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Optimal Hedging under Intertemporally Dependent Preferences

ERIC BRIYS, MICHEL CROUHY, and HARRIS SCHLESINGER*

ABSTRACT

This paper examines optimal hedging behavior in a market where preferences for current consumption are partly determined by the consumer's past consumption history. The model considers an individual exposed to price risk, who allocates wealth between consumption and futures contracts over a (continuous-time) finite planning horizon. The speculative component of the hedge ratio is shown to be smaller and the consumption path smoother than in models where preferences are separable over time. Some comparative-static properties of the hedge ratio are also examined.

CONTINUALLY CHANGING INTEREST RATE LEVELS coupled with rising price volatility have led to the creation of various futures markets. These markets have experienced a remarkable rate of growth throughout the world and have led to new hedging opportunities, which effect a better control of the risk exposures faced by market participants. This growing activity has fostered an abundant literature. Indeed, since Stein's (1961) and Johnson's (1960) seminal contributions, a fairly large body of research has dealt with optimal hedging behavior in futures and/or forward markets (see for instance Ederington (1979), Anderson and Danthine (1980), Losq (1982), Benninga, Zilcha, and Eldor (1984)). These articles assume that hedgers are monoperiodic expected-utility maximizers. A crux result is that the optimal hedge ratio can be divided into two components. The first one is named the pure hedge, which mainly depends on the covariance between futures prices and cash prices. The second component is qualified speculative as it depends on risk aversion and behavior of futures prices.

Relatively few papers on hedging have used a continuous-time framework, whereby the hedger maximizes the expected utility of intertemporal consumption subject to a wealth dynamics constraint. Stulz (1984) and Ho (1984), for instance, use a framework similar to Merton (1969) and bring additional insights into the optimal consumption and hedging rules. As in most contributions in continuous-time financial models, these insights are achieved at the cost of assuming preferences are additively separable over time. A given rate of consumption is supposed to provide the same level of satisfaction at any date regardless of one's

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past consumption experience (see for instance Samuelson (1969) and Merton (1969, 1971)). Such an assumption is obviously at odds with the often reported stable behavior of per capita consumption series (see Grossmann and Shiller (1981)). The complementarity of consumption in successive periods has been recognized by Hicks (1965) and by Ryder and Heal (1973) in the context of the theory of optimal growth. Sundaresan (1985) and Bergman (1985) have reframed Merton's initial contributions by relaxing the assumption of intertemporally independent preferences. Not surprisingly, such an improvement significantly changes the optimal consumption and portfolio choices.

The purpose of the present paper is to examine optimal hedging behavior in a market where current tastes depend, in part, on the consumer's level of past consumption. The intertemporal dependence of preferences is introduced in a parsimonious fashion as in Ryder and Heal (1973).

Section I presents the model and its basic assumptions. First-order conditions for optimal consumption and optimal hedging are derived and discussed in Section II. Section III presents closed-form solutions to the problem at hand for the case of isoelastic marginal utility. It is shown that the speculative component of the optimal hedge is smaller, ceteris paribus, in markets with "memory." As a consequence, the optimal hedging policy and optimal consumption policy are intertemporally dependent. In Section IV, some properties of the optimal hedge ratio are examined. A short conclusion summarizes the main findings and suggests further avenues of research.

I. The Model and Its Assumptions

Consider an investor whose total wealth is invested in a "spot" asset subject to price risk.[1] This investor can hedge his holding by selling (going short in) futures contracts. The futures price F is assumed to follow a geometric Brownian motion:

$$\frac{dF}{F} = \mu dt + \sigma dz, \tag{1}$$

where dz is the increment of a standard Wiener process: μ and σ are, respectively, the expected value and standard deviation of futures price changes per unit of time. When μ is positive, the futures market exhibits normal backwardation. When μ is negative, contango prevails. When μ is zero, the futures market is an unbiased estimator of future spot prices.

These conditions work in a similar manner to "premium loadings" in an insurance market. If we expect hedgers to be net short, then speculators in the market will be net long in futures contracts. Assuming downward sloping demand curves for both groups (hedgers and speculators), this leads to a futures price below the expected future spot price. In other words, speculators expect to earn a profit by buying futures contracts at prices below the expected spot price and cashing out at the expected future spot price. If we think of an insurer as

[1] To keep the analysis as simple as possible, the model does not incorporate a wealth allocation decision.

essentially "buying" a risky wealth position from a client and paying a certain (or at least "more certain") wealth position (initial wealth minus an insurance premium) in return, this hedging transaction is seen to be exactly what it is, a basic insurance transaction. "Buying" the risky wealth at a price lower than its expected value is insurance with a positive premium loading, and it corresponds to normal backwardation in the futures setting. Similarly, contango is easily shown to be analogous to insurance at a subsidized price. These insurance analogies will help us later in interpreting our results.

