Equilibrium Exchange Rate Hedging

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ABSTRACT

We assume a world like the one that gives the capital asset pricing model, but with many goods and many countries. We assume that investors in a given country have homothetic utility functions with the same weights, and a currency that has a sure end-of-period value using a price index with those weights. Siegel's paradox (derived from Jensen's inequality) makes investors want a positive amount of exchange risk. When average risk tolerance is the same across countries, every investor will hold the same mix of market risk (through the world market portfolio of all assets) and exchange risk (in a diversified basket of foreign currencies). In fact, the ratio of exchange risk to market risk is equal to the average investor's risk tolerance. We can write the ratio of exchange risk to market risk (and the fraction of the market's exchange risk that investors hedge) as depending on an average of world market risk premia, an average of world market volatilities, and an average of exchange rate volatilities. The weights in these averages are the same as the weights of the different countries in the currency basket. Given these averages, the ratio (and the fraction hedged) will not depend directly on exchange rate means or covariances. In equilibrium, we can use the ratio of exchange risk to market risk to measure average risk tolerance: in this model, risk tolerance is observable.

$$1 - \lambda = \frac{\mu_m - \sigma_m^2}{\mu_m - \frac{1}{2}\sigma_e^2}$$

λ = average risk tolerance; ratio of exchange risk to market risk;
$1 - \lambda$ = fraction of exchange risk hedged;
μ_m = average world market portfolio expected excess return;
σ_m^2 = average world market portfolio excess return variance;
σ_e^2 = average exchange rate variance.

I. The Capital Asset Pricing Model with Many Currencies

The capital asset pricing model, as derived by Sharpe (1964) and others, is a one-period model where investments have a joint normal distribution of payoffs. Assuming that the period is infinitesimal, investors who maximize expected utility will care only about the mean and variance of portfolio return, and

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investors can borrow or lend equilibrium amounts of the single good without fear of default.

Define the forward price of any asset as the current price plus interest to the end of the period, and excess return as the fractional difference between the asset's payoff and its forward price. Maximizing expected utility is equivalent to maximizing the mean excess return for a given variance or equivalent to minimizing the variance of excess return for a given mean.

To extend this model to a world with many currencies, let us assume, as in Grauer, Litzenberger, and Stehle (1976), that each investor has homothetic utility defined over some or all of a number of goods. This means that utility of end-of-period consumption is the product over goods of consumption of each good to a power. The powers are the weights in a price index for an investor with this utility.

We will define a group of investors with the same price index as a "country." We will define a "currency" for each country as having a known end-of-period value using the corresponding price index. An "exchange rate" will be the relative price of two currencies.

An asset may pay off in any good or combination of goods. An asset's return will depend on the currency from which investors view it.

Investors can borrow or lend in any currency. An exchange rate contract will be a combination of borrowing in one country and lending in another. The return on foreign borrowing or lending and the return on an exchange rate contract will depend on the currency used to figure the return. As before, the excess return is the payoff less the forward value, as a fraction of the forward value. Thus, the excess return on domestic lending is always zero.

We will assume that we know the joint distribution of asset payoffs and foreign currency values from any domestic currency point of view. This joint distribution will itself be the result of an equilibrium created after uncertainty about tastes and technology is resolved. As before, we will assume that the interval is infinitesimal and the distribution is joint normal.

Solnik (1974) developed the first equilibrium model of this kind. Fischer (1975), Grauer, Litzenberger, and Stehle (1976), Sercu (1980), Stulz (1981), Adler and Dumas (1983), and Trevor (1986) have generalized Solnik's model and have derived more results. For example, they find that all investors hold shares of the world market portfolio of all assets mixed with borrowing or lending in different currencies.

In fact, when investors know the end-of-period value of every currency, they find that every investor will hold a mix of a "universal logarithmic portfolio" with domestic lending. For example, see Adler and Dumas (1983, p. 952). The universal logarithmic portfolio may include foreign borrowing or lending along with the world market portfolio of all assets.

This means that every investor does the same amount of hedging in each foreign currency. Swiss, German, and Japanese investors all short the same amount of French francs against each unit of the world market portfolio that they hold.

We will assume, as these authors do, no taxes, trading costs, or other barriers to international investment or disinvestment. Thus, investors in our world will hold the world market portfolio hedged to a constant extent in each foreign currency.

To focus on the portfolios of investors with different domestic currencies, we will assume that each country has only one investor. Thus, investor i will be a typical investor with domestic currency i.

