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Corrections for Trading Frictions in Multivariate Returns

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ABSTRACT

When observed stock returns are obtained from trades subject to friction, it is known that an individual stock's beta and covariance are measured with error. Univariate models of additive error adjustment are available and are often applied simultaneously to more than one stock. Unfortunately, these multivariate adjustments produce non-positive definite covariance and correlation matrices, unless the return sample sizes are very large. To prevent this, restrictions on the adjustment matrix are developed and a correction is proposed, which dominates the uncorrected estimator. The estimators are illustrated with asset opportunity set estimates where daily returns have trading frictions.

A NUMBER OF UNIVARIATE nonsynchronous trading adjustments are available for beta, but no multivariate adjustment has been specifically designed for a general covariance matrix. For example, the univariate trading friction adjustment of Cohen, Hawawini, Maier, Schwartz, and Whitcomb (CHMSW) (1983) is a generalization of Scholes and Williams (1977). The CHMSW procedure adjusts beta to account for price-adjustment delays exceeding one period, where a portion of the true return is reflected in lagged returns. The procedure is extended and applied in Shanken (1987) to estimation of the daily multivariate covariance matrix of stocks' returns. Observed covariances significantly understated true covariances by a factor of two, suggesting that a daily covariance adjustment is necessary. Because of a large number of studies which employ multivariate daily stock returns, it is therefore important that the adjustment results in a good estimator of the covariance matrix.

While no evidence is available regarding other properties, it is shown in CHMSW that their univariate adjusted estimator has the desirable statistical properties of consistency and unbiasedness. Unfortunately, when the adjusted covariance estimator is applied in finite samples, the resulting adjusted covariance matrix may not be a covariance matrix and the implied adjusted correlation matrix, independently, may not be a correlation matrix because the adjustment does not restrict the covariance and correlation matrices to be positive definite.[1] When these matrices are not positive definite, functions of these matrices can produce very unreliable inference results.

This paper provides empirical examples of the violations of the positive definite

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[1] In this study, it is assumed that return covariance matrices are full rank and positive definite.

property by the adjusted covariance matrix. A theorem provides the restrictions that must be placed on the eigenvalues of a function of the unadjusted and adjustment matrices, such that the trading adjustment is proper. A natural shrinkage correction arises out of the theorem which overcomes the violations. The small sample properties of the corrected adjustment procedure are investigated and found to be superior to the CHMSW-adjusted and the unadjusted estimators. While the paper uses the CHMSW adjustment in its illustrations, the theorem of Section II applies to any covariance adjustment such as those derivable from Scholes and Williams (1977), Dimson (1979), Merton (1980), Fowler and Rorke (1983), and Lo and MacKinlay (1988). The approach also applies in robust covariance estimation (Gnanadesikan (1977, p. 127)), where the trimming of extreme observations may be thought of as an additive adjustment to the usual estimator of the covariance matrix.

Section I provides samples, of increasing size, illustrating the lack of positive definiteness induced by the CHMSW trading adjustment. Section II proves a theorem on positive definite matrices and provides a preferable corrected trading frictions adjustment. Section III examines some sampling properties of the new adjustment, and Section IV applies the corrected adjustment to asset opportunity set estimation with daily returns.

I. Market Illustrations of Improper Trading Friction Adjustments in Daily Stock Data

In order to illustrate the problems that can arise from the multivariate application of the CHMSW adjustment to covariance matrices, 29 NYSE stocks without missing returns in the July 3, 1962 to December 31, 1987 period were randomly selected from the daily CRSP file. The S&P 500 index was included to form a set of 30 assets. The overall time period was partitioned into an increasing sequence of six subperiods all beginning on July 3, 1962. Thus, the first sample of $T = 200$ returns begins on July 3, 1962 and ends 200 observations later; the second sample begins on July 3, 1962 and ends 400 returns later; and so on until the entire sample of 6409 returns are exhausted. Therefore, the subperiods correspond to accumulated sample data as the agent moves through time.

