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Default Risk and the Duration of Zero Coupon Bonds

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ABSTRACT

This paper applies a contingent claims approach to examine the duration of a zero coupon bond subject to default risk. One replicating portfolio for a default-prone zero coupon bond contains a long position in the default-free asset plus a short position in a put option on the underlying assets. The duration of the bond is shown to be a weighted combination of the duration of the default-free bond and the put option. The duration is less than maturity and is not an immunizing duration. The technique is then extended to subordinated debt.

DURATION IS A SUBJECT of much concern to researchers and bond portfolio managers. In the last five decades, the literature on duration has grown from a simple explication of its property as a measure of average maturity (Macaulay (1938)) to sophisticated strategies such as funding multiple liabilities (Fong and Vasicek (1984)). Duration, however, is not without its critics. For example, the simplest framework for a duration model, a world of parallel yield curve shifts, is not only an unlikely state of nature but is also inconsistent with equilibrium (Ingersoll, Skeleton, and Weil (1978)). Fortunately, however, Bierwag, Kaufman, and Toevs (1982) have shown that a given duration measure, even the simplest one, can be consistent with equilibrium, as well as disequilibrium, processes. Thus, it should not be too surprising that, in spite of its limitations, duration is still widely used today in bond portfolio management.

Although progress and, in some cases, solutions have been developed for a number of practical and theoretical problems in duration, many remain unsolved. For example, the effect of default risk on duration is currently unknown. It hardly seems wise to assume that default-free duration strategies will be appropriate for default-prone markets. In a world of contingent claims contracting, however, investors can separate default effects from interest rate effects. Given the large size of the market for default-prone bonds, it is obvious, nonetheless, that many investors choose to accept both risks.[1] Therefore, the extent to which default premia are affected by interest rates, and, thus, have a potential impact

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[1] For example, from 1984 to 1988 private domestic borrowing was nearly three times federal government borrowing (*Federal Reserve Bulletin*, June 1989).

on duration, is an important and unresolved issue affecting many investors and, accordingly, is the subject of this paper.

The focus is on zero coupon bonds, which owing to their simplicity are the obvious starting point for this problem. Section I presents the model of default risk; Section II derives and illustrates the duration of a default-prone zero coupon bond; Section III examines the issues of immunization and the effect of subordination; and Section IV provides the conclusions.

I. Modeling Default Risk

Early attempts to model default risk have typically taken the form of cross-sectional regressions of yields (Fisher (1959)) or probability-based models (Bierman and Hass (1975), Yawitz (1977)). With the development of contingent claims theory (Merton (1973), Black and Scholes (1973)), new insights into the role and pricing of default risk have been gained. Merton (1974) has shown that the risk structure of interest rates can be modeled in a contingent claims framework and yields many interesting results. The model developed here is an adaptation of Merton's model to the problem of default risk. In this section we briefly review the Merton model.

Consider the following notation: i = default-free interest rate, y = yield on bond subject to default risk, F = face value of debt, T = maturity, and A = value of issuer's assets. Note that the default-free rate, i, should not be interpreted as a deterministic interest rate. Rather, it defines the (stochastic) price of a one dollar default-free bond as $P(i,T) = e^{-iT}$. Of course, deterministic interest rates are assumed in most contingent claims models, but this is neither required nor advisable here, a point to be discussed more fully in Section II.

We also assume no taxes or transaction costs, and the issuer has outstanding only a single zero coupon debt instrument. The bond price is $B = Fe^{-yT}$. When the bond is due, the firm's assets are worth A_T. If $A_T \leq F$, the bond is in default and the bondholder receives the assets. If $A_T > F$, the bond is paid off and the bondholder receives F. In this manner, the stock is equivalent to a call option on the assets of the firm.

As insurance against default, let the bondholder purchase a European put option on the assets of the firm. The exercise price is F, and the put price is $p(A,F,T)$. If $A_T < F$, the firm defaults on the bond, but the bondholder exercises the put, bringing a payoff of A_T from the bond and $F - A_T$ from the put for a total payoff of F. If $A_T \geq F$, the bond is paid off and the put expires, yielding a payoff of F from the bond and zero from the put for a total of F. Since the portfolio containing the risky bond and put is riskless, its current value must be $FP(i,T)$, which we designated as R, so that $B + p(A,F,T) = R$. Isolating the bond price gives

$$B = R - p(A,F,T). \tag{1}$$

The risky bond is equivalent to a risk-free bond and a short position in a put on the firm's assets. The holder of a risky bond thus accepts the risk of the firm's assets and receives compensation in the form of the put premium. It should be noted that the replicating portfolio used here is only one of many combinations of securities and options that could reproduce the risky debt. However, this portfolio is particularly attractive because it does not require balancing through time or with changes in asset values.