The price of the spot asset at time t is denoted $S(t)$ and satisfies the cost of carry relationship,

$$S(t) \equiv F(t)e^{-b(T-t)}, \tag{2}$$

where b is the constant cost of carry over time and T the maturity of the futures contract.[2] Applying Itô's Lemma obtains the following price dynamics for S,

$$\frac{dS}{S} \equiv (\mu + b)dt + \sigma dz. \tag{3}$$

Note that under our assumptions, F and S are affected by the same Wiener process. In other words, the spot price and the futures price are perfectly correlated. Although such an assumption is not necessary to obtain our results, it simplifies the analysis considerably while allowing us to focus on the effects arising out of the intertemporally dependent nature of preferences.

The wealth constraint can now be written by denoting W, the individual's total wealth:

$$dW \equiv \{(\mu + b)W - c - \alpha W\mu\}dt + (1 - \alpha)\sigma W dz, \tag{4}$$

where c stands for consumption per unit time. The individual is assumed to hedge his spot position by selling $\$\alpha W$ of futures contracts, where α is the hedge ratio to be selected by the investor.

To capture the nonseparability of preferences over time, we assume that the hedger's utility function at time t, denoted $u(c(t), p(t), t)$, depends not only on the current consumption rate $c(t)$ but also on the weighted average of the past consumption rates, denoted by $p(t)$, where

$$dp \equiv \beta(c - p)dt. \tag{5}$$

Following Ryder and Heal (1973), β can be interpreted as a memory coefficient. We assume $\beta > 0$.[3]

The objective of the investor is then to maximize the expected utility of his intertemporal consumption over his hedging horizon, T, which coincides with the

[2] As in most futures markets the cost of carry b is positive. We assume that the spot asset does not yield any coupon and/or that the so-called convenience yield is negligible.

[3] Time complementarity is introduced along the lines suggested by Ryder and Heal (1973) and used more recently by Sundaresan (1985). More specifically p is an exponentially weighted average of past consumption rates, $p(0) = 0$. Under discrete time p would be written as: $p(t + 1) = \beta c(t) + (1 - \beta)p(t)$. If $\beta = 0$, then $p(t) = 0$ for all t.

maturity of the futures contract:

$$\max_{c,\alpha} E \int_0^T u(c(t), p(t), t)\, dt,$$

subject to (4), (5), and $W(0) = W_0 > 0$, where E denotes the expectation operator. We further require that the optimal controls, c and α, be chosen such that wealth remains positive at all times. For the sake of simplicity, we assume that the opportunity set is fixed and that there is no bequest motive.

II. The Equation of Optimality and Optimal Consumption and Hedging Rules

Let

$$J(W(t), p(t), t) \equiv \max_{c,\alpha} E \int_t^T u(c(s), p(s), s)\, ds$$

denote the indirect utility function. The state of the system at time t is completely described by the vector $(W(t), p(t))$. Applying Bellman's principle of dynamic programming yields the equation of optimality:

$$-J_t = \max_{c,\alpha} \{u(c, p) + [(\mu + b)W - \alpha\mu W - c]J_W + \beta(c - p)J_p + \tfrac{1}{2}(1 - \alpha)^2\sigma^2 W^2 J_{WW}\}, \quad (6)$$

where the subscripts represent partial derivatives.

The first-order conditions of optimality are:[4]

$$u_c(c^*, p) - J_W + \beta J_p = 0, \quad (7)$$

$$-\mu J_W + (\alpha^* - 1)\sigma^2 W J_{WW} = 0. \quad (8)$$

Equation (7) is the envelope condition for consumption. It differs from the traditional one because the marginal utility at a given date now depends on the future marginal utilities of the current consumption standard. Current consumption not only affects future utility through its effect on W, as in models without intertemporal dependence, but also through its effect on p. Note also that this condition differs from the one that obtains under a stochastic opportunity set and/or state-dependent preferences. The reason for this difference is that, under intertemporally dependent preferences, p is not an exogenous state variable, but changes deterministically over time. Indeed, differentiating (7) with respect to p yields:

$$J_{wp}(W, p, t) = u_{cc}(c, p)\frac{\partial c}{\partial p} + u_{cp}(c, p) + \beta J_{pp}(W, p, t).$$

Unless $\beta = 0$ or $J_{pp}(W, p, t) = 0$, the structure of the present problem differs from that encountered when stochastic opportunity sets and/or state-dependent preferences are introduced.[5]

[4] We assume interior solutions and require $[c(t), W(t)] \in \mathbb{R}_+^2\ \forall t$ for our solution to be valid.

