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Valuing Flexibility as a Complex Option

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ABSTRACT

This paper develops an approach for valuing flexible production systems using contingent claims pricing. Demand curves for our model's underlying assets (output products) may be downward sloping, in contrast with the standard option pricing assumption. Also, our marginal production(exercise) costs may be increasing. In addition, we allow for multiple products and a production capacity constraint. These elements of the model result in complex exercise decisions for the contingent claims which comprise the production system's value. We illustrate our approach by valuing a flexible system that produces two products which have profit margin functions with stochastic parameters.

DESPITE SIGNIFICANT ADVANCES OVER the last few decades, there continue to be complaints that quantitative capital budgeting techniques often understate the value of a project to the firm's shareholders.[1] One problem in many applications is that "managerial control" is often completely ignored or improperly valued. Managerial control refers to the ability of management to affect the course of a project by acting in response to the resolution of uncertainty over time.[2] A flexible project may allow for downside protection against unfavorable events (e.g., by abandoning the project) and may also introduce growth opportunities on the upside.

Contingent claims pricing techniques have been used to value a project's flexibility or "real options", a term coined by Myers (1977) to emphasize that the value of control opportunities is contingent on the value of underlying *real* assets. The real options literature contains various examples of switching between different operating states. These states may be discrete (e.g., an open or closed plant) or continuous (e.g., operating at different levels of capacity). The option to initiate a project (see Majd and Pindyck (1987), McDonald and Siegel (1986), or Pindyck (1988)) and the option to abandon a project (see Myers and Majd (1984)) both involve an irreversible decision which precludes switching back to the previous state. In contrast, the option to shut down a plant temporarily (see

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[1] Hayes and Abernathy (1980) as well as Hayes and Garvin (1982) are widely cited articles which express quite strong opinions in this regard.

[2] Trigeorgis and Mason (1987) discuss this issue under the label of "managerial flexibility" and describe several operating options which management may be able to exercise.

McDonald and Siegel (1985)) is a reversible decision. Brennan and Schwartz (1985) incorporate both types of decisions in their valuation of a copper mine.

In this paper, we examine the pricing of complex options that arise in valuing a flexible production system which allows the firm to switch its output mix over time.[3] Acquisition of such a flexible system may be substantially more expensive than that of an inflexible (dedicated) production facility. Consequently, valuing the tradeoff between flexibility and initial cost is an important issue for these types of investments.

For contingent claims on financial securities, it is standard to assume that the underlying security (e.g., common stock) is traded in a perfectly competitive market. This perfect competition assumption has generally been carried over to the real options literature. However, it is clear that many real asset markets are monopolistic or oligopolistic rather than perfectly competitive. This paper explicitly addresses that issue by allowing downward sloping demand curves for the underlying assets in our model.[4] We also allow for possibly increasing marginal production (exercise) costs. In addition, the model includes multiple products and a capacity constraint on the production system. These elements of our model result in complex exercise decisions for the contingent claims which comprise the value of the production system. These complications present a significant departure from the previous options literature and thus require that we develop a somewhat different valuation procedure.

In Section I, we present a valuation approach which allows for downward sloping marginal profit curves for the firm's products as well as for a capacity constraint on the production system. In Section II, we derive an analytic valuation expression for the case where there are two products and two underlying uncertainties. In Section III, we use that formula to calculate the value of a hypothetical flexible production system for a variety of input parameter values. Section IV interprets the basic valuation procedure and discusses several extensions.

I. The Basic Model

Assume that a firm is considering the purchase of a production facility which is perfectly flexible in that it can be costlessly switched to produce different combinations of k products. Product i can be produced instantaneously at time t at an average variable cost of $C_i(t)$ and will be instantaneously sold at a price of $P_i(t)$. We consider the case where the market for the firm's products may not be perfectly competitive and where the marginal cost of manufacturing a product

[3] Examples of flexible production systems include refineries, chemical plants, and flexible manufacturing system (FMS) installations. Some production systems have the flexibility to use different inputs. For example, Kulatilaka (1988) studies the valuation of a power generation plant which can be fired by either coal or oil. Our valuation procedure of product-flexible systems can be easily modified to value input-flexible (or process-flexible) systems.

[4] Two recent papers, Pindyck (1988) and He and Pindyck (1989), incorporate downward-sloping demand curves to examine capacity-choice decisions in a real options framework. Although that is a closely related issue to what is addressed here, those papers take a different analytical approach by focusing on an infinite-horizon model with continuous output quantity adjustments. These assumptions are also used in an earlier paper by McDonald and Siegel (1986) that contains a discussion of the investment decision of a monopolist which produces a single commodity.

may not be constant. Thus, the per-unit profit margin for product i at time t, $R_i(t) = P_i(t) - C_i(t)$, is allowed to be a function of $q_i(t)$, the production rate at time t. We assume for simplicity that the profit margin of product i is a linear function of the production rate, i.e., $R_i(t) = A_i(t) - B_i(t)q_i(t)$.[5] We assume that B_i is a positive constant and A_i satisfies the following stochastic differential equation:

$$dA_i = \alpha_i\, dt + \sigma_i\, dz_i;\ A_i(0) = a_i, \qquad i = 1, \cdots, k. \tag{1}$$

The drift and standard deviation parameters (α_i and σ_i) are assumed to be constant although, as will be discussed in Section IV, it is also possible to allow these parameters to be time-varying. z_i denotes the one-dimensional Standard Brownian motion affecting the profit margin of product i, with ρ_{ij} being the correlation between these stochastic processes for products i and j. The intercept variables, A_i, will be normally distributed at any point in time given the processes in (1). Such a distribution seems to be a reasonable approximation for describing profit margins.[6]

We assume that the production system has a finite life, T, and that adjustments in the production rates may occur only at discrete, preset points in time. The life of the production system, $[O, T]$, is split up into N periods of equal duration, τ.[7] At the beginning of each period, the firm chooses production rates for each of its products subject to a capacity constraint described below. These rates are sustained throughout that period. The production rate for the ith product during the period $[n\tau,(n + 1)\tau]$ is denoted by $q_i(n\tau)$. The sequence of these production rate selections over the lifetime of the system will be referred to as the firm's manufacturing (or production) program.