The unadjusted covariance matrix, $\mathbf{S}$, was computed for each subperiod as well as the nonsynchronous trading adjustment matrix, $\mathbf{K}$, an element of which is given by

$$k_{i,j} = \textstyle\sum_{l=1}^{3} \operatorname{cov}(\tilde{r}_{i,t}, \tilde{r}_{j,t-l}) + \textstyle\sum_{l=1}^{3} \operatorname{cov}(\tilde{r}_{i,t-l}, \tilde{r}_{j,t}), \quad \forall i, j = 1, 2, \cdots, 30, \quad \text{if } i \neq j, \tag{1a}$$

and

$$k_{i,j} = 0, \quad \text{if } i = j. \tag{1b}$$

The choice of $l = 3$ leads and lags, no adjustment to the diagonal of the covariance matrix, and stock grouping coincides with Shanken (1987). The

Table I

Sample Eigenvalues and Correlations

Eigenvalues and correlations, for Cohen et al. (1983) trading frictions adjusted and unadjusted covariance matrices, **V** and **S**, respectively, of 30 securities (29 randomly selected NYSE stocks and the S&P 500 index) computed from T daily returns beginning July 3, 1962 and ending December 31, 1987 when $T = 6409$.

Sample Size and Matrix Adjustment	No. of Negative Eigenvalues in Covariance Matrix	No. of Correlations Greater than 0.5	Maximum Correlation
$T = 6409$			
S, Unadjusted	0	4	0.643
V, Adjusted	0	44	0.827
$T = 2000$			
S, Unadjusted	0	2	0.553
V, Adjusted	1	25	0.932
$T = 1000$			
S, Unadjusted	0	2	0.554
V, Adjusted	1	21	1.008
$T = 800$			
S, Unadjusted	0	4	0.569
V, Adjusted	1	23	1.029
$T = 400$			
S, Unadjusted	0	4	0.626
V, Adjusted	4	65	1.116
$T = 200$			
S, Unadjusted	0	8	0.666
V, Adjusted	9	151	1.422

adjusted covariance matrix is then given by $\mathbf{V} = \mathbf{S} + \mathbf{K}$ for each of the three groups of 30 securities.[2]

The *unadjusted* covariance matrix, **S**, is by definition positive definite, which implies that all of its eigenvalues exceed zero. Also by definition, the *unadjusted* correlation matrix has elements with absolute values less than one. The *adjusted* covariance matrix, **V**, and *adjusted* correlation matrix must also have the properties described for their unadjusted counterparts; otherwise they are not a covariance matrix and a correlation matrix.

The eigenvalues and correlations of the adjusted and unadjusted sample covariance matrices were computed and tabulated for the six subperiod samples. Table I shows a count of the number of negative eigenvalues, the number of correlations exceeding 0.5 in magnitude, and the maximum correlation in each subperiod's adjusted and unadjusted covariance matrices. The negative eigenvalues prove that the adjusted matrices, at every sample size except $T = 6409$, are not positive definite. The correlations, from the adjusted matrices, show the

[2] All sample covariances are computed as maximum likelihood estimates, which have lower MSE than the unbiased estimates. The use of unbiased estimators can result in autocovariance functions which are not positive definite. See Priestley (1981, p. 323). In this study, sample sizes are large enough that there exists little difference between the estimates, and, therefore, the maximum likelihood estimators are nearly unbiased.

general increase that is expected from the trading adjustment, with substantially more correlations exceeding 0.5 than in the unadjusted case. The adjusted correlation matrices, from samples of size less than $T = 2000$, are not positive definite, as evidenced by the maximum correlations exceeding 1.0.

Thus, these samples illustrate the likely problems that arise from application of the univariate nonsynchronous trading adjustment in multivariate data. Because the resulting adjusted matrices are potentially not covariance matrices, it suggests that an incorrect adjustment takes place in finite samples. It is apparent from Table I that, in many studies, sufficient sample sizes may not be available to mitigate the matrix rank problem induced by the trading friction adjustment. Therefore, either the adjustment must be abandoned or some form of restriction on the adjustment is necessary in order to prevent computation problems while still maintaining desirable statistical properties of the covariance estimator.

The next section presents a theorem that places some limitations on general additive nonsynchronous trading adjustments and leads to improved adjustments and estimators.