II. Duration of a Zero Coupon Bond

The duration of a nontaxable default-free zero coupon bond is well known to equal its maturity. However, many zero coupon bonds are neither nontaxable nor default free. Little (1984) has examined the case of default-free taxable zero coupon bonds. This paper focuses on the second case, nontaxable zero coupon bonds subject to default risk, and shows that the interest sensitivity of the option component of the bond gives the risky zero coupon bond a duration less than its maturity. In order to incorporate the duration concept into the model of a non-default-free bond, it is necessary to consider the applicability of duration to an option.

Garman (1985) has examined the duration concept as it applies to options and shows that duration of an option is a legitimate measure of its interest sensitivity. The source of an option's duration is the bond component of the replicating portfolio. Because duration is an operation in comparative statics, it remains robust in the comparative statics framework of contingent claims theory. However, as Garman notes, comparative statics analysis requires that we impose restrictions on other variables that may in fact change while we assume them to be constant. Thus, comparative statics allows us only to examine the behavior of a model in isolation from certain other influences. However, the requirements of the model presented here are no more restrictive than those of Garman's models and are actually somewhat more general in permitting stochastic interest rates.

In Garman's model, options are priced according to the Black-Scholes (1973) model.[2] Black and Scholes assume a deterministic default-free interest rate r. While it is permissible, as Garman does, to determine the duration of an option by finding $-(\partial W/\partial r)W$, where W is the option price, there is a fundamental inconsistency in so doing. On one hand, we examine the dynamics of the option price when r changes, while, on the other, we employ a model in which r is assumed to be constant.[3] When addressing the duration problem, we should have a preference for models in which interest rates are assumed to be stochastic.

The problem is best handled by considering Merton's (1973) alternative derivation of the option pricing model. Merton assumes that, in addition to the underlying asset following a lognormal diffusion process, the price of the default-free bond, $P(i,T)$, also follows a lognormal diffusion process with different parameters. Owing to the possibility of term structure effects, bond returns are not assumed to be independent.

Merton solves the option pricing problem by constructing a three-asset hedge

[2] Garman also considers the case where options are priced by the Black (1976) model. In our analysis, forward options are not necessary or appropriate.

[3] Garman clearly recognizes this inconsistency (p. 311) and offers his approach as a useful approximation.

involving positions in the underlying asset, the option, and the default-free bond. Merton derives the general equilibrium conditions under which no asset will be dominant, and the resulting solution is his equation (40). In applying Merton's approach to the problem at hand, we now require the assumption that the firm's assets and the default-free bond price follow lognormal diffusion processes. The solution for the put price is then

$$p(A,F,T) = FP(i,T)[1 - N(d_2)] - A[1 - N(d_1)],$$

where

$$d_1 = \frac{\ln(A/F) - \ln P(i,T) + (\sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T},$$

and $N(\cdot)$ is the cumulative normal probability. The volatility $\sigma^2 T$, however, has a different interpretation from that of the standard Black-Scholes model. It is defined as

$$\sigma^2 T = \int_0^T [\sigma_A^2 + \sigma_p^2(t) - 2\rho\sigma_A\sigma_p(t)]dt,$$

where σ_A^2 is the variance of the logarithmic return on the underlying asset, $\sigma_p^2(t)$ is the local variance of the logarithmic return on the default-free bound, and ρ is the correlation between the returns on the asset and the bond.[4] The bracketed term is the instantaneous variance of the return on an asset defined as the ratio of the underlying asset price to the product of the exercise price and the default-free bond price. As Merton notes, the deterministic risk-free rate that appears in the Black-Scholes model (rT) has been replaced by $-\ln P(i,T)$, which equals iT, and the present value of the exercise price, Fe^{-rT}, is now given as $FP(i,T)$ or Fe^{-iT}.