The cost of purchasing a production system of capacity Q is given by $I_0(Q)$. The production system is abandoned as worthless at time T but may not be abandoned earlier. Since profit contributions for all products could potentially become negative, we allow the firm to temporarily suspend production by costlessly shutting down and restarting the operation. However, we assume that a fixed cost, $C_F(Q)$, per unit time is incurred regardless of whether the system is operating.

[5] A sufficient condition for linear profit margin functions is that the demand curves and average cost curves are linear. The assumption that the profit margin functions are linear can be relaxed as long as the resulting profit function is strictly concave. However, nonlinear profit margin functions may result in more laborious evaluations of the integrals in the intermediate steps of the solution procedure. In contrast, time-varying slopes can easily be incorporated.

[6] In some cases, it may be appropriate to introduce a Poisson jump term in the process described in (1). This would capture the effects of such events as competitive entries into a market. This type of approach is discussed in Trigeorgis (1986). Our solution procedure can be slightly modified in order to allow for diversifiable, normally distributed jumps. For each European option in our problem, we would need to calculate a set of option values, each using the variance and the correlation coefficient that correspond to a particular number of jumps which occur during the period until maturity. These option values are then weighted using the associated probabilities for the number of jumps from a Poisson probability distribution. This is a straightforward adaptation of the procedure developed by Merton (1976) for the case where an underlying stock price follows a lognormal diffusion process with lognormally distributed jumps.

[7] The assumption of equal length periods is used for simplicity and can be easily relaxed.

The securities market is assumed to be frictionless; trading may take place continuously; and the A_i variables are spanned by existing securities.[8] In particular, we require that portfolios of securities can be constructed such that their price processes, M_i, are given by

$$\frac{dM_i}{M_i} = \mu_{M_i}\,dt + \sigma_i dz_i, \qquad i = 1, \cdots, k. \tag{2}$$

These portfolios are used as proxies for the intercept variables in the construction of a portfolio which replicates the firm's manufacturing program.[9] We also assume that there exists a riskless asset whose price, D, follows the process $dD_t = rD_t dt$, with r assumed to be constant through time.[10]

The model has been structured so that the production selection decision in one period does not affect the value of producing in subsequent periods. Thus, we can value the manufacturing program by summing up the values of the options to produce in each period. The option to produce in $[n\tau, (n+1)\tau]$, which expires at time $n\tau$, has value $W^{n\tau}(A_1(t), \cdots, A_k(t), t)$ at time $t \leq n\tau$, where it is understood that $W^{n\tau}$ also depends on the production rates, $q_i(n\tau)$, which will be selected at $n\tau$ according to a rule known at time t. The optimal choice of these rates will be discussed shortly. The option value $W^{n\tau}$ must satisfy the following partial differential equation:

$$\frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{k} \rho_{ij}\sigma_i\sigma_j W_{ij}^{n\tau} + \sum_{i=1}^{k}(r - \delta_i)W_i^{n\tau} - rW^{n\tau} + W_t^{n\tau} = 0, \tag{3}$$

where $\delta_i = \mu_{M_i} - \alpha_i$, $i = 1, \cdots, k$, and subscripts on $W^{n\tau}$ denote partial derivatives with respect to the indicated argument(s). Such δ_i terms arise in real option models because the underlying real assets may not be traded in equilibrium markets—see McDonald and Siegel (1985). Equation (3) follows from constructing a portfolio of traded securities to replicate the option to produce in the $(n+1)$th period (i.e., the interval $[n\tau,(n+1)\tau]$).[11] The option value must also satisfy

[8] We assume that portfolios with infinite payouts in some states are not traded in order to preclude "doubling strategies" as discussed in Harrison and Kreps (1979).

[9] Note that we have constructed portfolios of traded securities such that the standard deviations of the rates of return of these portfolios are the same as the standard deviations of the intercept levels. This construction is performed on a unitless basis. Conceptually, one could also have divided the dA_i processes by $1 to eliminate their units.

[10] The assumption of a constant riskless interest rate could be relaxed by using alternative assumptions such as those in Merton (1973) or Bhattacharya (1981).

[11] Merton's (1974) approach can be used to derive equation (3). This involves constructing a portfolio which consists of 1) the option to produce ($W^{n\tau}$), 2) a short position of $W_i^{n\tau}$ in each of the i portfolios which proxy for the A_i values, and 3) borrowing $W^{n\tau} - \sum_{i=1}^{k} W_i^{n\tau}$ at the riskless interest rate r. This portfolio requires zero net investment, and it is straightforward to show using Itô's Lemma that its change in value is nonstochastic. In equilibrium, the instantaneous change in value of such a portfolio must be zero. Equation (3) follows directly from this condition.

Our basic solution procedure could be utilized in a discrete-time trading framework without the ability to construct a replicating portfolio for the manufacturing program, as long as all investors have exponential utility functions. This "discrete-time" option pricing result is developed in Brennan (1979) for a single normally distributed underlying variable and has been extended by Stapleton and Subrahmanyam (1984) for several underlying variables.

the following terminal boundary condition (A_i and q_i will be used hereafter to denote $A_i(n\tau)$ and $q_i(n\tau)$ in order to simplify notation):

$$W^{n\tau}(A_1, \cdots, A_k, n\tau) = \sum_{i=1}^{k} q_i(A_i\beta + \eta_i - B_iq_i\beta), \tag{4}$$

where

$$\beta = \frac{(1 - e^{-r\tau})}{r}, \qquad \eta_i = \left[\frac{r - \delta_i}{r}\right]\{\beta - \tau e^{-r\tau}\}, \qquad i = 1, \cdots, k.$$

The terminal condition in (4) represents the present value at time $n\tau$ of instantaneously producing a mix of q_i units of the ith product (for $i = 1, \cdots, k$) throughout the $(n + 1)$th period.[12] In order to obtain the maximum value of each option, and thus of the manufacturing program, we must determine the optimal production levels q_i^* as functions of A_i for $i = 1, \cdots, k$ and use these values in condition (4).