II. A Correction for Positive Definiteness

The adjusted covariance matrix, $\mathbf{V}$, is the result of the addition of the unadjusted matrix, $\mathbf{S}$, and the adjustment matrix, $\mathbf{K}$. The illustration of Section I was for the CHMSW trading friction adjustment, but the theorem and correction in this section are for any additive adjustment. The theorem presents a simple restriction on the size of the elements of the adjustment matrix only.

THEOREM: *a) There exists a scalar correction factor, f, such that the corrected matrix, $\mathbf{V}^* = \mathbf{S} + f\mathbf{K}$, is a proper covariance matrix and its associated correlation matrix is also proper, where*

$$0 < f \begin{cases} < \dfrac{1}{|\min d_i|}, & \text{if } \min d_i \leqq -1, \\ = 1, & \text{if } \min d_i > -1, \end{cases}$$

and where d_i, $i = 1, 2, \cdots, N$ are the eigenvalues of the matrix $\mathbf{KS}^{-1}$.

b) When no variance adjustment is present in the adjustment, $\mathbf{K}$,

$$0 < f \leqq \min\left[\frac{1}{|\min d_i|} - \epsilon, 1\right],$$

where ϵ is arbitrarily small.

Proof: See the Appendix.

The upper limit of $f = 1$ keeps the value of $f\mathbf{K}$ at its uncorrected univariate value when appropriate. Otherwise, the theorem shrinks the elements of the adjustment matrix until the eigenvalues of the adjusted matrix are restricted to positive values and the correlations are restricted to absolute values less than one.

Scholes and Williams (1977) conclude that the adjusted univariate variance estimator, for individual stocks, produces a slight overestimate of the true variance. If the adjustment matrix, $\mathbf{K}$, were to mitigate this bias, its diagonal elements would contain negative elements, the presence of which would cause the correction factor, f, to be even smaller. For portfolios, the unadjusted variance estimate is known to be too small and the diagonal elements of $\mathbf{K}$ would be positive.

Thus, a simple and tractable implementation of the basic CHMSW or any additive adjustment is proposed in multivariate settings. The procedure is implemented by computing the unadjusted covariance matrix, $\mathbf{S}$, the usual univariate trading adjustment, $\mathbf{K}$ in matrix form, and the matrix product, $\mathbf{KS}^{-1}$. The smallest eigenvalue of the latter matrix is identified, the largest allowable correction factor, f, calculated, and the corrected matrix, $\mathbf{V}^*$, computed. The implied correlations are checked; if some are found to exceed one, the value of f is reduced until the correlation limit is not violated.

It is known from the theorem that the mathematical properties of this new adjustment are improved, but the statistical properties are not known. The next section investigates this issue.

III. Sample Properties of the Corrected Covariance Matrix Estimator

If the adjusted covariance matrix, $\mathbf{V}$, is a consistent estimator, as is the case with the CHMSW adjustment, then the corrected covariance estimator, $\mathbf{V}^*$, also possesses the desirable property of consistency. The reason is that, if $\mathbf{V}$ is a consistent estimator of a positive definite matrix, then it must converge in probability to a positive definite matrix. The theorem and its proof show that, if the adjusted matrix is positive definite, the eigenvalues of $\mathbf{KS}^{-1}$ are all positive and the correction factor equals one. Thus, the corrected estimator, $\mathbf{V}^*$, converges to $\mathbf{V}$, a consistent estimator.[3]

For finite samples, a Monte Carlo study was conducted to determine the sampling properties of the corrected covariance matrix, $\mathbf{V}^*$, in comparison to the unadjusted estimator, $\mathbf{S}$, and the CHMSW adjusted estimator, $\mathbf{V}$. These details are addressed in the remainder of this section.

A. The Return Generating Process

True returns are assumed serially independent for a given asset as well as serially independent across assets at all nonzero lags. True returns are identically normally distributed across time,

$$\tilde{i}_t \sim N(\mu, \Sigma), \tag{2a}$$

where $\tilde{i}$ is the $(n \times 1)$ vector of true returns on the n assets, and μ and Σ are the true mean and covariance matrices, respectively. These matrices were parameterized in the Monte Carlo study by the covariance matrix and mean vector of a random sample of 30 CRSP stocks without missing daily returns in the interval July 3, 1962 to December 31, 1972.

