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On Viable Diffusion Price Processes of the Market Portfolio

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ABSTRACT

The assumption that the market portfolio follows a specified diffusion process implies, in a simple equilibrium framework, that the representative individual must have a certain utility function which is identified in the paper. Not every diffusion process is viable, i.e., can be "endogenized" to be the market portfolio's price process in such an equilibrium model. The paper provides necessary and sufficient conditions for viability which imply that viable diffusion processes constitute a rather restricted family.

IT IS COMMON IN financial economics, especially in the option pricing literature, to take a certain price dynamics of an "underlying asset" as a primitive assumption. The results are then seemingly "preference free," but it is known that, if one wishes to embed such a "partial equilibrium" model in a general equilibrium framework, this can be obtained only with certain individuals' preferences. For example, the Black-Scholes (1973) model in which the underlying asset follows a geometric Brownian motion is known to be consistent with utility functions which exhibit constant proportional risk aversion (CPRA). (See Rubinstein (1976), Breeden and Litzenberger (1978), Brennan (1979), Stapleton and Subrahmanyam (1984), and Bick (1987).) By this we mean that, if the market portfolio follows a geometric Brownian motion, then necessarily the utility function of the "representative individual" exhibits CPRA. Conversely, one can construct a simple economy with a stock and a riskless bond where identical individuals have CPRA-type utility functions and where the endogenous stock price (enumerated by the bond) follows a prespecified geometric Brownian motion. However, the utility function which corresponds in equilibrium to a general diffusion process has never been identified. This was the initial motivation for this paper. The problem, loosely stated, is as follows. Suppose we are told that in a simple equilibrium model as above the market portfolio follows a given diffusion process. What does it imply regarding individuals' preferences?

It is clear that the problem is not well formulated if a dividend process is also postulated (e.g., as in Merton (1973, 1977)). In this case, consistency with two processes, the price process and the dividend process, is to be required. This

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problem is solved in Bick (1987) for the special case of the Merton proportional dividend model, but in general it is, of course, more difficult.[1] In this paper we concentrate, therefore, on the case where there are no dividends and consumption naturally takes place only at a final date. Admittedly, such a setup cannot be claimed to represent real-world situations, but on the other hand even this easier problem has not been solved yet.

It turns out that the problem involves much more than identifying the associated utility function because the given process may be inconsistent with an equilibrium. We call a given price process "viable" (relative to the above framework) if there exists a utility function exhibiting risk aversion such that the optimal policy of the representative individual who is endowed with the total terminal output is not to trade at any time. In an equilibrium with a representative agent, the optimal policy ought to be not to trade, and, conversely, given a viable price process we show that one can define a model where this process represents equilibrium prices. The term "viable" is borrowed from Kreps (1981) and Harrison and Kreps (1979), where it is used in a somewhat different context. (See also Huang (1985).)

It should be mentioned that any arbitrage-free price system can be sustained in a competitive equilibrium with a representative agent, simply by taking the preferences which are defined by the price functional. (See Harrison and Kreps (1979).) This paper restricts further the set of "reasonable" price processes by imposing the requirement that preferences are represented by a von Neumann-Morgenstern utility function exhibiting risk aversion. Such a representation, which is standard in the finance literature, is possible under rather palatable assumptions.[2] Thus, our objective is to give a complete characterization of the diffusion processes which can be endogenized to be the price process of the market portfolio in a simple continuous-time economy where the representative agent has a concave von Neumann-Morgenstern utility function and where consumption takes place only at the final date. The methodology of the paper can most likely be extended to the case of multiperiod consumption, but this will not be done here.

The paper is organized as follows. First, in Section I, we study two discrete-time examples which help to clarify what makes a process viable and how the associated utility function can be computed. In Section II we analyze the continuous-time setup, focusing on diffusion price processes. We give several equivalent conditions on such processes which are necessary for viability, and we identify the associated utility function. This utility function can be expressed either in terms of the transition probability density (the marginal utility is the continuous-time analogue of the quotient of the elementary contingent claim price and the corresponding probability) or in terms of the diffusion coefficients. The Arrow-Pratt risk aversion measure is equal to the quotient of the local mean and the local variance. In the case where the process is time homogeneous, several other conditions are presented. One of them, which amounts to a simple test on

[1] Two more recent working papers which address related issues are Kim (1987) and Stapleton and Subrahmanyam (1988).

[2] See Huang and Litzenberger (1988), Chapter 1 or Duffie (1988), Chapter 5, and references therein.

the diffusion coefficients, is particularly appealing. We also give a condition on a utility function to be associated with a time homogeneous viable process. In Section III we show that the above conditions are also sufficient: for a given viable process, one can define a two-asset economy where this will be the relative price process of the risky asset. Section IV summarizes the paper.

I. Discrete-Time Examples

Before we tackle the continuous-time case, let us first gain some understanding of the problem by examining two discrete-time examples. In both examples, as well as in the ensuing continuous-time case, economic activity takes place in an exchange economy at times $t \in J$, where J is a discrete set or an interval on the real line with a maximal element (terminal date) T. There is one consumption good ("ripe crop") which may be consumed only at time T. It is produced by a finite number of production units whose output is entirely exogenous. The ownership of each production unit is represented by one equity share which is traded frictionlessly in the market. Also traded frictionlessly is a promissory note, termed "bond," which pays one unit of consumption at time T with certainty. Its net supply is zero. The bond serves as a numeraire; i.e., its price at any time is identically 1. The price of the aggregate equity at time t, termed "the market portfolio," will be denoted S_t.

We assume that there is one price-taker individual, interpreted as a representative "stand in" for a large number of identical individuals, or alternatively as a "consensus individual" in the sense of Rubinstein (1974). This individual is endowed with all the above stocks, and his/her preferences are represented by a state-independent von Neumann-Morgenstern utility function $U(\cdot)$ of date-T consumption defined on $I = (0, \infty)$ which is twice continuously differentiable and satisfies $U' > 0$ and $U'' \leq 0$. In the rest of the paper, a "utility function" will always mean a function with the above properties.

Suppose $\{S_t;\ t \in J\}$ is regarded by the individual as a positive stochastic process[3] defined on some underlying probability space. As we will see in Section III, this is related to the information observed by the individual. Since in this section we only discuss necessary conditions, there is no need to elaborate on this. Relations between random variables will always be understood to hold almost surely. Let E be the expectation operator, and let E_t be the expectation conditioned on the information at time t. (In the sequel E_t is always computed in terms of transition probabilities, and this is why we allow ourselves here to avoid excess notation and be somewhat imprecise.) We assume that $E[|U(S_T)|] < \infty$.