These optimal production levels are found by solving the following quadratic programming problem:

$$\max_{q_i} \sum_{i=1}^{k} q_i(A_i\beta + \eta_i - B_iq_i\beta) \tag{5}$$

subject to

$$q_i \geq 0, \qquad i = 1, \cdots, k,$$

$$\sum_{i=1}^{k} q_i/y_i - Q \leq 0.$$

The positive constants, y_i, denote potentially differing production yields from devoting one unit of the production capacity, Q, to each product for one unit of time. Let λ_i, $i = 1, \cdots, k$, be the non-negative Lagrange multipliers which correspond to the non-negativity constraints on the q_i, and let λ_{k+1} be the Lagrange multiplier for the capacity constraint in (5). The maximization problem involves a concave objective function and linear constraints. Thus, the following Kuhn-Tucker conditions are both necessary and sufficient and will be used to determine the optimal production quantities:

$$A_i\beta + \eta_i - 2B_iq_i^*\beta - \lambda_i + \lambda_{k+1}/y = 0, \qquad i = 1, \cdots, k. \tag{6}$$

The associated complementary slackness conditions are

$$\lambda_iq_i = 0, \qquad i = 1, \cdots, k, \tag{7}$$

$$\lambda_{k+1}\left[\sum_{i=1}^{k} q_i/y_i - Q\right] = 0.$$

[12] Each term in the summation of (4) is calculated as follows:

$$\int_{n\tau}^{(n+1)\tau} e^{-r(t'-n\tau)}\{q_i[A_i + (r - \delta_i)(t' - n\tau)] - B_iq_i^2\}\,dt'.$$

This integral discounts a stream of expected payouts under an equivalent probability measure using a risk-free rate. (See Cox and Ross (1976) as well as Harrison and Kreps (1979).)

II. An Analytic Solution for the Two-Product Case

To illustrate the solution procedure, we derive an analytic expression for the two-product case. When solving (6) and (7) simultaneously for two products, there are seven types of feasible solutions corresponding to whether each of the three Lagrange multipliers is zero or positive. (All three multipliers cannot be positive at the same time, and thus there is one fewer than 2^3 solution types.) These solutions correspond to regions within the two-dimensional A_1–A_2 space, as will be seen later. These seven regions are denoted below by Roman numerals with the corresponding Lagrange multipliers and optimal production quantities (as functions of the input parameters) for the $(n + 1)$th period:

$$
\begin{aligned}
&\text{I:} && \lambda_1 \neq 0,\ \lambda_2 \neq 0,\ \lambda_3 = 0:\ q_1^* = 0,\ q_2^* = 0,\\
&\text{II:} && \lambda_1 = 0,\ \lambda_2 \neq 0,\ \lambda_3 = 0:\ q_1^* = K_1A_1 + K_2,\ q_2^* = 0,\\
&\text{III:} && \lambda_1 \neq 0,\ \lambda_2 = 0,\ \lambda_3 = 0:\ q_1^* = 0,\ q_2^* = K_3A_2 + K_4,\\
&\text{IV:} && \lambda_1 = 0,\ \lambda_2 \neq 0,\ \lambda_3 \neq 0:\ q_1^* = Qy_1,\ q_2^* = 0,\\
&\text{V:} && \lambda_1 \neq 0,\ \lambda_2 = 0,\ \lambda_3 \neq 0:\ q_1^* = 0,\ q_2^* = Qy_2,\\
&\text{VI:} && \lambda_1 = 0,\ \lambda_2 = 0,\ \lambda_3 = 0:\ q_1^* = K_1A_1 + K_2,\ q_2^* = K_3A_2 + K_4,\\
&\text{VII:} && \lambda_1 = 0,\ \lambda_2 = 0,\ \lambda_3 \neq 0:\ q_1^* = K_5A_1 - K_6A_2 + K_7,\ q_2^* = y_2\left[Q - \frac{q_1^*}{y_1}\right],
\end{aligned}
\tag{8}
$$

where

$$K_1 = \frac{1}{2B_1},\quad K_2 = \frac{\eta_1}{2B_1\beta},\quad K_3 = \frac{1}{2B_2},\quad K_4 = \frac{\eta_2}{2B_2\beta},\quad K_5 = \frac{1}{2B'},\ K_6 = \frac{y_2}{y_1}K_5,$$

$$K_7 = \frac{K_5}{\beta}\left[\eta_1 - \frac{y_2}{y_1}\eta_2 + \frac{y_2^2}{y_1}2B_2Q\beta\right],\quad \text{and}\quad B' = \left[B_1 + \frac{y_2^2}{y_1^2}B_2\right].$$

To determine the value of the option $W^{n\tau}(a_1, a_2, 0)$, we use an "equivalent martingale" valuation approach based on the work of Cox and Ross (1976) and Harrison and Kreps (1979). This approach values assets by discounting their expected payouts under an equivalent probability measure using a risk-free discount rate. Let $f^*(A_1, A_2)$ denote the transformed probability density over which we will calculate the expected terminal value of the option. The transformed bivariate density function can be shown to be normally distributed, with means of the two variables at time $n\tau$ given by $\mu_i = a_i + (r - \delta_i)n\tau$, $i = 1, 2$, and variances equal to $\sigma_i^2 n\tau$, $i = 1,2$. The correlation between the two variables is the same as in the true distribution $f(A_1, A_2)$, but to simplify notation it will be denoted as ρ_1.