[3] I thank the referee for pointing out the consistency property of the corrected estimator.

In general, observed returns are assumed generated under the restrictions given in CHMSW plus those of Shanken (1987), the latter of which use three lagged values of true returns and time independent values of the delay weights, $\tilde{\gamma}_{i,l,t-l}$. Specific parameter values were determined which generated observed returns similar to those of Shanken (1987). That is, observed return, $\tilde{r}_{it}$, on each of n assets are generated according to the delay structure,

$$\tilde{r}_{it} = \textstyle\sum_{l=0}^{3} \tilde{\gamma}_{i,l,t-l}\tilde{i}_{i,t-l}, \ \forall i = 1, 2, \cdots, n, \tag{2b}$$

where

$$\tilde{\gamma}_{i,l,t-l} \sim N(E(\tilde{\gamma}_{i,l}), V(\tilde{\gamma}_{i,l})),$$

$$0.25 \leqq E(\tilde{\gamma}_{i,0}) \leqq 1.00, \forall i,$$

$$V(\tilde{\gamma}_{i,l}) = 0.0625, \forall i, l,$$

$$E(\tilde{\gamma}_{i,l}) = E(\tilde{\gamma}_{i,0})\Theta_i^l, \quad 0 < \Theta_i < 1.00, \forall i, l,$$

such that

$$\textstyle\sum_{l=0}^{3} E(\tilde{\gamma}_{i,l}) = 1,$$

and where the $\tilde{\gamma}_{i,l,t-l}$ are independent across assets, i, lags, l, and time periods, $t - l$. The $E(\tilde{\gamma}_{i,0})$ were randomly (uniform distribution) assigned to the 30 stocks and the values of the weights, Θ_i, determined, which satisfied the given restrictions.[4]

B. Adjusted Covariance and the Relative Importance of the Lags

A simulation with 100 replications was performed for various sample sizes of T daily returns generated according to (2). The contemporaneous, lag 1, lag 2, and lag 3 covariances were computed for all asset pairs. The adjusted contemporaneous covariance was calculated according to (1), using the observed contemporaneous and lag covariances.

Table II displays the sample covariance results. At $T = 400$, for example, the average of the (870 possible) observed covariance elements is 0.24, compared to the average parameter covariance element of 0.50. On the other hand, the CHMSW adjusted element average of 0.48 is close to the average parameter element value of 0.50, as expected. The contributions of the observed contemporaneous covariance and each of the three observed lag covariances to the adjusted covariance are 0.50, 0.30, 0.14, and 0.06, respectively.[5] Therefore, the

[4] For example, American Electric Power, Inc., has a randomly assigned value of $E(\tilde{\gamma}_{i,0}) = 0.486$. The value of Θ_i was 0.564, which then gave $E(\tilde{\gamma}_{i,1}) = 0.274$, $E(\tilde{\gamma}_{i,2}) = 0.154$, and $E(\tilde{\gamma}_{i,3}) = 0.087$.

[5] These contributions are computed as in Shanken (1987). For example, with $T = 400$, the lag 1 covariance, averaged across all possible covariance pairs, is 0.07. From (1), $2 \times 0.07 = 0.14$ is the lag 1 total contribution. The adjusted covariance, averaged across all possible covariance pairs, is 0.46 from Table I. The proportional contribution is therefore $0.14/0.46 = 0.30$. It is important to note that this is not equivalent to the lag 1 proportion to be expected from the average element. When the averages are computed across the proportions, the average proportional contributions of the contemporaneous, lag 1, lag 2, and lag 3 covariances are −0.163, 0.326, 0.516, and 0.245, respectively. An inference to be drawn from this is that proportions are quite different across elements and samples.

Table II

Simulation Covariance Magnitudes

Simulation covariance results from 100 replications of T returns, where the averages (in $\%^2$) are taken over the 870 covariance elements of the 30 × 30 unadjusted contemporaneous (lag 0) covariance matrix or the Cohen et al. (1983) adjusted covariance matrix.