The mathematical operations necessary to obtain the duration of the Merton model are no different from those required to obtain the duration of the Black-Scholes model. However, there are significant differences in the interpretation of the variables and in the application of the model as well. While the interest rate is obviously different, the variance is also somewhat different, as noted above. In fact, the variance of the default-free bond will play a role in determining the duration of the option. This result is intuitively appealing because it means that uncertainty in the default-free interest rate, the state variable in any duration model, will enter into the formula for duration. This is not a common finding in duration models.

To solve our duration problem, we apply the approach of Garman to the Merton model. That is, we treat duration as a comparative statics analysis of interest rate changes. Thus, the partial differentiation of (1) with respect to i gives

$$\partial B/\partial i = -RT + RT[1 - N(d_2)] = -RTN(d_2), \tag{2}$$

[4] Note that the local variance of the bond return is permitted to change over time.

using the comparative statics of the option pricing model.[5] Duration, D_B, is found as $-(\partial B/\partial i)/B$:

$$D_B = TN(d_2)(R/B). \tag{3}$$

In a similar vein we can express the duration as a weighted average of the durations of the default-free bond, D_R, and the put, D_P:

$$D_B = w_R D_R + w_p D_p, \tag{3a}$$

where $w_R = R/B$, $w_P = 1 - (R/B)$, $D_R = T$, and $D_P = TR(1 - N(d_2))/p(A,F,T)$. Equation (3a) reduces to equation (3).[6]

It is well known that, in the risk-neutral valuation relationships that are conveniently used in option pricing models, the Gaussian probability $N(d_2)$ is the probability of exercise of the call option. Thus, here it is the probability that the bond will not default. The duration of a default-free zero coupon bond can be seen as the maturity (T), multiplied by the present value of the face value $FP(i,T)$, divided by the price $FP(i,T)$. This, of course, reduces to simply the maturity. When the bond is subject to default, a similar interpretation can be given, but expected values replace certain values. The expected payoff to the holder of the risky bond, $E(B_T)$, is $F - E(p_T)$, where $E(p_T)$ is the expected value of the put at expiration. It is well known that $E(p_T) = F[1 - N(d_2)] - AP(i,T)^{-1}[1 - N(d_1)]$. Thus, $E(B_T) = FN(d_2) + AP(i,T)^{-1}[1 - N(d_1)]$. The first term, $FN(d_2)$, is $F\text{prob}(A_T > F)$, which is the value of a full payoff times the probability of a full payoff. The second term, $AP(i,T)^{-1}[1 - N(d_1)]$ is known to be $E(A_T | A_T < F)\ \text{prob}(A_T < F)$, which is the conditional expected value of a partial payoff times the probability of a partial payoff. Since the assets have zero duration, only the first term enters into the duration formula.[7] The duration is, thus, the maturity (T) multiplied by the present value at the default-free rate of

[5] See, e.g., Jarrow and Rudd (1983), pp. 119–120, adapted to the Merton model.

[6] A risk-free issue is a special case of the model. Define a risk-free issue as one in which $\sigma^2 = 0$ and $A > R$. In this case, the assets could be escrowed and would grow to asufficient value to pay off the debt. There exists a real number, k, such that $A = FP(k,T)$, where $k < i$ so that $A > FP(i,T)$. Using the value of $d_1 = (\ln(A/F) - \ln P(i,T) + (\sigma^2 T/2))/\sigma\sqrt{T}$, substitute $FP(k,T)$ for A and take the limit as $\sigma \to 0$:

$$\lim_{\sigma \to 0} \frac{T(i - k) + (\sigma^2/2)T}{\sigma\sqrt{T}} = \infty.$$

Then $N(d_1) \to 1$ and $N(d_2) \to 1$. Merton's (1974) equation (14) states that the yield, y, is

$$y = \frac{\ln\left\{\dfrac{F}{A(1 - N(d_1)) + RN(d_2)}\right\}}{T},$$

which in this case reduces to $y = i$. Substituting for y and $N(d_2)$ in equation (3) gives $D_B = T$. A second risk-free case, where bankruptcy is certain, is specified by having $k > i$. Then the limiting value of d_1 is $-\infty$, $N(d_1) \to 0$, $N(d_2) \to 0$, and D_B goes to zero. Yet a third, but trivial, risk-free case is an all-equity firm, whereupon $F = 0$. Then $N(d_2) \to 1$, $R = B$, and $D_B = T$.