To calculate the expected terminal value of the option, we must integrate over several areas in A_1–A_2 space. Each one (or in some cases two) of these areas corresponds to the regions in (8). The boundaries of these areas are defined in terms of the following coefficients:

$$\nu_1 = \frac{-\eta_1}{\beta}, \quad \nu_2 = \frac{-\eta_2}{\beta}, \quad \nu_3 = \frac{Qy_1 - K_2}{K_1}, \quad \nu_4 = \frac{Qy_2 - K_4}{K_3}, \quad \nu_5 = \frac{Qy_1 - K_7}{K_5},$$

$$\nu_6 = \frac{K_6}{K_5}, \quad \nu_7 = \frac{-K_7}{K_5}, \quad \nu_8 = \nu_3 - \frac{K_4 y_1}{K_1 y_2}, \quad \nu_9 = \frac{-K_3 y_1}{K_1 y_2}.$$

The expression for the expected terminal value of the production option is given below in (9) along with the region number to which each of the integrals corresponds. Let $Z_i(A_i, q_i^*) = q_i^*(A_i\beta + \eta_i - B_i q_i^* \beta)$, $i = 1, 2$. The first integral sign in each term shows the bounds for A_2, and the second is for A_1:

$$\begin{aligned}
&E^*\{W^{n\tau}(A_1, A_2, n\tau)\} \\
&= \int_{-\infty}^{\nu_2} \int_{\nu_1}^{\nu_3} Z_1(A_1, K_1A_1 + K_2) f^*(A_1, A_2)\, dA_1\, dA_2 \qquad \text{(II)} \\
&+ \int_{\nu_2}^{\nu_4} \int_{-\infty}^{\nu_1} Z_2(A_2, K_3A_2 + K_4) f^*(A_1, A_2)\, dA_1\, dA_2 \qquad \text{(III)} \\
&+ \int_{-\infty}^{\nu_2} \int_{\nu_3}^{\infty} Z_1(A_1, Qy_1) f^*(A_1, A_2)\, dA_1\, dA_2 \qquad \text{(IV)} \\
&+ \int_{\nu_2}^{\infty} \int_{\nu_5+\nu_6A_2}^{\infty} Z_1(A_1, Qy_1) f^*(A_1, A_2)\, dA_1\, dA_2 \qquad \text{(IV)} \\
&+ \int_{\nu_4}^{\infty} \int_{-\infty}^{\nu_7+\nu_6A_2} Z_2(A_2, Qy_2) f^*(A_1, A_2)\, dA_1\, dA_2 \qquad \text{(V)} \\
&+ \int_{\nu_2}^{\nu_4} \int_{\nu_1}^{\nu_8+\nu_9A_2} \{Z_1(A_1, K_1A_1 + K_2) + Z_2(A_2, K_3A_2 + K_4)\} \\
&\qquad \cdot f^*(A_1, A_2)\, dA_1\, dA_2 \qquad \text{(VI)} \\
&+ \int_{\nu_2}^{\nu_4} \int_{\nu_8+\nu_9A_2}^{\nu_5+\nu_6A_2} \Bigg\{ Z_1(A_1, K_5A_1 - K_6A_2 + K_7) \\
&+ Z_2\left(A_2, y_2\left[Q - \frac{K_5A_1 - K_6A_2 + K_7}{y_1}\right]\right)\Bigg\} f^*(A_1, A_2)\, dA_1\, dA_2 \qquad \text{(VII)} \\
&+ \int_{\nu_4}^{\infty} \int_{\nu_7+\nu_6A_2}^{\nu_5+\nu_6A_2} \Bigg\{ Z_1(A_1, K_5A_1 - K_6A_2 + K_7) \\
&+ Z_2\left(A_2, y_2\left[Q - \frac{K_5A_1 - K_6A_2 + K_7}{y_1}\right]\right)\Bigg\} f^*(A_1, A_2)\, dA_1\, dA_2 \qquad \text{(VII).}
\end{aligned} \tag{9}$$

A closed-form analytic solution to (9) can be obtained, and it involves standard normal density and distribution functions (univariate and bivariate). The analytic expression is rather long and thus is omitted here for brevity. (The solution can be obtained from the authors.) The integrals shown in the Appendix appeared frequently in the process of obtaining the analytic solution and are thus included in this paper for reference in solving similar problems.

The net present value at time zero of manufacturing the two products using an optimal manufacturing program is the sum of the present values for current and future production options minus the discounted stream of fixed costs and the initial investment. Recalling that $A_i(0) = a_i$, this net present value is given by

$$V(a_1, a_2, 0) = W^0(a_1, a_2, 0) + \sum_{n=1}^{N-1} W^{n\tau}(a_1, a_2, 0) - \frac{C_F(Q)(1 - e^{-rT})}{r} - I_0(Q), \tag{10}$$

where $W^0(a_1,a_2,0) = Z_1(a_1,q_1^*(0)) + Z_2(a_2,q_2^*(0))$, the value of producing during the first period. The values of $q_1^*(0)$ and $q_2^*(0)$ are found from (8) once the region within which a_1 and a_2 lie is identified. The latter is done by finding which bivariate integral bound in (9) includes the values of a_1 and a_2 and then reading off the corresponding region number shown at the right margin. (The only region not found in (9) is the area where $a_1 < \nu_1$ and $a_2 < \nu_2$, which corresponds to region I in (8)).

III. An Example

In this section, we illustrate the above methodology by calculating the Net Present Value (NPV) of manufacturing two products (e.g., two models of microcomputers) on a flexible system for a variety of input parameter values. The present values are calculated from equation (10) using the analytic solution obtained from solving (9). A mathematical software package (IMSL) is used to calculate bivariate cumulative normal distribution values.

Assume that a flexible production system can be built with different capacities in the range of $75 \leq Q \leq 125$ for an initial cost of $I_0(Q) = \$(7.5 + .025Q)$ million. The fixed cost per year is assumed to be one-tenth of the initial cost. The system is being considered for immediate purchase and would have a useful life of five years, after which it would be worthless.

Given the multiplicity of input parameters, there is a myriad of possible combinations which could be explored. In most of our analysis, we will consider cases with two relatively similar products which share many identical parameters. This restriction focuses attention on the value of switching the production mix when neither product has an a priori advantage in terms of either volatility or expected profitability. The initial values of the intercept terms for the profit margin functions of the two products (a_1,a_2) are assumed to be equal to 50. The drift of the stochastic process for each intercept term (α_1,α_2) is assumed to be 4.0. We let the slopes of the two profit margin functions (B_1,B_2) be equal to 0.0005. The instantaneous variances of the intercept terms (σ_1^2,σ_2^2) and the correlation between them (ρ_1) will take on different values, depending on the case being evaluated. However, we will generally assume that $\sigma_1 = \sigma_2 = \sigma$.

