Treasury bond futures, *Journal of Futures Markets* 5, 361–374.

Boyle, P., 1989, The quality option and timing option in futures contracts, *Journal of Finance* 44, 101–113.

Broadie, M. and S. Sundaresan, 1988, The pricing of timing and quality options: An application to Treasury bond and Treasury note futures markets, Working Paper, Graduate School of Business, Columbia University.

Carr, P., 1988, Valuing bond futures and the quality option, Working Paper, Graduate School of Management, University of California (Los Angeles).

Cheng, S., 1985, Pricing models for multiple-currency option bonds, Working Paper, Graduate School of Business, Columbia University.

———, 1986, Multi-asset contingent claims in a stochastic interest rate environment, Working Paper, Graduate School of Business, Columbia University.

Chicago Board of Trade, 1980, *Delivery Manual: U.S. Treasury Bonds and Notes* (Chicago Board of Trade, Chicago).

———, 1985a, *Chicago Board of Trade Conversion Factors* (Financial Publishing Company, Boston).

———, 1985b, *The Chicago Board of Trade's Municipal Bond Futures Contract* (Chicago Board of Trade, Chicago).

———, 1986, *CBOT U.S. Treasury Bond Futures* (Chicago Board of Trade, Chicago).

———, 1987, *The Delivery Process in Brief: Treasury Bond and Treasury Note Futures* (Chicago Board of Trade, Chicago).

Chowdhry, B., 1986, Pricing of multivariate contingent claims, Unpublished manuscript, Graduate School of Business, University of Chicago.

Cornell, B. and K. French, 1983, Taxes and the pricing of stock index futures, *Journal of Finance* 38, 675–694.

——— and M. Reinganum, 1981, Forward and futures prices: Evidence from the foreign exchange markets, *Journal of Finance* 36, 1035–1045.

Cox, J., J. Ingersoll, and S. Ross, 1981, The relation between forward prices and futures prices, *Journal of Financial Economics* 9, 321–346.

Elton, E., M. Gruber, and J. Rentzler, 1984, Intra-day tests of the efficiency of the treasury bill futures market, *The Review of Economics and Statistics* 66, 129–137.

French, K., 1982, The pricing of futures and forward contracts, Unpublished Ph.D. Dissertation, University of Rochester, New York.

———, 1983, A comparison of futures and forward prices, *Journal of Financial Economics* 12, 311–342.

Gay, G. and S. Manaster, 1984, The quality option implicit in futures contracts, *Journal of Financial Economics* 13, 353–370.

——— and S. Manaster, 1986, Implicit delivery options and optimal delivery strategies for financial futures contracts, *Journal of Financial Economics* 16, 41–72.

——— and S. Manaster, 1989, Equilibrium futures prices in the presence of implicit delivery options: Theory and applications, Unpublished manuscript, College of Business Administration, Georgia State University.

Hegde, S., 1988, An empirical analysis of implicit delivery options in the Treasury bond futures contract, *Journal of Banking and Finance* 12, 469–492.

———, 1989, On the value of the implicit delivery options, *Journal of Futures Markets* 9, 421–437.

Hemler, M., 1988, The quality delivery option in Treasury bond futures contracts, Unpublished Ph.D. Dissertation, University of Chicago.

Johnson, H., 1987, Options on the maximum or minimum of several assets, *Journal of Financial and Quantitative Analysis* 22, 277–283.

Kane, A. and A. Marcus, 1986a, The quality option in the Treasury bond futures market: An empirical assessment, *Journal of Futures Markets* 6, 231–248.

——— and A. Marcus, 1986b, Valuation and optimal exercise of the wild card option in the Treasury bond futures market, *Journal of Finance* 41, 195–207.

Kolb, R., G. Gay, and J. Jordan, 1982, Are there arbitrage opportunities in the Treasury-bond futures market?, *Journal of Futures Markets* 2, 217–229.

Livingston, M., 1987, The delivery option on forward contracts, *Journal of Financial and Quantitative Analysis* 22, 79–87.

Margrabe, W., 1978, The value of an option to exchange one asset for another, *Journal of Finance* 33, 177–186.

———, 1982, A theory of the price of a claim contingent on N asset-prices, Working Paper, School of Government and Business Administration, George Washington University, Washington, D.C.

Rendleman, R. and C. Carabini, 1979, The efficiency of the Treasury bill futures market, *Journal of Finance* 34, 895–914.

Resnick, B. and E. Hennigar, 1983, The relationship between futures and cash prices for U.S. Treasury bonds, *Review of Research in Futures Markets* 2, 282–299.

Richard, S. and M. Sundaresan, 1981, A continuous time model of forward prices and futures prices in a multigood economy, *Journal of Financial Economics* 9, 347–371.

Stulz, R., 1982, Options on the minimum or the maximum of two risky assets: Analysis and applications, *Journal of Financial Economics* 10, 161–185.