[7] See Garman (1985, p. 314) for a discussion of the point regarding the zero duration of the assets. This applies to Garman's case involving the duration of options on spot commodities, rather than on forward contracts.

the face value ($R = FP(i,T)$) times the probability of solvency ($N(d_2)$), divided by the price (B). Thus, the duration of a default-prone bond is equivalent to that of a default-prone bond that pays off in full with probability $N(d_2)$ and pays nothing with probability $1 - N(d_2)$. That a risk-neutral interpretation is appropriate is well known from contingent claims theory. Investor risk preferences are embedded in the value of the underlying instrument, the assets of the firm.

Even though $0 \leq N(d_2) \leq 1$, $R/B > 1$ suggests that duration might be greater than T. However, it is easy to show that this is impossible. Note that $D_B > T$ if and only if $N(d_2)R/B > 1$. However, $N(d_2)R/B$ can be expressed as $N(d_2)FP(i,T)/\{A(1 - N(d_1)) + P(i,T)N(d_2)\}$. It follows that this expression exceeds unity only if $N(d_1) > 1$, which we know is false. Thus, T is the limiting duration.

To examine the behavior of duration in light of changes in the underlying variables, a comparative statics analysis was performed. Unfortunately, the derivatives cannot be generally signed. Accordingly, some numerical examples were developed to assist in identifying the behavior of duration. However, these should be used only as examples and not as generalizations.

Figure 1 illustrates duration as a function of Merton's quasi-debt ratio, R/A, of the firm. Duration tends to decline steeply at low debt ratios and levels off at higher debt ratios. This reflects the fact that at high debt ratios the put has little time value, as default is likely, and the put price is primarily driven by the difference between the asset value and the face value of the bonds. Bonds of

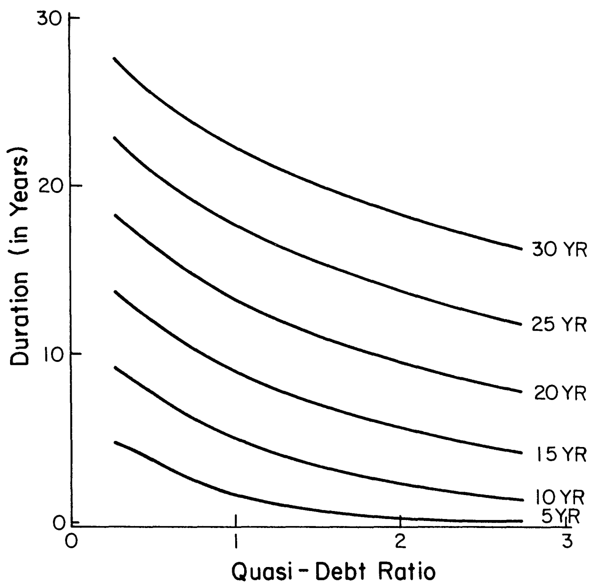

Figure 1. The relationship between duration and the quasi-debt ratio. The quasi-debt ratio is defined as the present value, discounting at the default-free rate, of the face value of the debt divided by the value of the assets. Each curve represents the duration of debt with the indicated maturity. The computations assume a default-free rate of 6%, a standard deviation of 25%, a face value of the debt of 500, and asset values ranging from 100 to 1000.

different maturities behave similarly in that regard. For what might be loosely considered a "normal" quasi-debt ratio, say 0.5, the duration is from about 3.5 for 5-year bonds to about 25.5 for 30-year bonds.

Figure 2 illustrates duration as it relates to the level of the default-free interest rate. For these examples, duration increases with the level of the interest rate as it does for default-free coupon bonds. However, for short-maturity bonds, duration increases at a much lower rate, but the durations of all bonds asymptotically approach the maturity, T. At an interest rate of about 7 percent, duration of a 5-year bond is about 4.5, while duration of a 30-year bond is about 27.8.

Figure 3 presents duration as a function of the volatility. For standard deviations up to around 0.15, duration almost equals maturity. As volatility increases, duration drops off rapidly but then stabilizes at a volatility of about 0.6–0.7. At $\sigma = 0.6$, duration of a 5-year bond is about 3.3, while duration of a 30-year bond is about 19.3. The much larger difference between maturity and duration is due to the greater time value and, hence, greater interest sensitivity of the put, arising from the effects of both maturity and volatility.