We will use the following symbols:

f_{wi}	= fraction of world wealth held by investor i;
f_{mi}	= fraction of world assets held by investor i;
f_{li}	= lending by investor as a fraction of world wealth or assets;
s_j	= total gross short position in currency j as a fraction of gross short position in all currencies; also weight on currency j in averages μ_m, σ_m^2, σ_e^2, μ, and σ;
w_{ij}	= short position by investor i in currency j as a fraction of wealth;
w_{im}	= fraction of investor i's wealth in the market;
λ_i	= Lagrange multiplier representing risk tolerance for investor i (ratio of variance to mean for his portfolio excess return);
λ	= constant representing average risk tolerance;
μ_{ij}	= mean excess return for investor i from lending in currency j;
μ_{im}	= mean excess return for investor i on the market portfolio;
μ_i	= mean excess return on investor i's portfolio;
μ_m	= weighted average over all currencies of the market's mean excess return as seen from that currency;
μ	= average over investors of mean excess portfolio return when all investors have the same risk tolerance;
σ_{ijk}	= covariance between excess returns on lending currencies j and k for investor i;
σ_{ijm}	= covariance between excess returns on lending currency j and on the market for investor i;
σ_{ij}^2	= variance of excess return on lending currency j for investor i;
σ_{im}^2	= variance of excess return on the market for investor i;
σ_e^2	= weighted average over all pairs of currencies of variance of the exchange rate;
σ_i^2	= variance of excess return on investor i's portfolio;
σ_m^2	= weighted average over all currencies of the market's variance as seen from that currency;
σ_n^2	= variance of the weighted average excess return on the market;
σ^2	= average over investors of the variance of excess portfolio return when all investors have the same risk tolerance.

The investor's problem is

$$\text{minimize} \quad \sigma_i^2 = w_{im}^2 \sigma_{im}^2 - 2 w_{im} \textstyle\sum_j w_{ij} \sigma_{ijm} + \sum_{jk} w_{ij} w_{ik} \sigma_{ijk} \tag{1}$$

$$\text{subject to} \qquad \mu_i = w_{im}\mu_{im} - \textstyle\sum_j w_{ij}\mu_{ij}. \tag{2}$$

Taking derivatives of equation (1) subject to (2) and using Lagrange multipliers λ_i, we have

$$w_{im}\sigma_{im}^2 - \textstyle\sum_j w_{ij}\sigma_{ijm} = \lambda_i\mu_{im}, \tag{3}$$

$$w_{im}\sigma_{ijm} - \textstyle\sum_k w_{ik}\sigma_{ijk} = \lambda_i\mu_{ij}. \tag{4}$$

Let us try a solution to (3) and (4) of the form

$$w_{im} = \lambda_i/\lambda = f_{mi}/f_{wi}, \tag{5}$$

$$w_{ij} = \lambda_i s_j(1 - \lambda)/\lambda = f_{mi}s_j(1 - \lambda)/f_{wi}, \tag{6}$$

where λ is a constant. Then (3) and (4) become

$$\sigma_{im}^2 - (1 - \lambda)\textstyle\sum_j s_j\sigma_{ijm} = \lambda\mu_{im}, \tag{7}$$

$$\sigma_{ijm} - (1 - \lambda)\textstyle\sum_k s_k\sigma_{ijk} = \lambda\mu_{ij}. \tag{8}$$

Siegel (1972, 1975) applies Jensen's inequality to exchange rates and shows that μ_{ij} and μ_{ji} do not add to zero. (This is sometimes called "Siegel's paradox.") In fact, they satisfy

$$\mu_{ij} + \mu_{ji} = \sigma_{ij}^2. \tag{9}$$

Similarly, Jensen's inequality implies that

$$\sigma_{ijm} + \sigma_{jim} = \sigma_{ij}^2, \tag{10}$$

$$\sigma_{ijk} + \sigma_{jik} = \sigma_{ij}^2. \tag{11}$$

Adding to (8) the result of reversing subscripts i and j in (8), using (9)–(11), and simplifying, we have

$$\textstyle\sum_k s_k = 1. \tag{12}$$

Thus, we have verified our trial solutions for w_{im} and w_{ij}.

Multiplying equation (6) by f_{wi} and summing over i, we have

$$s_j = \textstyle\sum_i f_{wi}w_{ij}/(1 - \lambda). \tag{13}$$

Thus, $(1 - \lambda)s_j$ is the total short position in currency j as a fraction of world wealth. Summing over j, we find that $1 - \lambda$ is the total short in all currencies as a fraction of world wealth.