The value of δ_i, and hence μ_{M_i}, must be specified for each product. These values can typically be obtained from observable financial market data. For purposes of our example, we utilize μ_{M_i} values which are consistent with the CAPM. We assume an excess return on the market equal to .08, a risk-free rate of .08, and standard deviation of the market portfolio equal to .20. Since $\delta_i = \mu_{M_i} - \alpha_i$, with μ_{M_i} (in our example) determined by the CAPM, δ_i values will depend on σ_i and the correlation with the market. The correlation of A_1 (the intercept term of product one's profit margin function) with the market portfolio is assumed to be .25, while the correlation of A_2 with the market portfolio is allowed to vary as indicated in the legends to Tables I–IV.

In our calculation for Table I, we assume that the production mix can be adjusted semiannually and that the flexible system has capacity $Q = 100$ and $y_1 = y_2 = 1000$. That is, we are considering a system with an initial cost of \$10 million and annual capacity of 100,000 units. Table I shows the NPV of manufacturing two products on a flexible system for different variances and correlation coefficients of the profit margin intercept terms. Clearly this NPV is quite sensitive to the parameter values associated with the two products. Consider the case of $\rho_1 = .25$ and $\sigma = 10$. With these parameters, the manufacturing program using the flexible system has a negative NPV. However, if $\sigma = 25$ (e.g., due to large demand uncertainty), then the value of the production program increases dramatically to \$1,467,700. In general, the manufacturing program using the flexible system is seen to be especially valuable when the A_i's are negatively correlated and their variances are large.

We can calculate the value of flexibility by comparing the value of the future profit stream from a flexible system with that for a set of dedicated systems, each of which produces only one product.[13] Table II provides the results of such calculations in a comparison between the flexible system described above and a set of two dedicated systems. Each of the dedicated systems was chosen to have half the capacity and half the operating cost, C_F, of the flexible system. That is, the total capacity and operating costs of the flexible and "inflexible" alternatives were chosen to be the same. This comparison places the dedicated systems at a modest disadvantage since the optimal total capacity of k dedicated systems will generally exceed that of a flexible system capable of producing all k products. Also, the optimal capacity generally differs across dedicated systems as a function of the correlation between the A_i's and the market portfolio. A comparison based on optimal capacities would result in differing capacity combinations for each entry in Table II. We chose to avoid this rather cumbersome and potentially confusing approach since it added relatively little insight.

We also allowed the dedicated systems to be partially flexible in that their production rates could be adjusted each period (semiannually in our example).

[13] The procedure for valuing a dedicated system is a special case of the procedure used for valuing the flexible system. The problem is simpler since there are fewer regions over which to integrate and the regions are one-dimensional.

Table I

Net Present Value of a Flexible Production System

The production system NPV (in thousands of dollars) is calculated using a contingent claims approach and assuming a five-year system life with semiannual product mix adjustments. The production system has an initial cost of \$10 million and an annual fixed operating cost of \$1 million and can produce either or both of two products with a total annual capacity of 100,000 units. The instantaneous revenue for each product (indexed by i) depends on both the production rate $q_i(t)$, which is optimally determined by our model, and an underlying normal diffusion process (denoted by A_i) with constant drift and standard deviation parameters α_i and σ_i. For purposes of this example, $\alpha_1 = \alpha_2 =$ \$4 per year and $\sigma_1 = \sigma_2 = \sigma$, which takes on various dollar values per annum as indicated below. The correlation between the A_i processes is denoted by ρ_1. The riskless annual rate of interest used in the calculations is 8%. Expected rates of return on replicating portfolios were assumed to be consistent with the CAPM using an excess annual return on the market of 8%, with a standard deviation for the market portfolio of 20%. The correlation between process A_1 and the market portfolio is .25, while that for A_2 varies across the seven columns. For the respective columns (from left to right), that correlation is −.25, −.15, −.10, 0, .10, .15, and .25.

σ	$\rho_1 = -1$	$\rho_1 = -.5$	$\rho_1 = -.25$	$\rho_1 = 0$	$\rho_1 = .25$	$\rho_1 = .5$	$\rho_1 = 1$
5	250.0	98.5	22.6	−101.5	−224.5	−298.2	−443.4
10	912.8	504.5	309.7	24.7	−251.7	−427.7	−770.4
15	1,909.3	1,170.1	865.7	446.8	43.4	−228.5	−846.2
20	2,944.7	1,923.6	1,590.8	1,111.3	648.9	334.2	−688.6
25	3,898.5	2,729.2	2,441.3	1,947.4	1,467.7	1,167.5	−355.4

This seemed a fairer comparison and clearly more realistic than assuming a fixed production rate for the entire five-year horizon. However, it should be emphasized that, in practice, many NPV valuations of production facilities ignore the ability to adjust production rates on dedicated systems in response to uncertainty resolution over time. The limited flexibility provided by adjusting production rates can be quite valuable and generally should not be ignored.[14]

Using the results in Table II, we can directly compare the benefits of production flexibility versus any incremental costs of the flexible system as compared to the dedicated systems. For example, if $\sigma = 20$ and $\rho_1 = 0$, then a flexible system should be purchased only if the present value of its purchase cost does not exceed that of the combined dedicated systems by more than \$1 million. It should be noted that the value of flexibility generally increases as the time horizon is lengthened. For example, when $\sigma = 15$ and $\rho_1 = 0$, the value of flexibility as shown in Table II is approximately \$.65 million. This value was computed for $T = 5$ years. For $T = 8$ (with semiannual adjustments), the value would be \$1.54 million. The comparable value for $T = 10$ would equal \$2.24 million.

