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On Arbitrage-Free Pricing of Interest Rate Contingent Claims

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ABSTRACT

Unlike most interest rate claim models, the Ho-Lee model utilizes full information on the current term structure. Unfortunately, the model has a major deficiency in that negative interest rates can occur. This article modifies the model such that interest rates are well behaved.

Ho and Lee (1986) have developed an ingenious discrete time approach for pricing bonds and interest rate contingent claims. Unlike other discrete time models, their model incorporates all information on the yield curve and produces consistent call and put prices.[1] Unfortunately, a major deficiency exists in their model. Specifically, the constraints they impose on movements of the entire discount function are not sufficient to ensure that interest rates remain nonnegative. The purpose of this note is to adjust the model such that interest rates are well behaved over time and never go negative.

The problem with the Ho and Lee model has been recognized by Heath, Jarrow, and Morton (1988), who provide a continuous time model in which negative interest rates do not occur. The problem has also been recognized by Pedersen, Shiu, and Thorlacius (1988), who have generalized the two-parameter model of Ho and Lee by introducing more parameters that allow the evolution of the discount function over the binomial lattice to be more flexibly managed. This note focuses on the two-parameter Ho-Lee model and shows that the range of one of the two parameters needs to be curtailed to obtain an economically meaningful model.

In the next section we provide an example which illustrates the failings of the Ho and Lee model. Then we provide the necessary adjustments to the two-parameter model. Finally, we provide further examples and additional comments.

I. The Arbitrage-Free Model

Ho and Lee impose arbitrage-free and path-independent constraints on the discount function over a binomial lattice. The arbitrage constraint precludes the

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[1] A direct application of binomial option models can produce inconsistent results in that call and put prices may not satisfy parity relationships. (See Bookstaber, Jacob, and Langsam (1986), Cox, Ross, and Rubinstein (1979), Pitts (1985), and Rendelman and Bartter (1980).) Furthermore such models typically do not use information from the yield curve. For discussion of some preference-dependent models, see Brennan and Schwartz (1979), Cox, Ingersoll, and Ross (1985), Courtadon (1982), Dothan (1978), Richard (1978), and Vasicek (1977).

possibility of riskless profits being generated over any single period of the binomial lattice. With no single-period arbitrage opportunities available, multiple-period arbitrage opportunities are closed off as well. This is best illustrated by considering a two-period portion of the binomial lattice with no single-period arbitrage opportunities available. If a two-period arbitrage opportunity exists, then a dynamic trading strategy can be constructed that generates nonnegative cash flows in all attainable states after two periods (with strictly positive cash flows in at least one state) and does not require any initial investment. To see that such a strategy is impossible, note that, since the cash flows in the second period are strictly nonnegative, to avoid single-period arbitrage, the portfolio values in the first period must be nonnegative. However, nonnegative portfolio values in the first period, together with no single-period arbitrage opportunities, imply that the current portfolio value must be nonnegative. This contradicts the fact that the two-period strategy is an arbitrage strategy, and we therefore conclude that the lack of single-period arbitrage opportunities eliminates any two-period riskless arbitrage possibilities.[2]

The restrictions imposed by Ho and Lee imply that all bonds can be priced consistently by a model characterized by a "risk-neutral" probability parameter, π, and an "interest rate spread" parameter, δ. Unfortunately, as the following example illustrates, the evolution of the discount function over the binomial lattice may not be constrained to a monotone nonincreasing function bounded in the interval [0, 1].

Example 1: The current yield curve for the first five periods is $r_0(1) = 9.531\%$, $r_0(2) = 8.6178\%$, $r_0(3) = 7.696\%$, $r_0(4) = 7.2321\%$, $r_0(5) = 7.2321\%$. From this yield curve the discount function, shown in the first node of Figure 1, was constructed.[3] The evolution of the relevant discount function over the first three periods using the Ho-Lee model is shown for $\pi = 0.4$ and $\delta = 0.8$. Note that, for this selection of π and δ, the prices of some pure discount bonds exceed one, indicating negative interest rates in the lattice.