Investor i's share of the total short position is $(1 - \lambda)f_{mi}$. Assuming that gross lending is equal to gross borrowing, investor i's gross lending in his own currency must be $(1 - \lambda)s_i$. His net lending is the difference between these two, or $(1 - \lambda)(s_i - f_{mi})$.

However, investor i's net lending must also be his fraction of world wealth minus his fraction of world assets. Thus,

$$f_{wi} - f_{mi} = (1 - \lambda)(s_i - f_{mi}), \tag{14}$$

$$s_i = (f_{wi} - \lambda f_{mi})/(1 - \lambda). \tag{15}$$

The weights s_i depend on the average risk tolerance for the world and on total wealth and total assets held by investors in each country (or on total wealth and risk tolerance for investors in each country). The weights do not depend on means, variances, or covariances, except insofar as these are used in estimating world risk tolerance.

Multiplying (7) by s_i and summing, we have

$$\sum_i s_i \sigma_{im}^2 - (1 - \lambda) \sum_{ij} s_i s_j \sigma_{ijm} = \lambda \sum_i s_i \mu_{im}. \tag{16}$$

Multiplying (10) by s_i and s_j and summing, we have

$$\sum_{ij} s_i s_j \sigma_{ijm} = \tfrac{1}{2} \sum_{ij} s_i s_j \sigma_{ij}^2. \tag{17}$$

Combining (16) and (17), we have

$$\sigma_m^2 - \tfrac{1}{2}(1 - \lambda)\sigma_e^2 = \lambda \mu_m, \tag{18}$$

$$\lambda = (\sigma_m^2 - \tfrac{1}{2}\sigma_e^2)/(\mu_m - \tfrac{1}{2}\sigma_e^2), \tag{19}$$

$$1 - \lambda = (\mu_m - \sigma_m^2)/(\mu_m - \tfrac{1}{2}\sigma_e^2). \tag{20}$$

Note that the weight s_i has two roles: s_i is the fraction of total gross short currency position that every investor holds in currency i, and s_i is the weight on the mean and variances for currency i in forming the averages μ_m, σ_m^2, and σ_e^2.

Similarly, λ has two roles: λ is the world average risk tolerance, and $1 - \lambda$ is the total short position that each investor holds in currencies to hedge his asset portfolio.

Multiplying equation (5) by f_{wi} and summing, we see that λ is the world average risk tolerance using weights f_{wi}:

$$\lambda = \sum_i f_{wi} \lambda_i. \tag{21}$$

Thus, λ is the one average that uses weights other than s_i. Since the λ_i's are positive, λ will be positive.

Adler and Dumas (1983, p. 952) give a result that leads directly to the fact that optimal hedging depends on this average of investor risk tolerances.

Another result of Jensen's inequality is as follows:

$$\sigma_m^2 - \sigma_n^2 = \tfrac{1}{2}\sigma_e^2. \tag{22}$$

Since σ_n^2 is a variance, this means

$$\sigma_m^2 \geq \tfrac{1}{2}\sigma_e^2. \tag{23}$$

Since λ is positive, (19) and (23) imply

$$\mu_m \geq \tfrac{1}{2}\sigma_e^2. \tag{24}$$

In other words, average exchange rate variance sets a lower bound on both the average over investors of the market's variance and the average over investors of the market's return.

Note that the short position in each currency depends on the wealth and risk tolerance of investors in that currency. It does not depend on the distribution of asset locations across countries or on any means, variances, or covariances except (through equation (20)) certain average means and variances.

This is similar to the original capital asset pricing model. In that model, the optimal portfolio for any investor could depend on expected excess returns and covariances among excess returns on all available assets. In equilibrium, though, the optimal portfolio for any investor is a mix of the market portfolio with borrowing or lending. The expected returns and covariances cancel one another, so they do not affect the investor's optimal holdings.

In the special case where risk tolerances are the same across countries, we can figure the average variance and mean for an investor's portfolio. Multiplying (1) and (2) by s_i and summing, setting f_{mi} equal to f_{wi}, and using (9)–(11), we have

$$\sigma^2 = \sigma_m^2 - \tfrac{1}{2}(1 - \lambda)\sigma_e^2, \tag{25}$$

$$\mu = \mu_m - \tfrac{1}{2}(1 - \lambda)\sigma_e^2. \tag{26}$$

Similarly, we can figure an average variance and mean when all investors hedge by an amount that is not optimal. The results look just like (25) and (26), where we take λ to be the actual amount left unhedged rather than the optimal amount.

Equations (5) and (6) imply that every investor mixes one part market risk with λ parts exchange risk. Then he chooses an amount of this mix to hold based on his risk tolerance. Since λ is always positive, every investor takes some exchange risk. In other words, the optimal amount of exchange rate hedging is always less than 100%.