We will need the following fact: a necessary condition for an equilibrium is that

$$S_t = E_t[U'(S_T)S_T]/E_t[U'(S_T)] \tag{1}$$

[3] Our analysis will also be valid with minor modifications in the case that S_t may be negative and $I = (-\infty, \infty)$, provided that negative wealth is somehow meaningful. We will use this fact in the examples, but the main results will be derived under the more demanding assumptions that $I = (0, \infty)$ and zero is an inaccessible barrier for S_t.

for any $t \in J$. This well known first-order condition is obtained by rearranging the equality

$$0 = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} E_t[U((1-\varepsilon)S_T + \varepsilon S_t)] = E_t[U'(S_T)(S_t - S_T)]. \tag{2}$$

This equality holds because in equilibrium the optimal strategy ought to be not to trade. Hence, it is suboptimal for the individual to replace the market portfolio at time t by a portfolio of $1 - \varepsilon$ units of the market portfolio and εS_t bonds and hold it to time T. There are two subtle points in this derivation. First, the differentiation inside the expectation needs to be justified. Second, $U(\cdot)$ is not assumed to be defined outise $(0, \infty)$; hence, (2) is legitimate only if there exist $\delta > 0$ such that for each $\varepsilon \in (-\delta, \delta)$ the random variable $(1 - \varepsilon)S_T + \varepsilon S_t$ is positive almost surely. In general this need not be satisfied. Instead of giving a general rigorous proof here, we will comment on this issue later when (1) is applied.

Example 1: Suppose that the price of the market portfolio follows a binomial process as in Cox, Ross, and Rubinstein (1979), where in our case the interest rate is zero. The rate of return at each node is $u - 1$ or $d - 1$ (where $d < 1 < u$), and assume for simplicity that the current price is $S_0 = 1$. Consider first a three-date case ($t = 0, 1, 2$) where the probabilities of "up" in the next period as viewed from the nodes 1, u, and d are q, q_1, and q_2, respectively. (See Figure 1.) We assume that $q, q_1, q_2 > (1 - d)/(u - d) \equiv p$.

Now, we may fix $U'(ud) = 1$, and at time $t = 1$ at node u equation (1) becomes[4]

$$u = [q_1U'(u^2)u^2 + (1 - q_1)ud]/[q_1U'(u^2) + (1 - q_1)].$$

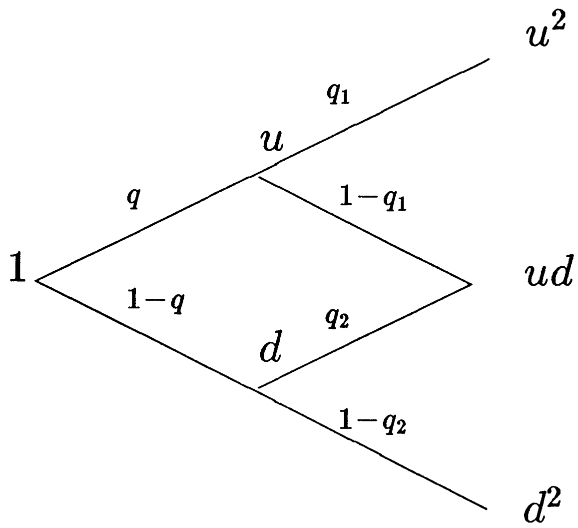

Figure 1. A two-period binomial price process with state-dependent probabilities.

[4] It is not difficult to see that equation (2) goes through, as the conditional expected utility under the ε-perturbation is $q_1U(u^2 + \varepsilon(u - u^2)) + (1 - q_1)U(ud + \varepsilon(u - ud))$. Also notice that in the latter expression the arguments of $U(\cdot)$ are strictly positive for ε in a sufficiently small neighborhood of the origin.

Rearranging gives $U'(u^2) = (1 - q_1)(1 - d)/q_1(u - 1)$. Similarly, the first-order condition at node d becomes $U'(d^2) = q_2(u - 1)/(1 - q_2)(1 - d)$. These are necessary conditions for the existence of an equilibrium in the above framework which sustains the price process as in Figure 1. However, such an equilibrium need not exist. This is because there is another requirement which is imposed by the first-order condition at $t = 0$. Here equation (1) gives

$$1 = \frac{qq_1U'(u^2)u^2 + [q(1-q_1) + (1-q)q_2]U'(ud)ud + (1-q)(1-q_2)U'(d^2)d^2}{qq_1U'(u^2) + [q(1-q_1) + (1-q)q_2]U'(ud) + (1-q)(1-q_2)U'(d^2)}.$$

Recalling that $U'(ud) = 1$ and substituting the above expressions for $U'(u^2)$ and $U'(d^2)$, we obtain that $q(1 - q_1) = (1 - q)q_2$.

A similar result can be obtained in the n-period binomial model. If the price process $\{S_j\}$ is sustained by an equilibrium (in a framework as above), then necessarily:

(B1) All trajectories with the same terminal value have the same probability.

Thus, a general binomial model with arbitrary probabilities at each node may be inconsistent with an equilibrium as above.

Before turning to the second example, let us give an alternative formulation of (B1). It will seem more cumbersome, but it will help us later to see the analogy between the discrete-time and continuous-time cases. For any event G (which, with the obvious identification, may be viewed as a set of trajectories), let $P(G)$ be its probability, let 1_G be the random variable which is 1 if G occurs and zero otherwise, and let $c(G)$ be the price of the security which pays 1_G at the terminal time n. (More exactly, $c(G)$ is the cost of the strategy in the market portfolio and in lending/borrowing which produces the payoff 1_G.) If G includes only one element (i.e., one trajectory), say with j "ups," then one can show that[5] $c(G) = p^j\,(1 - p)^{n-j}$. If G_j denotes the set of all trajectories with exactly j "ups," then clearly, for each $G \subset G_j$, $c(G) = mp^j\,(1 - p)^{n-j}$ and $P(G) = mq^j\,(1 - q)^{n-j}$, where m is the number of trajectories in G. Using these facts, it is left to the reader to verify that (B1) is equivalent to the following property[6]:

(B2) For each j, $c(G)/P(G) = c(G_j/P(G_j)$ for all $G \subset G_j$.

That is, for a fixed j, the price of 1_G divided by the probability of G is the same for all events G for which S_n can only take the value u^jd^{n-j}.