Table II clearly illustrates that the value of flexibility increases as the correlation, ρ_1, decreases and the variance, σ, increases. The last column in the table shows that there is no value to flexibility when the stochastic parameters (A_i's) are perfectly correlated and have the same variance. However, it is easy to illustrate that flexibility does have some limited value when $\rho_1 = 1$ if $\sigma_1 \neq \sigma_2$. For

[14] This could be viewed as the basic message in Brennan and Schwartz (1985). The point is also illustrated in a monopoly context in McDonald and Siegel (1986).

Table II

Present Value of Production Flexibility

The Present Values (in thousands of dollars) displayed below represent the Present Values of future profits from a flexible production system minus that for two dedicated (single-product) production systems, each of which has one-half the total capacity and annual fixed operating cost of the flexible system. The instantaneous revenue for each product (indexed by i) depends on both the production rate $q_i(t)$, which is optimally determined by our model, and an underlying normal diffusion process (denoted by A_i) with constant drift and standard deviation parameters α_i and σ_i. For purposes of this example, $\alpha_1 = \alpha_2 = \$4$ per year and $\sigma_1 = \sigma_2 = \sigma$, which takes on various dollar values per annum as indicated below. The correlation between the A_i processes is denoted by ρ_1. The flexible production system has an annual fixed operating cost of \$1 million and can produce either or both of two products with a total annual capacity of 100,000 units. A five-year life is assumed with semiannual product mix adjustments. Semiannual adjustments are also made for the output of each dedicated system. The riskless annual rate of interest used in the calculations is 8%. Expected rates of return on replicating portfolios were assumed to be consistent with the CAPM using an excess annual return on the market of 8%, with a standard deviation for the market portfolio of 20%. The correlation between process A_1 and the market portfolio is .25, while that for A_2 varies across the seven columns. For the respective columns (from left to right), that correlation is −.25, −.15, −.10, 0, .10, .15, and .25.

σ	$\rho_1 = -1$	$\rho_1 = -.5$	$\rho_1 = -.25$	$\rho_1 = 0$	$\rho_1 = .25$	$\rho_1 = .5$	$\rho_1 = 1$
5	215.0	159.3	131.1	102.9	75.6	49.6	0
10	763.5	728.8	439.0	338.6	245.7	161.0	0
15	1,452.8	980.4	808.3	651.8	507.5	363.9	0
20	1,996.9	1,314.8	1,149.5	1,000.7	863.0	708.2	0
25	2,315.6	1,551.9	1,463.6	1,362.5	1,266.3	1,154.2	0

example, if $\sigma_1 = 10$ and $\sigma_2 = 20$, then a value of flexibility can be calculated, which turns out to be approximately \$50,000.

Table III illustrates the effect on the flexible system's NPV when the frequency at which product mix decisions are made is changed. This effect is examined in seven cases. In columns 3, 4, and 5, we hold the correlation coefficient constant while changing the variance. In columns 2, 4, and 6, the correlation coefficient takes on different values while the variance is unchanged. The first and last columns correspond to more extreme cases where flexibility is very valuable and only slightly valuable, respectively.

The NPV for $N = 1$ indicates the value of production when there is no flexibility, i.e., when a decision is made at the beginning of the time horizon as to the product mix which will be used throughout the life of the system. The value of the manufacturing program can be seen to increase dramatically as one switch point is added halfway along the time horizon (i.e., $N = 2$). The present value continues to increase as the product mix is adjusted more frequently. This increase is more pronounced when the variance of the stochastic parameters are large and the correlation between these parameters is low. For example, when $\sigma = 25$ and $\rho_1 = 0$, the NPV increases by approximately \$3.3 million as N increases from 1 to 20. For $\rho_1 = 1$ and $\sigma = 10$, the comparable improvement in NPV is less than \$.2 million.

Table III

The Effect of Adjustment Frequency on System NPV

N denotes the number of points during the five-year life of the flexible production system at which the product mix can be adjusted. For each of the specified values of N, the Net Present Value (in thousands of dollars) for that production system is calculated using the indicated stochastic process parameters. As previously, the standard deviation of the underlying process for each product is assumed to equal σ, which takes on various dollar values per annum as indicated below. The correlation between the two A_i processes is denoted by ρ_1. The flexible production system has an initial cost of \$10 million and an annual fixed operating cost of \$1 million and can produce either or both of two products with a total annual capacity of 100,000 units. For each product, the underlying stochastic process (denoted by A_i) is a normal diffusion with a constant drift parameter of \$4 per year. The riskless annual rate of interest used in the calculations is 8%. Expected rates of return on replicating portfolios were assumed to be consistent with the CAPM using an annual excess return on the market of 8%, with a standard deviation for the market portfolio of 20%. The correlation between process A_1 and the market portfolio is .25, while that for A_2 varies across the seven columns. For the respective columns (from left to right), that correlation is −.25, −.15, 0, 0, 0, .15, and .25.

N	$\rho_1 = -1$, $\sigma = 20$	$\rho_1 = -.5$, $\sigma = 15$	$\rho_1 = 0$, $\sigma = 5$	$\rho_1 = 0$, $\sigma = 15$	$\rho_1 = 0$, $\sigma = 25$	$\rho_1 = .5$, $\sigma = 15$	$\rho_1 = 1$, $\sigma = 10$
1	118.3	−227.7	−210.5	−680.2	−1,138.6	−1,123.6	−933.3
2	1,767.1	544.3	−152.1	−89.7	527.7	−684.4	−866.1
3	2,280.3	804.7	−131.6	125.7	1,110.9	−507.2	−831.1
4	2,525.5	934.9	−121.1	237.6	1,407.2	−411.8	−810.4
5	2,668.4	1,013.1	−114.7	306.2	1,586.3	−352.4	−796.9
10	2,944.7	1,170.1	−101.5	446.8	1,947.4	−228.5	−770.4
15	3,034.0	1,222.5	−97.0	494.7	2,068.7	−185.7	−756.7
20	3,078.1	1,248.7	−94.8	518.8	2,129.6	−164.0	−751.2
60	3,165.1	1,301.2	−90.1	567.5	2,251.5	−120.1	−739.9
1500	3,206.8	1,326.8	−87.6	591.3	2,310.4	−98.5	−734.1

It is important to note that the marginal increase from adding an additional switch point decreases as the number of these points increases. As N becomes very large, the value of the manufacturing program will rapidly converge to its value under continuous adjustment.[15] It is interesting to note that the increase in present value from switching "daily" ($N = 1500$) rather than monthly ($N = 60$) is quite small—in fact, only about half the increase obtained from switching monthly rather than quarterly.