[5] For more details the reader is referred to Ingersoll (1987, equation (55), p. 283 and equation (78), p. 289).

Equation (8) gives the optimal hedge ratio and can be rewritten as:

$$\alpha^* = 1 - \frac{\mu}{\sigma^2}\left[\frac{-J_W}{WJ_{WW}}\right]. \tag{9}$$

The first term of (9) is the pure hedge component that is equal to one. This should come as no surprise given our perfect correlation of spot and futures prices assumption. The second term is the speculative component of the futures position. When the futures market is unbiased ($\mu = 0$), the investor fully hedges his spot asset. Indeed, the futures market provides an actuarial coverage and full coverage becomes optimal "insurance" (see Mossin (1968) and Smith (1968)). When normal backwardation prevails in the futures markets ($\mu > 0$), the hedger asks for less than full coverage, exactly as in an insurance market with "loaded" prices. The speculative component is negative and depends on the hedger's risk aversion. This is seen by noting that the second term in (9) has a factor of $-J_W/WJ_{WW}$, which is a measure of relative risk tolerance over the investor's planning horizon. A lower level of risk tolerance (higher degree of risk aversion) would imply a hedge ratio closer to a pure hedge, $\alpha^* = 1$. If the futures price obeys contango, more than full coverage is optimal. In this case, the speculative component is positive.

These results are obviously not new and can be traced back to the early works of Johnson and Stein. What is new, however, is that the speculative hedge now depends on the standard of consumption $p(t)$ via the indirect value function. It is interesting to note that α^* can be negative in the case of normal backwardation. In this case, the investor goes long in futures contracts and is, in essence, a speculator rather than a hedger. This is sometimes referred to as a "Texan hedge," in tribute to the famous market squeeze of the Hunts in the silver market.

III. Closed Form Solutions under Isoelastic Marginal Utility

To gain additional insights and explicit solutions to the optimal consumption and hedging rules, we will assume that the hedger's preferences are characterized by isoelastic marginal utility. This utility function is fairly common in the literature for models with intertemporally separable preferences (see, for example, Merton (1969, 1971) and Ingersoll (1987)). More recently, Sundaresan uses this specification in an infinite-horizon optimal-consumption model with intertemporally dependent preferences. In particular, consider the following instantaneous utility function:

$$u(c, p) \equiv \frac{(c-p)^\gamma}{\gamma} \qquad (\gamma < 1, \gamma \neq 0),$$

where $(1-\gamma)$ is the coefficient of relative risk aversion.

For isoelastic marginal utility the optimal consumption rate is:[6]

$$\begin{aligned} c^*(t) &= p(t) + [g(\beta, t)](bW(t) - p(t)) \\ &= [1 - g(\beta, t)]p(t) + [g(\beta, t)]bW(t), \end{aligned} \tag{10}$$

[6] The indirect utility function is given by $J(W, p, t) = [\gamma(b+\beta)]^{-1}g(\beta, t)^{\gamma-1}(bW - p)^\gamma$.

where

$$g(\beta, t) \equiv a/\{[1 - e^{a(t-T)}](b + \beta)\}$$

$$a \equiv \left(b + \frac{1}{2}\frac{\mu^2}{\sigma^2}\frac{1}{1-\gamma}\right)\left(\frac{-\gamma}{1-\gamma}\right).$$

Note that the optimal consumption at time t is to consume more than the past consumption standard, $p(t)$. We also see that the optimal consumption is a linear combination of current wealth and past consumption. The case of intertemporally separable preferences follows in our model by setting $\beta = 0$, in which case $p(t)$ is also zero. Note also that $g(\beta, t)$ is monotonically decreasing in β. The current consumption path when $\beta > 0$ will be less volatile than in the case of intertemporally separable preferences (cf. Merton (1969, 1971) and Ingersoll (1987)). This "smoothing" of consumption over time, as well as other properties of the optimal consumption path, are discussed by Sundaresan (1985) and will not be pursued here.