Example 2: Consider a setup as in the beginning of the section, but now we assume that the market is open for trade only at $t = 0$. Assume that there exist two continuous probability density functions, $g(\cdot)$ and $f(\cdot)$ defined on $(0, \infty)$ such that $g(\cdot)$ is the probability density of S_T and $f(\cdot)$ is a pricing function for the derivative securities of S_T. By this we mean that, for every real continuous fraction $h(\cdot)$ on $(0, \infty)$ such that $\int_0^\infty |h(y)|f(y)dy < \infty$, the payoff $h(S_T)$ is traded (possibly with zero net supply) and its price at $t = 0$ is $\int_0^\infty h(y)f(y)dy$. In analogy to a well known discrete-state result,[7] this entails

[5] See Huang and Litzenberger (1988), Chapter 8.

[6] This is motivated by Breeden and Litzenberger (1978), Section V.

[7] See, e.g., Huang and Litzenberger (1988), Section 5.5

$$U'(x) = af(x)/g(x), \tag{3}$$

where $a \equiv E[U'(S_T)]$ is a constant. Indeed, let $x > 0$ and let $c(x)$ be the price at $t = 0$ of the derivative security (a European call option) paying $(S_T - x)^+$ at time T (where $\alpha^+ \equiv \max(0, \alpha)$). Then $c(x) = \int_x^\infty (y - x)f(y)dy$. On the other hand, the first-order condition is[8] $c(x) = a^{-1} \int_x^\infty U'(y)(y - x)g(y)dy$. Now compare the two expressions for $c(x)$ and then differentiate twice with respect to x, and equation (3) will follow.

The main point in this example is that, if all derivative securities of S_T are traded, then the pricing functions together with the probability density function completely determine the utility function. This will extend to the continuous-time case, with the difference that here we regarded the pricing function as given, while in the continuous-time case it is determined "by arbitrage."

II. Viability of Diffusion Price Processes

We turn now to the continuous-time case. Let $\{S_t;\ t \in [0, T]\}$ be a positive diffusion evolving according to

$$dS_t = \alpha(t, S_t)dt + \beta(t, S_t)dz_t, \tag{4}$$

where $\{z_t\}$ and $\alpha(t, S)$ and $\beta(t, S)$ are defined for $t \in [0, T]$ and $S > 0$ and satisfy Lipschitz and growth conditions (in S).[9] We will assume that $\{S_t\}$ has a transition probability density function (representing transition probability from time-state (t, w) to time-state (τ, y)) which will be denoted $g(t, w, \tau, y)$. The transition probability density function of the diffusion process $\{Y_t\}$ satisfying $dY_t = \beta(t, Y_t)d\hat{z}_t$, where β is as in equation (4) and $\{\hat{z}_t\}$ is a standard Wiener process, will be denoted $f(t, w, \tau, y)$. Let us also denote $k(t, w, \tau, y) \equiv f(t, w, \tau, y)/g(t, w, \tau, y)$.

We will say that a stochastic process $\{S_t; \in [0, T]\}$ is a "viable market-portfolio price process" (in an economy with terminal consumption) or, in short, a viable process if there exists a utility function (which will be called "the utility function associated with the price process") such that $\{S_t\}$ is the equilibrium price process of the market portfolio in the economy delineated in the beginning of Section I.[10] In particular, the individual's optimal strategy must be not to trade at any time.

In this section we will present necessary conditions for viability. (As we shall see in the next section, they are also sufficient.) Note that, if the process is viable, then in particular it is obvious that there do not exist "doubling strategies" (see Harrison and Kreps (1979)) that yield arbitrage profits. In what follows we

[8] This equality, which says that $c(x) = E[U'(S_T)(S_T - x)^+]/E[U'(S_T)]$, is derived as in equation (2), using the fact that it is suboptimal to purchase ε call options financed by short selling of bonds. A more rigorous proof can be given exactly as in Bick (1987), Proposition 2.

[9] See Duffie (1988), Chapter IV. There exist alternative sets of assumptions on α and β that ensure existence and uniqueness of a solution $\{S_t\}$ to (4) and the validity of the other results that we will use later. See, e.g., Friedman (1975), Chapters 5 and 6 and Karatzas and Shreve (1988), Chapter 5.

[10] For the definition of continuous time equilibrium, see Duffie (1986, 1988) and Duffie and Huang (1985).

will characterize viability in terms of the transition density function and also in terms of the diffusion coefficients. We will need the following mathematical result:

PROPOSITION 1: *Suppose $\{S_t;\ t \in [0, T]\}$ is a stochastic process as above and $V(\cdot)$ is a positive continuous function such that*[11]

$$H(t, S) \equiv \int_0^\infty V(y)\, g(t, S, T, y)dy < \infty, \tag{5}$$

$$G(t, S) \equiv \int_0^\infty V(y)y\, g(t, S, T, y)dy < \infty. \tag{6}$$

for any $S > 0$ and t. Then:

(i) *the following conditions are equivalent:*
  (a) $S = G(t, S)/H(t, S)$,
  (b) *$H(t, S)$ satisfies $\alpha(t, S)H + \beta^2(t, S)\partial H/\partial S = 0$ or, equivalently,*

$$H(t, S) = F(t)\exp(-\textstyle\int_1^S [\alpha(t, \eta)/\beta^2(t, \eta)]d\eta) \tag{7}$$

  for some function $F(t)$,
  (c) *$k(t, S, t_1, S_1) = H(t_1, S_1)/H(t, S)$ for all $t < t_1$ and $S, S_1 > 0$,*
  (d) *$k(t, S, T, x) = V(x)/H(t, S)$,*
  (e) *there exists a positive function $\psi(t, S)$ defined for $t \in [0, T]$ and $S > 0$ such that $k(t, S, T, x) = V(x)\psi(t, S)$;*

(ii) *if the equivalent conditions* (a)–(e) *are satisfied, this implies that*

$$V(S) = \gamma\cdot\exp(-\textstyle\int_1^S [\alpha(T, \eta)/\beta^2(T, \eta)]d\eta), \tag{8}$$

where γ is a positive constant or, equivalently, $-V'(S)/V(S) = \alpha(T, S)/\beta^2(T, S)$.

The proof is given in the Appendix.

It is clear that, if a function $V(\cdot)$ satisfying (a)–(e) exists, then it can be unique at most to within a positive constant multiple. Part (ii) of the proposition asserts that it is indeed unique in this respect. It is also important to realize that, for an arbitrary diffusion process as in equation (4), the function $V(\cdot)$ from equation (8) is the "candidate" to satisfy (a)–(e), but this property may or may not be satisfied.