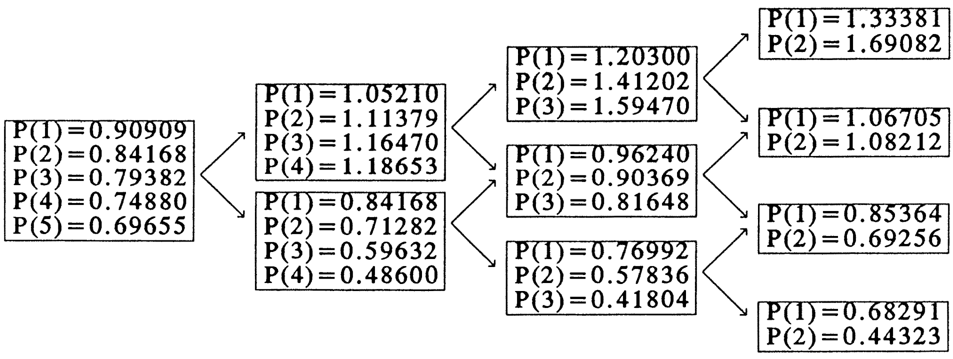

Figure 1. The discount function on the binomial lattice. $P(t)$ is the price of a pure discount bond with t periods to maturity. Given the initial prices of pure discount bonds, the Ho-Lee model, with $\pi = 0.4$ and $\delta = 0.8$, was used to generate prices of discount bonds at all other vertices.

[2] For an alternative proof of this result, see Pedersen, Shiu, and Thorlacius (1988).

[3] The pure discount bond prices were computed as $P_0(t) = \exp[-r_0(t)t]$.

Discount Function Shape Constraint: To correctly apply the two-parameter Ho and Lee model, a discount function shape constraint that forces the function to be a monotonically nonincreasing function bounded in the interval [0, 1] at all nodes in the lattice needs to be added. To achieve this constraint we adopt the notation of Ho and Lee and begin with their equation (22), namely

$$P_i^{(n)}(t) = \frac{P(t+n)\,h(t)h(t+1)\cdots h(t+n-1)}{P(n)\,h(1)h(2)\cdots h(n-1)}\,\delta^{t(n-i)} \quad \text{for } i = 0, 1, 2, \cdots, n, \tag{1}$$

where

$$h(t) = \frac{1}{\pi + (1-\pi)\delta^t} \quad \text{for } t \geq 1 \tag{2}$$

and

$$h(t) = 1 \text{ for } t = 0.$$

The additional constraint that is necessary at any time n and state i is

$$1 = P_i^{(n)}(0) \geq P_i^{(n)}(1) \geq P_i^{(n)}(2) \cdots \geq P_i^{(n)}(T)$$

or, equivalently,

$$P_i^{(n)}(t)/P_i^{(n)}(t+1) \geq 1,\ t = 0, 1, 2, \cdots, T-1. \tag{3}$$

From equation (1) the ratio of t-period bond prices in adjacent states, i and $i+1$, in period n is given by

$$\frac{P_{i+1}^{(n)}(t)}{P_i^{(n)}(t)} = \frac{1}{\delta^t}.$$

In particular, for $t = 1$, we have

$$P_{i+1}^{(n)}(1) = P_i^{(n)}(1)/\delta \geq P_i^{(n)}(1). \tag{4}$$

Hence, to avoid negative interest rates based on bonds with one period to maturity, the constraint $P_n^{(n)}(1) \leq 1$ must be added for each time period in the lattice. Substituting equation (1) into the constraint (for $t = 1$ and $i = n$), we obtain

$$\frac{P(n+1)}{P(n)}\,\frac{1}{\pi + (1-\pi)\delta^n} \leq 1 \text{ for } n = 0, 1, 2, \cdots, T-1, \tag{5}$$

from which we require

$$\delta \geq \left[\frac{P(n+1)/P(n) - \pi}{1-\pi}\right]^{1/n} \text{for } n = 0, 1, 2, \cdots, T-1. \tag{6}$$

Define

$$\delta_1(k) = \left[\frac{P(k+1)/P(k) - \pi}{1-\pi}\right]^{1/k} \text{ for } k = 0, 1, 2, \cdots, T-1.$$

Then the spread parameter δ must be constrained such that $\delta \geq \delta_1$, where $\delta_1 = \max_k[\delta_1(k)]$.

δ_1 has a simple economic interpretation. As δ decreases from 1 to δ_1, the volatility of interest rates increases. δ_1 is the lowest feasible value that the spread parameter can take to preclude negative interest rates based on single-period bonds.