III. Some Other Issues

A. Duration and Immunization

We have obtained a formula, equation (3), for duration of a zero coupon bond subject to default risk. The question naturally arises as to whether the bond can be immunized. The default-prone bond is equivalent to a default-free bond and

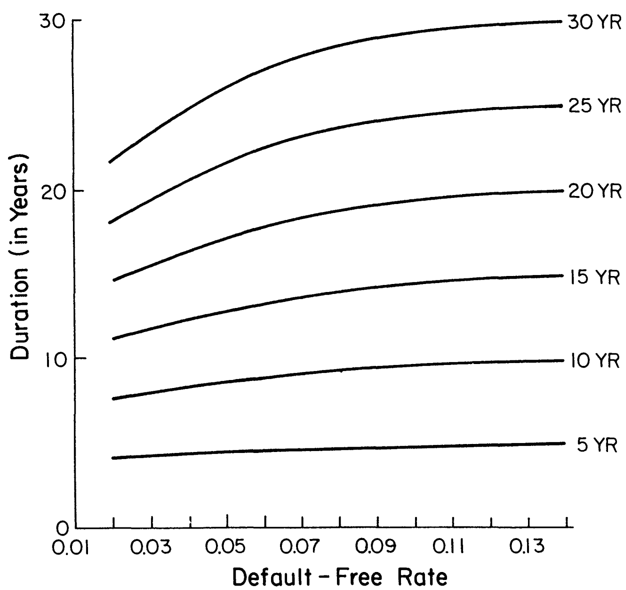

Figure 2. The relationship between duration and the default-free rate. Each curve represents the duration of debt with the indicated maturity. The computations assume a standard deviation of 25%, assets valued at 800, and a face value of the debt of 500.

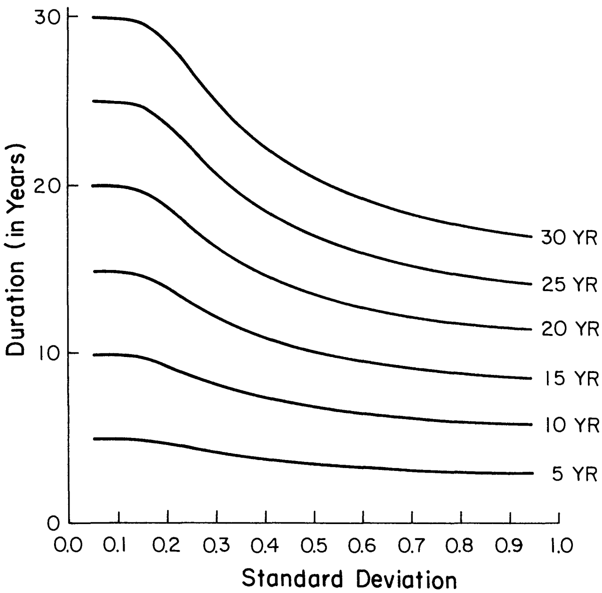

Figure 3. The relationship between duration and the standard deviation. The standard deviation is defined as the instantaneous standard deviation of the ratio of the underlying asset price to the product of the exercise price and the default-free bond price. Each curve represents the duration of debt with the indicated maturity. The computations assume a default-free rate of 6%, assets valued at 800, and a face value of the debt of 500.

a short position in a put on the firm's assets. An investor who purchases the default-prone bond and then sells it prior to maturity will find that, if interest rates increase (decrease), the default-free component will decrease (increase) his or her wealth, while the short put will increase (decrease) his or her wealth. From equation (3), however, there is no holding period, short of maturity, that immunizes since the investor will always be worse (better) off if interest rates increase (decrease). This is the same as the result for default-free bonds, implying that the default-free bond component dominates the put component in terms of interest sensitivity.