COROLLARY: *If $\{S_t\}$ is a viable process and $U(\cdot)$ is the associated utility function, then $V \equiv U'$* satisfies the equivalent conditions (a)–(e), *in particular, $\alpha(T, S) \geq 0$ for all S.*

Proof: The fact that the individual's optimal policy is not to trade entails equation (1). This was proved informally in the beginning of Section I, and a more rigorous proof which is applicable in this setup is given in Bick (1987, Proposition 2). The proof utilizes the fact that call options on S_T with all possible exercise prices can be produced by dynamic trading strategies. (See Merton (1977) and Duffie (1988), Section 22.) Equation (1) implies that property (a) is

[11] In what follows, the number T will be held fixed; hence, the dependence on T will usually be omitted from the notation.

satisfied. The fact that $\alpha(T, S) \geq 0$ follows from Proposition 1(ii) and the assumption that $V' = U'' \leq 0$. □

It follows in particular that, if the process is viable and f and g are known, then the associated $U(\cdot)$ can be identified: f/g (evaluated at T) must be of the form $V(x) \cdot \psi(t, S)$, and then necessarily $V(x) = U'(x)$ to within a constant multiple. This property, which is also equivalent to the statement that $(\partial/\partial x)\ln k(t, S, T, x)$ is only a function of x, need not be satisfied in general, and thus we immediately see that a general process need not be viable.

Example 3: To illustrate this, consider the Ornstein-Uhlenbeck process $dS_t = -\mu S_t dt + \sigma dz_t$, where μ and σ are positive constants.[12] This process takes values in the entire real line and has a negative drift for $S_t > 0$. This is an easy example that condition (a) need not be satisfied, and thus, even if we allow negative wealth and do not require risk aversion (i.e., α may be negative), this process will not be viable. Indeed,

$$g(t, S, T, x) = (\pi\kappa(\tau))^{-1/2} \exp\{-[x - Se^{-\mu\tau}]^2/\kappa(\tau)\},$$

where $\tau = T - t$ and $\kappa(\tau) = \sigma^2\mu^{-1}(1 - e^{-2\mu\tau})$. (See Karlin and Taylor (1981), p. 218.) $f(t, S, T, x)$ is the transition density function of the ordinary Brownian motion $dY_t = \sigma dz_t$; thus,

$$f(t, S, T, x) = (2\pi\tau)^{-1/2}\sigma^{-1} \exp\{-(S - x)^2/2\sigma^2\tau\}.$$

The reader can now verify without difficulty that $(\partial/\partial x)\ln f/g$ is a function of t and S. Hence, the process is not viable.

In general, $f(t, S, T, y)$ is a pricing function. That is, if $h(\cdot)$ is any continuous function on $(0, \infty)$ satisfying $\int_0^\infty |h(y)| f(t, S, T, y)dy < \infty$, then the value at time-state (t, S) of a self-financing strategy paying $h(S_T)$ at time T is $\int_0^\infty h(y)f(t, S, T, y)dy$.[13] Thus, our continuous-time setup may be viewed as a continuum of two-date economies, all of which are as in Example 2 in Section I. This is because at any time-state (t, S) the individual's optimal strategy is not to trade over the time interval $[t, T]$, and this will obviously not change if we do not allow him/her to trade after time t. Hence, $U'(\cdot)$ can be computed as in equation (3) in Example 2, which is rewritten as the condition (d) for this case. Since this can be done for any t and S, $U'(\cdot)$ may be "overdetermined."

Example 4: To see a case where the viability conditions are satisfied, consider a stochastic process $\{S_t\}$ of the form $S_t = X_t^{-1} - C$, where C is an arbitrary constant (or $C \leq 0$, if we require $S_t > 0$)[14] and where $\{X_t\}$ is a positive process of the form $dX_t = h(t, X_t)dz_t$, where h satisfies Lipschitz and growth conditions. Denoting $\beta(t, S) = -(S + C)^2 h(t,(S + C)^{-1})$, it follows from Itô's Lemma that

$$dS_t = (S_t + C)^{-1}\beta^2(t, S_t)dt + \beta(t, S_t)dz_t.$$

[12] See footnote 3.

[13] This well known result follows from Cox and Ross' (1976) "risk-neutral valuation" or directly from the fact (see Appendix) that f is a fundamental solution of the PDE $f_t + \frac{1}{2}\beta^2 f_{SS}$, which is the same PDE that is satisfied by the value of any self-financing strategy as above. (See Merton (1977) and Duffie (1988), Section 22, specialized to the case of "zero interest rate.")

[14] See footnote 3.

To check that condition (a) is satisfied with $V(S) = 1/(S + C)$, we use the fact that $\{X_t\}$ is a martingale:

$$\begin{aligned} H(t, S) &= E[(S_T + C)^{-1} \mid S_t = S] = E[X_T \mid X_t = (S + C)^{-1}] = (S + C)^{-1}, \\ G(t, S) &= E[(S_T + C)^{-1} S_T \mid S_t = S] \\ &= E[X_T(X_T^{-1} - C) \mid X_t = (S + C)^{-1}] = 1 - C(S + C)^{-1}, \end{aligned}$$

and this readily implies the desired result. In fact, by a similar argument, equation (1) will be satisfied even if $\{X_t\}$ is a general martingale, not necessarily a diffusion. Conversely, by reversing the argument, it can be shown that, if (1) is satisfied with $U'(S) = 1/(S + C)$ and a stochastic process $\{S_t\}$ such that $S_t > -C$ for all t, then necessarily $X_t \equiv 1/(S_t + C)$ is a positive martingale.

Going back to the general case, it is noteworthy that, in equation (8), where the associated utility function is calculated from the diffusion coefficients, only their values at time T are important. This is the case because an arbitrarily small interval $[t, T]$ is sufficient to determine $U(\cdot)$. In equilibrium, the optimal policy at time t must be not to trade at $[t, T]$, and the same analysis applied above for $[0, T]$ is also valid for $[t, T]$, implying the same conclusions regarding $U(\cdot)$.

Also note that, if the utility function $U(\cdot)$ is associated with a viable price process with the diffusion coefficients $\alpha(t, S)$ and $\beta(t, S)$, then it is also "associated" in the sense of equation (8) with any other process with diffusion coefficients of the form $\lambda^2(t, S)\alpha(t, S)$ and $\lambda(t, S)\beta(t, S)$, where λ is an arbitrary function. However, the latter process need not be viable, as will be evident from Proposition 3 (condition (P6)) below.