In our model, we have assumed that there are no costs associated with adjusting the product mix. It may be more plausible to incorporate a cost for each adjustment point.[16] In this case, the product mix would never be adjusted

[15] Since we use an analytic formula which can quickly yield numerical results, NPVs can be readily obtained for large values of N (e.g., $N = 1500$). Thus, we can easily obtain quite accurate approximations to the limiting continuous adjustment case.

[16] If there is a switching cost only when a product adjustment actually takes place, then it may not be optimal to make adjustments to the product mix at every decision point. Such conditional switching costs can be incorporated into our model, though they add a great deal of complexity (e.g., see Triantis (1988)). However, we can obtain a lower bound for the present value in this case by assuming that there is a fixed cost for each decision point, whether or not an adjustment occurs. This lower bound will be a good approximation to the optimum for situations where the cost of the actual adjustment is small compared to the fixed cost of periodic review.

continuously. In fact, we can easily determine the optimal frequency of review such that the cost of an additional decision point would exceed the benefit from the extra adjustment. For example, if $\sigma = 25$ and $\rho_1 = 0$ and the fixed cost per adjustment point is \$1,000, it turns out that the optimal number of adjustment points is $N = 60$. Since this review cost is not unrealistically large, it may be inappropriate to model operating flexibility using continuous adjustments, as has been typically done in other papers.

We can also use our model to determine the optimal capacity for a flexible production system. Table IV presents NPVs for production programs on systems with different capacities. These NPVs are shown for the same seven cases as in Table III. As one would expect, we can see in comparing the middle three columns

Table IV

NPV of Flexible Production Systems with Differing Capacities

In this example, the total annual capacity for a two-product flexible production system is indexed by Q and allowed to vary between 75,000 and 125,000 units. For each of the specified values of Q, the Net Present Value (in thousands of dollars) for that production system is calculated using the indicated stochastic process parameters. The initial cost of a given system is a function of capacity and equal to \$(7.5 + .025Q) million. The annual fixed operating cost for that system is one-tenth of its initial cost. As previously, the standard deviation of the underlying process for each product is assumed to equal σ, which takes on the various dollar values per annum indicated below. The correlation between the two A_t processes is denoted by ρ_1. For columns where purchase of the flexible production system is warranted, the NPV corresponding to the optimal capacity is underlined. We assume a five-year system life with semiannual product mix adjustments. For each product, the underlying stochastic process (denoted by A_t) is a normal diffusion with a constant drift parameter of \$4 per year. The riskless annual rate of interest used in the calculations is 8%. Expected rates of return on replicating portfolios were assumed to be consistent with the CAPM using an annual excess return on the market of 8%, with a standard deviation for the market portfolio of 20%. The correlation between process A_1 and the market portfolio is .25, while that for A_2 varies across the seven columns. For the respective columns (from left to right), that correlation is −.25, −.15, 0, 0, 0, .15, and .25.

Q	$\rho_1 = -1$, $\sigma = 20$	$\rho_1 = -.5$, $\sigma = 15$	$\rho_1 = 0$, $\sigma = 5$	$\rho_1 = 0$, $\sigma = 15$	$\rho_1 = 0$, $\sigma = 25$	$\rho_1 = .5$, $\sigma = 15$	$\rho_1 = 1$, $\sigma = 10$
75	2,106.9	499.9	−765.6	−248.7	911.0	−920.8	−1,381.1
95	2,893.7	1,134.1	−131.9	393.9	1,822.9	−287.3	−804.8
100	2,944.7	1,170.1	−101.5	446.8	1,947.4	−228.5	−767.4
101	2,948.1	1,172.0	−101.4	452.7	1,967.9	−221.1	−764.5
102	<u>2,949.2</u>	<u>1,172.3</u>	−103.3	457.2	1,987.1	−214.9	−763.0
103	2,948.1	1,171.1	−106.9	460.4	2,005.0	−210.0	−762.8
104	2,944.9	1,168.6	−112.1	462.4	2,021.8	−206.1	−763.8
105	2,939.5	1,164.7	−118.9	<u>463.3</u>	2,037.6	−203.3	−765.9
106	2,931.8	1,159.6	−127.2	463.0	2,052.2	−201.6	−769.1
107	2,921.7	1,153.2	−137.1	461.6	2,065.8	−200.9	−773.4
108	2,909.1	1,145.6	−148.4	459.1	2,078.3	−201.2	−778.8
109	2,893.9	1,136.9	−161.0	455.6	2,089.8	−202.5	−785.3
110	2,875.8	1,127.1	−175.0	451.0	2,100.2	−204.8	−792.7
115	2,740.9	1,062.4	−263.2	413.4	2,138.2	−230.2	−844.4
120	2,526.6	975.6	−377.1	353.7	<u>2,153.7</u>	−276.9	−917.5
125	2,235.4	871.5	−510.8	274.7	2,148.9	−342.4	−1,009.0

that ceteris paribus greater variance results in a greater optimal capacity. (Where the maximum NPV is negative, it is optimal not to purchase a system.) Also, the three columns which have $\sigma = 15$ indicate that the optimal capacity of purchased systems is smaller for lower correlation coefficients.

The example presented above has illustrated the use of our valuation model for comparing alternative production systems and selecting the most appropriate system for a given set of product parameters. While we have looked at a particular system under a limited number of parameter values, it is straightforward to examine alternative scenarios. This type of valuation procedure should prove to be an important tool for evaluating investments in flexible production systems. The results obtained from our example support many of the descriptive notions regarding flexibility which are often cited. However, the model allows more precise analysis. For example, given the costs of periodic review for a system, a firm can determine the optimal frequency of product mix adjustment.