PROPOSITION 1: *If all single-period pure discount bonds are priced below one, and if no single-period arbitrage opportunities exist, then, to avoid riskless arbitrage, the price of all pure discount bond prices of all maturities must be priced below one.*

Proof: We prove the result by contradiction. Suppose that, at a particular vertex i in period n, a bond of maturity t, $t > 1$, was priced with $P_i^{(n)}(t) > 1$. Consider the following strategy. Go short this bond, pocket the amount in excess of one unit, and invest the one unit by rolling it over at the one-period rate in each future state until maturity in period $t + n$. The investment process will certainly accumulate to more than one since all single-period interest rates are nonnegative. Hence, multiperiod arbitrage opportunities will be present. However, as discussed earlier, the exclusion of single-period arbitrage opportunities implies that no multiperiod arbitrage opportunities exist. Hence, the price of the original t-period bond, $P_i^{(n)}(t)$, cannot exceed one.

PROPOSITION 2: *If all single-period pure discount bonds are priced below one, and if no single-period riskless arbitrage opportunities exist, then at each node of the vertex the discount function must be monotone nonincreasing.*

Proof: We prove this result by contradiction. If at vertex i in period n $P_i^{(n)}(t') > P_i^{(n)}(t'')$ with $t' > t''$, then a multiperiod arbitrage opportunity exists. Specifically, one would sell the long-term expensive bond, use the proceeds to buy the short-term bond, and pocket the difference. At time t'', the one-unit face value of the bond would be received and would be invested at the one-period rate in each future state until time t', at which time the payment of one unit would be made. Again, since single-period interest rates are nonnegative, this strategy will yield arbitrage profits. Since no single-period arbitrage possibilities exist, no multiperiod arbitrage opportunities are available, and we thus conclude that the discount function at any node must be a monotonic nonincreasing function.

II. Option Prices and Feasible Discount Functions

The constraints of interest rate behavior imposed by Ho and Lee ensure that put-call parity conditions hold and no arbitrage opportunities result. Unfortunately, without the additional constraints just discussed, the early exercise feature on an American option can become valuable when it should not be.

Example 2: Consider a three-period call and put option with strike price \$85,000 on the previous five-period discount bonds. Using the same initial discount function as example 1, the minimum value of δ for $\pi = 0.4$ was computed as $\delta_1 = 0.9696$. Table I compares the value of American and European call options for one feasible value of δ ($\delta = 0.97$) and one infeasible value ($\delta = 0.80$). Note that in both cases put-call parity conditions hold for the European options. However,

Table I

Bond and Option Prices for Different Delta Values

Using the Ho-Lee model with $\pi = 0.4$, the values of three-period European and American call and put options with strike price \$85,000 are computed. The underlying bond is a five-period discount bond with face value \$100,000 and current price \$69,665.86.

Delta[a] Value	Bond Price	American Option Call	American Option Put	European Option Call	European Option Put
0.97	69,655.86	2,784.57	15,344.13	2,784.57	604.45
0.80	69,655.86	15,088.63	21,941.23	12,448.05	10,267.93

[a] The price of an American call exceeds that of its European counterpart because interest rates are negative when the delta value is 0.8.

only for feasible delta values will the price of the American option equal that of its European counterpart. For $\delta = 0.8$, the difference in call prices reflects the early exercise premium attributable to negative interest rates.

To implement the binomial model, Ho and Lee recommend that the parameters π and δ be estimated using an implied parametric approach. Specifically, if the price of a contingent claim were available, then the values of π and δ should be restricted to those values that equate the theoretical price with the observed price of the claim. With more than one contingent claim available, it is common to select the estimates $\hat{\pi}$ and $\hat{\delta}$, which minimize the sum of squared error between theoretical and observed prices. With the additional nonnegative interest rate constraint imposed, the search of the least squares estimate $\hat{\delta}$ must be restricted to intervals $[\delta_{\min}(\pi),1]$, where $\delta_{\min}(\pi)$ is the lowest delta value that ensures that interest rates never become negative.

III. Conclusion

The arbitrage-free option pricing model of Ho and Lee is the first interest rate contingent claim model that uses all information from the yield curve and produces consistent prices based solely on arbitrage arguments. Extensions of this model to multi-parameter models and to continuous time models have recently been considered (Heath, Jarrow, and Morton (1988), Kishimoto (1988), and Pedersen, Shiu, and Thorlacius (1988)). With the additional constraint discussed here, the stochastic evolutions of interest rates are constrained to be nonnegative. This constraint involves curtailing the range of one of the two parameters. Obtaining estimates of the parameters π and δ based on implied parametric methods should explicitly take into account the restrictions imposed on the parameters.

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