Another way to characterize viability follows from the following mathematical proposition:

PROPOSITION 2: *Suppose $\{S_t;\, t \in [0, T]\}$ is a stochastic process as in equation* (4) *(with the ensuing notation). Then the following conditions are equivalent:*

(P1) *There exists a positive continuous function $V(\cdot)$ satisfying equations* (5) *and* (6) *such that some (hence all) of the conditions* (a)–(e) *in Proposition 1 are satisfied.*

(P2) *For any $0 \leq t_0 < t_1 < \cdots < t_n \leq T$ and $x_0, x_1, \cdots, x_n > 0$,*

$$k(t_0, x_0, t_1, x_1)k(t_1, x_1, t_2, x_2) \cdots k(t_{n-1}, x_{n-1}, t_n, x_n) = k(t_0, x_0, t_n, x_n). \quad (9)$$

The proof is given in the Appendix.

The economic meaning of condition (P2) becomes clear when it is compared to property (B2) in Example 1 in Section I. Let us take for simplicity $n = 2$ in (P2). Informally, let G' be the event that $S_{t_2} \in |x_2, x_2 + dx_2]$ and let G'' be the event that $S_{t_1} \in [x_1, x_1 + dx_1]$ and $S_{t_2} \in [x_2, x_2 + dx_2]$. Then the right-hand side, namely $f(t_0, x_0, t_2, x_2)/g(t_0, x_0, t_2, x_2)$, is the cost at time-state (t_0, x_0) of the security paying $1_{G'}$, at time t_2 divided by the probability of G'. The left-hand side, namely $f(t_0, x_0, t_1, x_1)f(t_1, x_1, t_2, x_2)/g(t_0, x_0, t_1, x_1)g(t_1, x_1, t_2, x_2)$, is the same thing for G''.

Next, let us focus on the particular case where in equation (4) $\alpha(t, S) = q(t)a(S)$ and $\beta(t, S) = q(t)^{1/2}b(S)$, where $q(t)$ is a continuous positive function. That is,

$$dS_t = q(t)a(S_t)dt + q(t)^{1/2}b(S_t)dz_t. \quad (10)$$

The case $q(t) \equiv 1$ gives a time-homogeneous process, and in general (10) is "the same" as the corresponding time-homogeneous case after a suitable deterministic time change. (See Øksandel (1985).) For such a process we can give some additional characterizations of property (P1). Note that, if (P1) is satisfied, then, by the previous proposition, the corresponding $V(\cdot)$ is given to within a constant multiple by

$$V(S) = \exp(-\textstyle\int_1^S [a(\eta)/b^2(\eta)]d\eta). \tag{11}$$

We will also assume that a and b are differentiable functions. In what follows primes denote derivatives.

PROPOSITION 3: *Consider the process* $\{S_t\}$ *as in equation* (10) *and the function* $V(S)$ *as in equation* (11). *Define* H *as in equation* (5), *and suppose that the inequalities in equations* (5) *and* (6) *are satisfied. Then each of the following conditions is equivalent to* (P1):

(P3) $H(t, S) = F(t)V(S)$, *where* $F(\cdot)$ *is some positive function such that* (*necessarily*) $F(T) = 1$.
(P4) $H(t, S) = \exp(\lambda \int_t^T q(\tau)d\tau) \cdot V(S)$ *for some constant* λ.
(P5) $\tfrac{1}{2}b^2(S)V''(S) + a(S)V'(S) = \lambda V(S)$ *for some constant* λ.
(P6) *There exists a constant* K *such that*

$$\frac{d}{dS}\left[\frac{a(S)}{b^2(S)}\right] + \left[\frac{a(S)}{b^2(S)}\right]^2 = \frac{K}{b^2(S)}. \tag{12}$$

The proof is given in the Appendix.

Thus, if a time-homogeneous process $\{S_t\}$ as in (10) is viable, properties (P3)–(P6) are satisfied with $V \equiv U'$. The derivative of the associated utility function can now be written as in (11), and the Arrow-Pratt risk aversion function is $-U''(S)/U'(S) = a(S)/b^2(S)$.

Property (P6) is of particular interest as it amounts to a simple test on the diffusion coefficients. We note that it does not involve $q(\cdot)$ from equation (10). The economic reason for this is that, as it was pointed out, if $q(\cdot)$ is replaced by 1 in equation (10), this gives "the same process" to within a deterministic time change. It is clear that, if it is optimal not to trade, this will not change when time is counted at a different pace.

Example 5: Suppose $dS_t = \mu S_t^\gamma dt + \sigma S_t^\nu dz_t$, where $\mu > 0$, $\sigma \neq 0$, and γ and ν are constants.[15] In the previous notation,

$$(a/b^2)^2 + (a/b^2)' = \mu^2\sigma^{-4}S^{2\gamma-4\nu} + \mu\sigma^2(\gamma - 2\nu)S^{\gamma-2\nu-1}.$$

In order for equation (12) to be satisfied, this expression needs to be of the form K/b^2, i.e., $K\sigma^{-2}S^{-2\nu}$ for some constant K. In general, this is not satisfied unless

(i) $\gamma = \nu = 0$. This is a Brownian motion[16]: $dS_t = \mu dt + \sigma dz_t$. From equation (11) we obtain that, to within a constant multiple, $U'(S) = \exp(-\mu\sigma^{-2}S)$.

[15] Instead of discussing the issue of existence of a solution to this stochastic differential equation, let us assume that we have already restricted ourselves to a set of parameters for which the process is well defined. This will not affect the results because, as we shall see below, this process is not viable except for some special cases which are well defined.

[16] See footnote 3.

(ii) $\gamma = \nu = 1$. This is a geometric Brownian motion: $dS_t = \mu S_t dt + \sigma S_t dz_t$. Equation (11) gives $U'(S) = S^{-\mu/\sigma 2}$.

(iii) $\mu = \sigma^2$ and $\gamma = 2\nu - 1$. The process is $dS_t = \sigma^2 S_t^{2\nu-1} dt + \sigma S_t^{\nu} dz_t$ and here $U'(S) = S^{-1}$. Such a process is obtained (by Itô's Lemma) if $S_t = 1/X_t$, where X_t satisfies $dX_t = -\sigma X_t^{2-\nu} dz_t$. This is a situation as in Example 4 (with $C = 0$). Since we must require that X_t is positive, it follows that the viability condition is satisfied only for $\nu \leq 1½$.[17]

In all other cases the process is not viable.