IV. Concluding Remarks

We can restate our valuation procedure as follows. Given that the contingent claim (i.e., the manufacturing program) can be replicated by existing securities, we can use an equivalent martingale valuation for each European option to manufacture an optimal product mix during a given period. In valuing each option, the multidimensional space of possible values for the underlying state variables is divided into several regions. The regions are differentiated by the rule for exercising the option.

A simple version of this procedure is often used when valuing a standard European call option. In that case there are two regions: in one region, the stock price exceeds the exercise price at maturity and the option is exercised; in the other region, the call is not exercised. In our model, the exercise decision is much more complicated due to multiple products, downward-sloping marginal profit curves, flexibility, and a capacity constraint. In particular, the exercise decision (i.e., the choice of product mix) is made to maximize profits subject to the capacity constraint. Since there is a different exercise decision for each realization of the stochastic parameters, there is an infinite number of possible exercise decisions. However, these decisions follow a small number of rules where the optimal production quantities can be expressed as (closed-form) functions of the stochastic parameters. Each of these rules holds in a different region of the state space.

After determining the optimal control action for each region, we calculate the discounted expected value of profits for each region using an equivalent probability measure and a risk-free discount rate. We sum over the feasible regions to obtain the option value for each period and then sum over the periods to obtain the overall value for the manufacturing program.

Several of our assumptions could be relaxed and the basic valuation approach would still hold. For example, we could easily allow the parameters of the processes for the A_i to be time-varying (but nonstochastic). In such a situation, the distribution of the A_i's at each adjustment point would still be normally distributed. However, in calculating the present value of the option which expires at $n\tau$, we would use the average values for α_i, σ_i, and ρ_{ij} over the interval $[0,n\tau]$ in place of the constant parameters used above in our model. It would also be necessary to recalculate η_i using a procedure analogous to that described in footnote 12. Incorporating such time-varying parameters may be useful in modeling the drift and variance of demand for some products. For example, the drift term could be constructed to follow a product life cycle. Also, for a new product, the variance parameter could be specified so that it decreased as the product matured.

We could also allow a variety of nonlinear demand curves without adding major complications. However, increasing the number of products rapidly increases the number of feasible regions and consequently the number of integrals to be evaluated. As mentioned earlier, accurate representation of conditional switching costs may also be incorporated into the model, although this may add a great deal of complexity. In this respect, the richness of the model one constructs is limited by one's willingness to perform the integrations from which the valuation is obtained. Of course, numerical integration techniques can be used to obtain accurate approximations for most models. In this regard, Monte Carlo simulation would be particularly well-suited for the more complicated versions of our valuation problem.[17]

In summary, we have considered a departure from the commonly assumed price-taking behavior of firms and allowed for multiple products and a capacity constraint. We have shown how a flexible production system can be valued in such an environment. While we have been concerned with studying the effects of product flexibility on the value of a firm's manufacturing program, modeling other real asset valuation problems using a sequence of complex options is an interesting area for further research and implementation.

Appendix

In deriving the solution of (9), many of the integrals were reduced to simpler ones which are listed below. The integrals below may be referenced when solving similar valuation problems. These integrals can be solved by using familiar tricks such as integration by parts, substitution of variables, and completing squares. We let $n(z)$ denote the standard normal density function, $N(d)$ denote a standard cumulative normal, and $N_2(h_1,h_2,\rho)$ denote a standard bivariate cumulative normal.

$$\int_a^b z\ n(z)\ dz = n(a) - n(b). \tag{A1}$$

$$\int_a^b z^2\ n(z)\ dz = an(a) - bn(b) - N(a) + N(b). \tag{A2}$$

[17] See Boyle (1977) for an illustration of the use of this computational technique in valuing European options.

Let $\kappa_1(s) = \dfrac{h - \rho s}{[1 - \rho^2]^{1/2}}$, $\kappa_2(s) = \dfrac{s - h\rho}{[1 - \rho^2]^{1/2}}$.

$$\int_a^b n(z)N(\kappa_1(z))\,dz = N_2(h, b; \rho) - N_2(h, a; \rho). \tag{A3}$$

$$\int_a^b z\,n(z)\,N(\kappa_1(z))\,dz$$
$$= n(a)\,N(\kappa_1(a)) - n(b)\,N(\kappa_1(b)) + \rho\,n(h)\,\{N(\kappa_2(a)) - N(\kappa_2(b))\} \tag{A4}$$

$$\begin{aligned}\int_a^b z^2\,n(z)\,N(\kappa_1(z))\,dz &= N_2(h, b, \rho) - N_2(h, a, \rho) - b\,n(b)\,N(\kappa_1(b))\\ &+ a\,n(a)\,N(\kappa_1(a)) + \rho n(h)\,[[1 - \rho^2]^{1/2}\{n(\kappa_2(b)) - n(\kappa_2(a))\}\\ &- h\rho\{N(\kappa_2(b)) - N(\kappa_2(a))\}].\end{aligned} \tag{A5}$$

$$\int_a^b n(z)\,n(\kappa_1(z))\,dz = n(h)\,[1 - \rho^2]^{1/2}\{N(\kappa_2(b)) - N(\kappa_2(a))\}. \tag{A6}$$

$$\begin{aligned}\int_a^b z\,n(z)\,n(\kappa_1(z))\,dz &= n(h)\{(1 - \rho^2)[n(\kappa_2(a)) - n(\kappa_2(b))]\\ &+ [1 - \rho^2]^{1/2}h\rho[N(\kappa_2(b)) - N(\kappa_2(a))]\}.\end{aligned} \tag{A7}$$

In addition, we made use of the following identity (which can be found in Abramowitz and Stegun (1972)):

$$N_2(h_1, h_2, \rho) + N_2(-h_1, h_2, -\rho) = N(h_2). \tag{A8}$$

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