Instead, we turn our attention to the optimal hedge:

$$\alpha^*(t) = 1 - \frac{\mu}{\sigma^2(1-\gamma)} + \frac{p(t)\mu}{\sigma^2(1-\gamma)bW(t)}. \tag{11}$$

Properties of the optimal hedge will be discussed in the following section. For our solution in (10) and for our indirect utility function to make any economic sense, we must have $c - p > 0$. This will follow if we can show that $bW(t) - p(t) > 0 \; \forall t$, because it is easily seen that $g(\beta, t) > 0$ for $\gamma \neq 0$.

To prove this, define $y(t) \equiv bW(t) - p(t)$. Thus, $dy(t) = bdW(t) - dp(t)$. Using the dynamics of W and p, as given in equations (4) and (5), together with the optimal consumption and hedging rules from (10) and (11), we obtain:

$$\frac{dy(t)}{y(t)} = m(t)dt + ndz, \tag{12}$$

where

$$m(t) \equiv b - (b + \beta)g(\beta, t) + \mu^2/[\sigma^2(1-\gamma)]$$

$$n(t) \equiv \mu/[\sigma(1-\gamma)].$$

Stochastic integration of (12) yields:[7]

$$y(t) = \frac{y(0)\exp\{At + \mu z/[\sigma(1-\gamma)]\}\{1 - \exp[a(t-T)]\}}{1 - \exp(-aT)}, \tag{13}$$

where

$$A \equiv (b + \tfrac{1}{2}\mu^2/\sigma^2)(1-\gamma)^{-1},$$

$z \equiv$ normal variate with mean zero and variance t.

Because $p(0) = 0$ and $W(0) > 0$, it follows that $y(t) = bW(t) - p(t) > 0 \; \forall t < T$. Thus, $c(t)$ remains positive as does $W(t)$, because $W(t) > (p(t)/b) > 0 \; \forall t$ as well. Thus our solutions as given in (10) and (11) are well defined.

[7] Detailed computations are available from the authors.

It is also worth noting that $y(t)$ converges to zero as $t \to T$. Thus, $W(T) > 0$ unlike in Merton (1969).[8] The reason for this is that the individual needs to be certain that $c(t) > p(t)$. To ensure this the futures market plays a crucial role. Indeed, from the optimal hedge ratio (11), we see that $\alpha^*(t) \to 1$ as $t \to T$. Thus, by adjusting his futures position, the individual controls and progressively eliminates the source of randomness. In the course of doing so, $W(T)$ will be positive and $c(T) > p(T)$ almost surely.[9] Fluctuations in wealth are accommodated by the futures position, instead of being directly transmitted to the consumption function as in Merton (1969). The futures market thus enables the individual to monitor his randomness exposure.

IV. Properties of the Optimal Hedge Ratio

Some properties of the optimal hedge ratio are noteworthy. Under intertemporally independent preferences and biased markets, the optimal hedge ratio would be limited to the first two terms of the right-hand side of (11). In the presence of time complementarity, however, the speculative component of the optimal hedge is modified by a third term. This additional term has a nontrivial impact on the optimal hedging policy because the latter is no longer time invariant. The level of the hedge now depends on the consumption standard $p(t)$, the cost of carry b, and wealth $W(t)$, even though we are considering isoelastic preferences and a nonstochastic opportunity set. In biased markets past consumption obviously interacts with current hedging.

At this point, a caveat is in order. We note that the third term on the right-hand side of equation (11) does not imply that the investor is attempting to hedge against changes in $p(t)$, as would be the case if $p(t)$ were some exogenous state variable. As we mentioned previously, $p(t)$ changes deterministically over time, and there is no need for investors to use asset positions to hedge against it.[10] However, the presence of this term does affect the optimal hedge against price risk whenever $\mu \neq 0$. Indeed, because $bW - p > 0$, we see that this third term serves to lessen the intensity of the second term, and the hedge is thus closer to a pure hedge.