Note that we may also go in the reverse direction and view equation (12) as an ordinary differential equation in $\phi(S) \equiv a(S)/b^2(S)$ with the aid of which we can identify viable processes. If $K = 0$, the solution is $\phi(S) = 1/(S + C)$, where C is a constant. The associated utility function, as it follows from equation (11), is $U(S) = \ln(S + C)$. This is the case described in Example 4. For given $K \neq 0$ and $b(\cdot)$, equation (12) is a Ricatti equation from which we can find $\phi(S)$, and then we take $a(S) = \phi(S)b^2(S)$. Provided that the process $dS_t = a(S_t)dt + b(S_t)dz_t$ is well-defined, this process satisfies the viability condition (12).

Equation (12) can be translated to a condition on the utility function. This is of interest if we start with a utility function and ask ourselves whether it is associated with some viable price process.

PROPOSITION 4: *If a utility function* $U(\cdot)$ *is associated with a viable process of the form* (10), *then necessarily*[18]

$$\operatorname{sgn}(A'(S) + A^2(S)) = l, \tag{13}$$

where $A(S) \equiv -U''(S)/U'(S)$ *is the Arrow-Pratt risk aversion function and* l *is a constant* (± 1 *or* 0) *not depending on* S. *If* $l \neq 0$, *then the coefficients* a *and* b *in equation* (10) *are necessarily of the form* $b(S) = \gamma|A'(S) + A^2(S)|^{-1/2}$ *and* $a(S) = b^2(S)$, *where* γ *is some constant.*

The proof is immediate from Proposition 3 and the discussion following it.

Remark: It is sometimes more convenient to work with the risk tolerance function $R(S) \equiv 1/A(S) = -U'(S)/U''(S)$. It is easy to see that $[1 - R'(S)]/R^2(S) = A'(S) + A^2(S)$; thus, equation (13) is equivalent to $\operatorname{sgn}(1 - R'(S)) = l$ or $\operatorname{sgn}(\ln(R'(S))) = l$. If $l \neq 0$, the above equations for b and a become $b(S) = \gamma R(S)/|1 - R'(S)|^{1/2}$, $a(S) = \gamma^2 R(S)/|1 - R'(S)|$.

Example 6: Consider the HARA-class of utility functions, i.e., the utility functions $U(\cdot)$ for which $R(S) \equiv -U'(S)/U''(S) = \theta_1 S + \theta_2$ for some constants θ_1 and θ_2. We observe that $1 - R'(S) = 1 - \theta_1$ is a constant and in particular it has a constant sign. Assuming $\theta_1 \neq 1$, it follows that the corresponding price process is[19]

$$dS_t = \theta^2(\theta_1 S + \theta_2)dt + \theta(\theta_1 S + \theta_2)dz_t,$$

[17] The existence and uniqueness of solutions of stochastic differential equations of this type are discussed in Karatzas and Shreve (1988), pp. 291–293 and Section 5.5. The fact that X_t is positive, provided that X_0 is a positive constant, is not stated explicitly, but it follows from the proof of Theorem 5.4.

[18] Where $\operatorname{sgn}(0) = 0$ and $\operatorname{sgn}(x) = x/|x|$ for $x \neq 0$; i.e., this is the sign of x.

[19] See footnote 3.

where θ is any constant. (This is $\gamma/|1 - R'(S)|^{1/2}$ in the above notation.) The case $\theta_1 = 1$, i.e., $U(S) = \ln(S + C)$ where C is a constant, was described in Example 4.

III. Sufficiency of the Conditions

In the previous section we saw several equivalent properties of stochastic processes which turned out to be necessary for viability. In this section we will see that they are also sufficient in the following sense: if a stochastic process $\{S_t\}$ satisfies these properties, then one can define an economy with two securities (one riskless and one risky) such that the given process will be the equilibrium relative price of the risky asset.

Let $\{S_t;\ t \in [0, T]\}$ be a stochastic process as in the beginning of Section II (with the ensuing notation of g and f). Here we require that α is a strictly positive function, that β is strictly positive for all t and S or strictly negative for all t and S, and that α and β satisfy Lipschitz and growth conditions. The Wiener process $\{z_t\}$ is assumed to be defined on a probability space $(\Omega, \mathscr{F}, P)$. Now suppose that the equivalent properties (P1) and (P2) are satisfied. In the following model, $\{S_t\}$ will be the equilibrium price of "the market portfolio." Consider an exchange economy with one representative individual where economic activity takes place at the time interval $[0, T]$. The process $\{S_t\}$ is observable by the individual, and it represents the magnitude of an "an unripe crop," which is entirely exogenous. This crop is ripe for consumption only at time T, and thus the probability distribution of its magnitude at that time is continuously revised by the individual. (More formally, the temporal resolution of uncertainty is modeled by the filtration $\{\mathscr{F}_t\}$ generated by $\{S_t\}$, and $\mathscr{F}_T = \mathscr{F}$.) The individual is an expected utility maximizer, and the utility function $U(\cdot)$ satisfies $U' = V$, where V is as in equation (8), and in particular $U' > 0$, $U'' < 0$. Two securities are traded continuously and frictionlessly: one equity share ("stock"), representing ownership of the entire crop, and a promissory note ("bond") with zero net supply which pays one unit with certainty at time T. The individual, who is endowed with the total shock, believes that the relative price of the stock (when the bond is taken as the numeraire) will be equal to the exogenous process S_t at any time t and in particular that equation (4) also represents the stock price process.

The individual's choice space consists of all self-financing strategies in the bond and the stock. This can be defined formally as in Pliska (1986) or Duffie (1988) Section 16, with the obvious modification to our setting, and we will not elaborate upon this here. Furthermore, every absolutely integrable payoff (i.e., an element of $L^1(P)$) can be produced by such a strategy. (See Duffie (1988), Section 22.) We will first show that the price system is arbitrage free. It is well known[20] that it is sufficient to prove that there exists an equivalent probability measure Q under which $\{S_t\}$ is a martingale. Indeed, define Q via $dQ = \{V(S_T)/E[V(S_T)]\}dP$ or, equivalently, $E^Q[Y] = E[YV(S_T)]/E[V(S_T)]$, where Y is any random variable, E^Q is the expectation with respect to Q, and E is the expectation with respect to P. It is easy to see that Q is a probability measure which is

[20] See Harrison and Kreps (1979) and Harrison and Pliska (1981).

equivalent to P. To see that $\{S_t\}$ is a martingale with respect to Q, fix t and let X be any $\mathscr{F}_t$-measurable random variable. Then, denoting $m = 1/E[V(S_T)]$,

$$\begin{aligned} E^Q[XS_t] &= mE[XS_tV(S_T)] = mE[XS_tE[V(S_T) \mid \mathscr{F}_t]] \\ &= mE[XS_tH(t, S_t)] = mE[XG(t, S_t)] \\ &= mE[XE[S_TV(S_T) \mid \mathscr{F}_t]] = mE[XS_TV(S_T)] = E^Q[XS_T], \end{aligned}$$

where the first and the last equalities follow from the definition of E^Q, the second and the sixth equalities follow from the tower property of the conditional expectation and the F_t-measurability of X, and the other equalities follow from the definition of H and G and from property (P1) (condition (a)). By the definition of conditional expectation, this proves that $S_t = E^Q[S_T \mid \mathscr{F}_t]$, and our assertion that the price system is arbitrage free was proved.