A default-free bond is immunized only if held to maturity. If the default-prone bond is held to maturity and does not default, then the investor receives the promised default-free return, plus the risk premium. If the bond defaults, the investor receives A_T, which is bounded from below at zero. It is incorrect to interpret this outcome as a failure of the immunization strategy. There are two effects operating: 1) an interest rate effect, which drives the immunization problem, and 2) a value effect, arising from the stochastic behavior of the underlying assets and which drives the default problem. Investors who assume default risk can easily trade away that risk and reduce the problem to one of immunizing a default-free bond.[8]

[8] For example, the holder of risky debt could buy a put option. Alternatively, the risky debt could be replicated with appropriate combinations of riskless debt and the risky asset. Thus, risky debt and a short position in the risky asset (in the correct proportions) can be used to reduce the problem to the straightforward case of immunizing a default-free zero coupon bond. I thank a referee for suggesting this explanation.

B. The Effect of Subordination

Using the Black and Cox (1976) framework for subordinated bonds, we can easily extend this model to account for the role of subordination. Let F^S be the face value of senior bonds and F^J be the face value of junior bonds. The issues mature simultaneously. Of course, the senior bonds are unaffected by subordination. The junior bond payoffs are

$$\begin{array}{cl} 0 & \textit{if} \quad A_T < F^S, \\ A_T - F^S & \textit{if} \quad F^S \leq A_T \leq F^J + F^S, \\ F^J & \textit{if} \quad A_T > F^J + F^S. \end{array}$$

A portfolio consisting of the junior bonds, a short call on the assets of the firm at exercise price F^S, and a long call on the assets of the firm at exercise price $F^S + F^J$ has a payoff of zero in all states. Letting B^J be the market value of the subordinated debt, we have

$$B^J = -c(A,F^S + F^J,T) + c(A,F^S,T). \tag{4}$$

Then letting $R^S = F^S P(i,T)$ and $R^J = F^J P(i,T)$, duration, D^J, is found as $-(\partial B^J/\partial i)/B^J$:

$$D^J = T[(R^J/B^J)N(d_{2,SJ}) + (R^S/B^J)(N(d_{2,SJ}) - N(d_{2,S}))], \tag{5}$$

where

$$d_{2,SJ} = [\ln(A/(F^S + F^J)) - \ln P(i,T) + (\sigma^2/2)T]/\sigma\sqrt{T}$$

and

$$d_{2,S} = [\ln(A/F^S) - \ln P(i,T) - (\sigma^2/2)T]/\sigma\sqrt{T}.$$

Note that (5) is more general than (3), for, if there is no senior debt, $F^S = 0$, $R^S = 0$, and $N(d_{2,SJ}) = N(d_{2,S})$. Then (5) reduces to (3).

Subordinated debt can be viewed as a long position in the assets and short positions in the stock and the senior debt. Working from this approach, the duration of subordinated debt can be expressed in terms of the duration of the senior debt:

$$D^J = T[(R^J + R^S)/B^J]N(d_{2,SJ}) - D^S(B^S/B^J), \tag{5a}$$

where D^S is the duration and B^S is the value of the senior debt. Note that the duration of the junior debt contains a term that adjusts for the duration of the senior debt. Substituting equation (3) for D^S gives

$$D^J = T[(R^J + R^S)N(d_{2,SJ}) - R^S N(d_{2,S})]/B^J. \tag{5b}$$

Like equation (3) for senior debt, equation (5b) can be interpreted in a risk-neutral world. Duration is the maturity weighted by the present value of the expected full payoff of the total debt minus the present value of the expected full payoff of the senior debt, divided by the price. In other words, the duration of the junior debt is the maturity weighted by the present value of the expected full payoff of the junior debt, $(R^J + R^S)N(d_{2,SJ}) - R^S N(d_{2,S})$, divided by the price of the junior debt. This interpretation, like that of the senior debt, is consistent with what we know duration of a zero coupon bond to be—maturity weighted by the present value of the payoff.

IV. Conclusions

The duration of a zero coupon bond subject to default risk does not equal its maturity. A default-prone zero coupon bond can be replicated by a portfolio of a default-free bond and a short put on the firm's assets. The duration of such a bond is a weighted combination of the duration of the default-free bond and the duration of the put. Although the duration is not immunizing, it retains its character as a measure of price volatility in response to interest rate changes. It was shown here that such bonds have durations lower than their maturities and, thus, are less sensitive to interest rates than their default-free counterparts. The model was then extended to subordinated zero coupon bonds. For both senior and junior debt, a risk-neutral interpretation shows that duration is still the maturity weighted by the payoff, but where the latter is expressed as an expected rather than certain value.

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