Table I summarizes some comparative static properties of the optimal hedging policy. This table describes how the speculative component of the optimal hedge reacts to changes in various parameters. The first two partial derivatives are easily interpreted. Both a lower cost of carry and a higher degree of (instantaneous) relative risk aversion cause the speculative component of the optimal hedge to decrease. The effects of changes in either the level of wealth, W, or in the past consumption index, p, may be less apparent. The utility function we are considering exhibits constant relative risk aversion only as a relative of $(c - p)$

[8] Whereas $c(t)/W(t) \to \infty$ as $t \to T$ in Merton's model, $[c(t) - p(t)]/[bW(t) - p(t)] = g(t) \to \infty$ in our model with intertemporally dependent preferences. This does not, however, imply that $c(T) - p(T)$ is infinite. Indeed, substituting (13) into the original consumption equation (10) shows that $c(t) - p(T)$ is positive and finite.

[9] We are grateful to an anonymous referee for bringing this point to our attention.

[10] This is similar to equilibrium in stochastic-opportunity-set models, where a dual-beta CAPM collapses to a single-beta CAPM whenever the state variable (representing the stochastic-opportunity-set) evolves deterministically over time (see Ingersoll (1987)).

Table I

Comparative Static Properties of the Optimal Hedge Ratio under Isoelastic Marginal Utility

The hedge ratio in our model, α^*, consists of a pure hedge (equal to one) plus a "speculative component" that is negative when the drift of futures prices, μ, is positive (normal backwardation), and positive when the drift is negative (contango). The effects of changes in the cost of carry, b, the instantaneous degree of relative risk aversion, $1 - \gamma$, the wealth level, W, and the weighted index of past consumption p, are shown.

	$\mu > 0$	$\mu < 0$	Description
$\frac{\partial \alpha^*}{\partial b}$	<0	>0	Larger speculative component
$\frac{\partial \alpha^*}{\partial (1-\gamma)}$	>0	<0	Smaller speculative component
$\frac{\partial \alpha^*}{\partial W}$	<0	>0	Larger speculative component
$\frac{\partial \alpha^*}{\partial p}$	>0	<0	Smaller speculative component

and only at an instant in time. A more canonical measure of risk aversion is the inverse of the "risk tolerance" factor in equation (9). Define $R_J \equiv -WJ_{WW}/J_W$. Thus R_J denotes a measure of relative risk aversion over the planning horizon, assuming the investor behaves optimally. We see from equation (9) that increases in R_J decrease the speculative component of the hedge ratio, as should be expected. Now, given the utility function we are considering, it is straightforward to show that $R_J = (1 - \gamma)bW/(bW - p)$. From this, it follows that $\partial R_J/\partial W < 0$ and $\partial R_J/\partial p > 0$. It is through these indirect effects on relative risk aversion that we obtain the third and fourth comparative static results above.

When W rises at a point in time, a decreased level of risk aversion causes the speculative component of the hedge ratio to increase. Thus α^* is farther from one (perfectly hedged) than it would be otherwise. When p increases at a point in time, the consumer needs to increase consumption to maintain utility and to adjust the hedge ratio so that it is closer to one. This also implies that the aforementioned "Texan hedge" (long spot and long futures) is less likely than it would be otherwise.

V. Conclusion

This paper has shown that the interdependence of preferences over time has a nontrivial impact on the optimal hedging policy. The total hedge is closer to the pure hedge than the one traditionally derived in the literature (at least in biased markets and under isoelastic marginal utility). That is, the speculative component of the hedge ratio is always smaller in our model. For our particular specification of consumer "memory" and consumer tastes, the consumer needs to ensure himself that his consumption path is always rising. Thus the consumer in our model is seen to behave more prudently than would otherwise be the case.

Our result is easily extendable to an infinite time horizon, although this would necessitate a different story in terms of defining "futures contracts." It is also possible to derive an analytic solution for the case of constant absolute risk aversion, where instantaneous utility is given as

$$u(c, p) = -e^{(rp-sc)}, \quad r > 0, \quad s > 0.$$

Unfortunately, the solution under this specification is likely to lead to a negative level of wealth (see Sundaresan (1985)). Thus most of our results derived in this paper are limited to the case of isoelastic marginal utility.

In spite of this limitation, our results should be of interest for both theoretical and empirical research on forward/futures markets. As might be expected, optimal consumption exhibits an endogenous smoothing behavior that fits with the observed regularity of per capita consumption statistics. In particularly, the optimal consumption path was shown to be less sensitive to fluctuations in random wealth for higher values of the "memory coefficient," β.

Further avenues of research are of course possible and include such additions as a production decision, a stochastic opportunity set, and a random basis. A better picture of the optimal hedging behavior would certainly result from such an effort. Hopefully, the results derived here are a useful start toward such an end.

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