Because $\{S_t\}$ is a martingale under Q, so is the value of any self-financing strategy. Thus, the value at time 0 of a self-financing strategy which produces $\varphi(S_T)$ at time T is $E^Q[\varphi(S_T)] = \int_0^\infty \varphi(y)f(0, S_0, T, y)dy$.[21] (The last equality follows from the definition of E^Q and Property (P1), condition (d).) Before addressing the individual's optimization problem, we recall that, in a discrete-state complete market, if two states have the same ratio of price[22] to probability, then the optimal portfolio choice is such that the consumption is the same in both states. (See, e.g., Huang and Litzenberger (1988), Section 5.5.) In our continuous-time setup, equation (9) says that, loosely speaking, two trajectories with the same initial time-state and the same terminal time-state have the same ratio of price to probability. Hence, the optimal consumption of an expected utility maximizer (who consumes only at the end) is such that it does not depend on the stock price path; i.e., it depends only on S_T. This can probably be made more exact, but we will content ourselves here with this rather brief argument.

The individual optimization problem may be regarded as a one-period optimization problem. (See Pliska (1986) and Cox and Huang (1987)].) This is because a choice of an optimal strategy is tantamount to the selection of the random terminal payoff that it will produce. It follows from the previous discussion that we can limit our attention to payoffs which are functions of S_T, and thus the individual's objective at $t = 0$ is to maximize $E[U(S_T + \varphi(S_T))]$, where the optimization is on all measurable functions $\varphi(y)$ defined on $(0, \infty)$ such that $y + \varphi(y) > 0$ almost everywhere and such that $\varphi(S_T)$ costs zero at $t = 0$, i.e., $\int_0^\infty \varphi(y)f(0, S_0, T, y)dy = 0$. To show that the optimal choice is $\varphi \equiv 0$, it is clearly sufficient to take a nonzero φ as above and to show that $E[U(S_T + \nu\varphi(S_T))]$ is a decreasing function of $\nu \in (0, 1]$. Indeed, denote $\kappa(y, \nu) \equiv U'(y + \nu\varphi(y))/U'(y)$. The fact that $U'(\cdot)$ is decreasing function entails that $\kappa(y, \nu) \leq 1$ *if* $\varphi(y) \geq 0$ *and* $\kappa(y, \nu) > 1$ if $\varphi(y) < 0$.

$$\begin{aligned} &\int_{\varphi^{-1}[0,\, \infty)} \varphi(y)f(0, S_0, T, y)\kappa(y, \nu)dy \leq \int_{\varphi^{-1}[0,\, \infty)} \varphi(y)f(0, S_0, T, y)dy \\ &= -\int_{\varphi^{-1}(-\infty,\, 0)} \varphi(y)f(0, S_0, T, y)dy \\ &< -\int_{\varphi^{-1}(-\infty,\, 0)} \varphi(y)f(0, S_0, T, y)\kappa(\nu, y)dy, \end{aligned} \tag{14}$$

[21] See also footnote 13.

[22] I.e., the price of the elementary security which pays 1 if the state occurs and zero otherwise.

where the inequalities follow from the above properties of κ and the equality follows from the fact that $\int_0^\infty \varphi(y)f(0, S_0, T, y)dy = 0$. Now to conclude the argument,

$$\begin{aligned}\frac{d}{d\nu}E[U(S_T + \nu\varphi(S_T))] &= \frac{d}{d\nu}\int_0^\infty U(y + \nu\varphi(y))g(0, S_0, T, y)dy \\ &= \int_0^\infty \varphi(y)\kappa(\nu, y)U'(y)g(0, S_0, T, y)dy \\ &= H(0, S_0)\int_0^\infty \varphi(y)\kappa(\nu, y)f(0, S_0, T, y)dy < 0,\end{aligned}$$

where the second equality follows from differentiation under the integral sign and the definition of κ, the third equality follows from property (P1) (condition (d)), and the inequality follows from (14).[23]

Informally stated, the equilibrium is thus as follows. The individual believes that the relative stock price will be equal to S_t (the given exogenous process) at any time t. As a result, the optimal policy is not to trade at any time. Evolution of the stock price according to $\{S_t\}$ will confirm the individual's beliefs and will clear the market, and an equilibrium of plans, prices, and price expectations is obtained.

IV. Summary

The paper may be regarded as a study of the question: which assumptions on the price process of the market portfolio "make sense"? The paper identifies these processes (and the corresponding utility functions of the representative individual). Admittedly, this is done in a very simple framework, perhaps too simple to enable us to draw conclusions about the real world. Nevertheless, it demonstrates that the viable processes can constitute a rather restricted set,and it paves the way for the study of this issue within more complicated frameworks.

Appendix

We will summarize first some known mathematical results. They will be used below to prove the propositions which appear in the body of the paper. Let $\{S_t\}$ be a diffusion process defined on some underlying probability space and satisfying

$$dS_t = \alpha(t, S_t)dt + \beta(t, S_t)dz_t, \tag{A1}$$

where $\{z_t\}$ is a standard Wiener process where α and β are suitably well behaved functions defined for $t \in [0, T]$ and $S > 0$.[24] Let $g(t, S, \tau, x)$ be the transition probability density function of this process (representing transition probability from time-state (t, S) to time-state (τ, x)). Then g is a fundamental solution of

[23] More formally, the differentiation under the integral sign, and the conclusion $E[U(S_T)] \geq E[U(S_T + \varphi(S_T))]$, can be justified if $\varphi(\cdot)$ has a compact support. Then the general case for $\varphi \in L^1(P)$ can be obtained by a limiting argument.