II. A Simple Example

To understand why investors will bear some exchange risk in equilibrium, let us study a simple symmetric world with two currencies and two assets. Each investor starts with the same amount of each asset.

Assume that each unit of asset 1 has a sure payoff of one unit of currency 1, and each unit of asset 2 has a sure payoff of one unit of currency 2. The exchange rate between the two currencies will be 2:1 or 1:2 with equal probability.

Since everything is symmetric, the assets must have a one-for-one relative price at the start. Thus, investors can trade claims so that each investor holds a riskless position as measured in his own currency.

However, the payoff from holding the other asset is substantial. The expected payoff from holding one unit of the other asset unhedged is the average of 2 and .5, or 1.25. By trading a unit of one asset for a unit of the other, each investor trades an asset with an expected payoff of 1 for an asset with an expected payoff of 1.25.

We can also say that each investor trades an asset with a zero mean excess return and no risk for an asset with a positive mean excess return and some risk. Starting from a position of zero risk and zero mean excess return, every investor will want to make this trade. The only question is how much he will want to trade. Eventuallly, the added risk will more than offset the added mean excess return.

The gains from taking currency risk come entirely from Siegel's paradox. This result carries over to more complex cases as well: in general, investors hedge the currency risk in their stock portfolios by less than 100% because of Siegel's paradox.

III. The Limiting Case

Note that, as exchange rate volatility approaches zero, equations (19) and (20) approach

$$\lambda = \sigma_m^2/\mu_m, \tag{27}$$

$$1 - \lambda = (\mu_m - \sigma_m^2)/\mu_m. \tag{28}$$

Taking $1 - \lambda$ as the fraction hedged for each investor, we see that the fraction hedged approaches a limit that is neither zero nor one.

As exchange rate volatility declines, the cost of too much hedging or too little hedging declines, but the optimal amount of hedging approaches a limit given by equation (28).

IV. Interpreting the Weights

In the special case where risk tolerances are the same across currencies, the weights s_i in equation (15) are just the fractions of wealth held by investors in each country:

$$s_i = f_{wi} = f_{mi}. \tag{29}$$

In this case every investor holds a share of the world market portfolio without borrowing or lending, except the borrowing and lending implied by hedging. Every investor's hedge portfolio is short currencies in proportion to the fractions given by (29).

Thus, in this case we can say that each investor hedges each of his investments by the fraction $1 - \lambda$. The fraction hedged is universal. For further discussion of this case, see Black (1989).

When risk tolerances are not the same across countries, each investor mixes a portfolio of stocks and currencies with domestic borrowing or lending. His lending is

$$f_{li} = f_{wi} - f_{mi}. \tag{30}$$

From (5), (6), (15), and (30), we have

$$w_{ij} = (1 - \lambda) w_{im} f_{wj} + \lambda w_{im} f_{lj}. \tag{31}$$

In other words, the currency hedges of each investor consist of short positions in proportion to market value weights as before, plus a portfolio of "cross hedges" consisting of the domestic lending and borrowing positions of conservative and aggressive investors.

V. Siegel's Paradox

Equation (9) expresses Siegel's paradox. However, what if we replace it with (32), where the means add to zero?

$$\mu_{ij} + \mu_{ji} = 0. \tag{32}$$

Then equation (12) becomes (33):

$$\textstyle\sum_k s_k = 1/(1 - \lambda). \tag{33}$$

Using equation (6), we see that, when risk tolerances are equal across countries, every investor hedges 100% of his exchange risk rather than a fraction $(1 - \lambda)$.

Thus, we see that investors keep some exchange risk in their portfolio entirely because of Siegel's paradox.

VI. Observables

In the original capital asset pricing model, the expected excess return on the market μ_m depends on average risk tolerance. We cannot observe μ_m directly, and we normally do not think of risk tolerance as observable, so I take μ_m to be unobservable.

We can in principle observe borrowing and lending, so we can observe relative risk tolerance, but we cannot observe average risk tolerance.

That changes in this multicurrency version of the capital asset pricing model. Assuming that we can observe security holdings and borrowing and lending and forward positions, we can observe both μ_m and average risk tolerance. We just look at the average fraction of their currency risk that people leave unhedged.

Since we can observe relative risk tolerance too, we can observe every investor's risk tolerance.

All this happens because the reward for currency risk is fixed by the mathematics of Siegel's paradox, while the reward for equity risk depends on risk tolerance; thus, the optimal mix of the two risks depends on risk tolerance too.

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