[24] The ensuing results will be valid with minor modifications if S_t is not necessarily positive and α and β are defined for $-\infty < S < \infty$, but we focus on the positive case which is parallel to the analysis in the body of the paper.

the "backward equation"[25]:

$$g_t + \frac{1}{2}\beta^2(t, S)g_{SS} + \alpha(t, S)g_S = 0. \tag{A2}$$

That is, g solves the above partial differential equation (PDE) with boundary condition $g(T, S, T, x) = \delta(S - x)$, where $\delta(\cdot)$ is the Dirac δ-function. Another way to say the same thing is as follows: if F is any solution of the above equation, i.e.,

$$F_t + \frac{1}{2}\beta^2(t, S)F_{SS} + \alpha(t, S)F_S = 0, \tag{A3}$$

with the boundary condition $F(T, S) = Q(S)$, then necessarily

$$F(t, S) = \textstyle\int_0^\infty Q(x)g(t, S, T, x)dx = E[Q(S_T) \mid S_t = S], \tag{A4}$$

where E is the expectation operator. Conversely, every function of the form (A4) satisfies equation (A3). Thus, g spans in the sense of (A4) all the solutions of equation (A3). The above results can be found in Karlin and Taylor (1981), Section 15.5 or Schuss (1980), Chapter 5.[26]

Proof of Proposition 1:

(i) (a) $\Rightarrow$ (b): G satisfies equation (A3): $G_t + \frac{1}{2}\beta^2 G_{SS} + \alpha G_S = 0$. Now substitute $G = SH$ and use the product rule of differentiation. Then utilize the fact that H also satisfies equation (A3), i.e., $H_t + \frac{1}{2}\beta^2 H_{SS} + \alpha H_S = 0$, and the desired result will follow.

(b) $\Rightarrow$ (c): Let $\lambda(t, S, t_1, S_1) \equiv f(t, S, t_1, S_1)H(t, S)/H(t_1, S_1)$. We will prove that $\lambda(t, S, t_1, S_1) = g(t, S, t_1, S_1)$ by showing that λ satisfies the PDE (A3) with the boundary condition $\lambda(t_1, S, t_1, S_1) = \delta(S - S_1)$. Indeed, it is straightforward to verify the first part by employing the facts that H satisfies equation (A3) and property (b) and that f satisfies (in analogy to (A2)) $f_t + \frac{1}{2}\beta^2 f_{SS} = 0$. As $t \to t_1$,

$$\lambda(t_1, S, t_1, S_1) = \delta(S - S_1)H(t_1, S)/H(t_1, S_1) = \delta(S - S_1),$$

where the last equality follows from the definition of $\delta(\cdot)$.[27]

(c) $\Rightarrow$ (d): Take $t_1 = T$, $S_1 = x$.

(d) $\Rightarrow$ (e): Trivial.

(e) $\Rightarrow$ (d): To show that $\psi(t, S) = 1/H(t, S)$, multiply the two sides of the equality in (e) by $g(t, S, T, x)$ and integrate on x.

(d) $\Rightarrow$ (a): Multiply the two sides in (d) by $xg(t, S, T, x)$ and integrate on

[25] In what follows, subscripts of g, F, f, G, H, and $\bar{H}$ mean partial differentiation.

[26] These are applications-oriented texts. For a more detailed analysis, see the references from footnote 9.

[27] Or, equivalently, show that $\int_0^\infty Q(y)\lambda(t, S, t_1, y)dy$ is equal to $Q(S)$ for $t = t_1$.

x. Then use the fact that $\int_0^\infty xf(t, S, T, x) = S$. (This is (A4) applied to $F(t, S) \equiv S$ and to the process $dY_t = \beta(t, Y_t)d\hat{z}_t$, whose transition density function is f.[28])

(ii) Substitute $t = T$ in equation (7). □

Proof of Proposition 2:

(P1) $\Rightarrow$ (P2): To see that equation (9) is valid, use condition (c) to replace each $k(t_i, x_i, t_j, x_j)$ on both sides of (9) by $H(t_i, x_i)/H(t_j, x_j)$.

(P2) $\Rightarrow$ (P1): We will show that property (e) is satisfied. Let $\tau_j = T\cdot j/(j + 1)$, $j = 1, 2, \cdots$. Fix j and recall that T is regarded as fixed. Equation (9) gives

$$k(t, S, T, x) = k(t, S, \tau_j, 1)k(\tau_j, 1, T, x) \equiv \psi_j(t, S)V_j(x).$$

This is valid only for $t < \tau_j$. Repeating the same argument for $j + 1$, $k(t, S, T, x) = \psi_{j+1}(t, S)V_{j+1}(\mathrm{x})$, which is valid for $t < \tau_{j+1}$. However, it is not difficult to see that the two decompositions must be the same (to within a constant multiple of V_{j+1}) for any $t < \tau_j$ and $S, x > 0$. Since $\tau_j \to T$, concluding the argument to define $V(x)$ and $\psi(t, S)$ is straightforward. Property (e) is thus established, and the inequalities in equations (5) and (6) follow as in the implications (e) $\Rightarrow$ (d) $\Rightarrow$ (a) in the proof of Proposition 1. □

Proof of Proposition 3:

(P1) $\Leftrightarrow$ (P3): For the process as in equation (10), the conditions of equations (7) and (8) imply (P3), and vice versa.

(P3) $\Rightarrow$ (P4), (P5): H satisfies equation (A3), i.e., $H_t + \frac{1}{2}qb^2H_{ss} + qaH_s = 0$. Substituting $H = F(t)V(S)$ and rearranging gives

$$-F'(t)/F(t)q(t) = \left[\frac{1}{2}b^2(S)V''(S) + a(S)V'(S)\right]/V(S)$$

for any t and S. Thus, both sides must be equal to a constant λ. This implies that $F(t) = \exp(\lambda\int_t \mathrm{q}(\tau)\mathrm{d}\tau)$ and that (P5) holds.

(P5) $\Rightarrow$ (P4): Denote the expression on the right-hand side of (P4) by $\bar{H}(t, s)$. Use (P5) to show that $\bar{H}(t, S)$ satisfies $\bar{H}_t + \frac{1}{2}qb^2\bar{H}_{SS} + qa\bar{H}_S = 0$. However, $H(t, S)$ satisfies the same PDE and $H(T, S) = \bar{H}(T, S) = V(S)$. Hence, necessarily $H(t, S) = \bar{H}(t, S)$.

(P5) $\Leftrightarrow$ (P6): Equation (11) entails that $V' = -Va/b^2$ and $V'' = V[(a/b^2)^2 - (a/b^2)']$. Substitute this in (P5) and equation (12) will follow (with $K = -2\lambda$). The reverse implication is obtained by reversing the argument.

(P4) $\Rightarrow$ (P3): Trivial. This finishes the proof. □

[28] Or see the last assertion in the proposition in Duffie (1988), Section 21c.

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