Sample Size, T	Average of Unadjusted Covariance Elements	Average of Adjusted Covariance Elements	Proportion of Adjusted Covariance Due to Lag Covariances			
			Lag 0	Lag 1	Lag 2	Lag 3
Parameter	0.50	0.50	1.00	0.00	0.00	0.00
2000	0.24	0.50	0.49	0.29	0.15	0.08
1000	0.24	0.51	0.48	0.29	0.15	0.08
800	0.24	0.51	0.48	0.29	0.15	0.08
400	0.24	0.48	0.50	0.30	0.14	0.06
200	0.24	0.47	0.51	0.29	0.13	0.06
100	0.24	0.41	0.58	0.29	0.11	0.02

lagged covariances are important in the simulated population, with the first lag contributing 30% and all three lags contributing 50% of the adjusted covariance. These simulation results are quite comparable to Shanken's (1987, Tables I and II) observations of actual market data, where the adjustment increased the unadjusted covariance by about 100% on average and the first lag contributes about 33% to the adjusted covariance.[6]

The remainder of Table II contains similar results for sample sizes of 100, 200, 800, 1000, and 2000 daily returns.

C. *Sampling Distributions of the Covariance Estimators*

In each replication of the simulation, the unadjusted covariance matrix, **S**, the CHMSW adjustment matrix, **K**, and the adjusted matrix, **V**, were computed from the T return observations. The eigenvalues of the matrix, $\mathbf{KS}^{-1}$, were calculated and the correction factor, f, obtained according to the theorem developed in Section II. Given the factor, the corrected covariance matrix, $\mathbf{V}^*$, was then computed. These matrices permitted the calculation of the three estimators' sample biases in each replication.

For each of the 870 covariance elements, the (mean) bias, sampling variance, and mean squared error (MSE) were determined from the 100 replications of the simulation. The standard errors of the estimates of the bias, averaged across all 870 covariance elements for each matrix estimator, were computed. Also, the squared biases and MSE were averaged across the 870 covariance elements for

[6] We replicated Shanken's (1987) covariance estimates over the period July 3, 1962 to December 31, 1972 and obtained virtually identical results. Also, we extended the covariance estimation to the more recent period, January 2, 1975 to December 31, 1984. The adjustment was less pronounced, increasing the unadjusted covariance by 83% on average, where the first lag contributed 32% to the adjusted covariance. The trading friction bias appears slightly less severe in more recent data, which may reflect the general increase in trades experienced by the equity markets.

each matrix estimator. The results are tabulated in Table III, for sample sizes ranging from $T = 100$ to $T = 2000$ daily returns.

With, for example, $T = 800$ returns, the required correction factor averaged $\bar{f} = 0.79$ across the 100 simulation replications, with a standard deviation, $s_f = 0.12$. The standard errors of the bias estimates are very small, indicating the adequacy of the simulation in detecting nonzero bias. The adjusted estimator has insignificant bias (0.005) in the average element, and, of the 435 distinct covariance elements, there are 255 positively and 180 negatively biased elements. The corrected estimator shows slightly larger squared bias (0.01) than the adjusted estimator (0.00), but there are many more elements with negative bias (375) than positive bias (60), resulting in a significant negative bias (−0.049) in the average element. The unadjusted estimator displays a very large, significant, squared bias of 0.11 and has 433 negatively biased elements and two positively biased elements. Thus, on the basis of bias alone, the CHMSW estimator is preferable; however, its undesirable nonpositive definite property remains. When sampling variance is considered together with the squared bias in the MSE criterion, the unadjusted estimator MSE of 0.12 dominates, followed by the corrected estimator MSE = 0.13 and the adjusted estimator with MSE = 0.18. There is relatively little difference between the MSE of the unadjusted and corrected estimators. Thus, on the basis of MSE, the unadjusted estimator is best, followed very closely by the corrected estimator, and the CHMSW estimator is worst.

The adjusted estimator is known to be unbiased at all sample sizes when unbiased estimators of the lag covariance elements are employed in the adjustment, whereas the unadjusted estimator is known to be biased at all sample sizes. These properties are supported by the reported simulation standard errors, except at small ($T = 100$, 200) sample sizes. The squared bias in the corrected estimator declines with sample size, reducing to 0.00 at $T = 1000$ and $T = 2000$, from 0.08 at $T = 100$. The corrected estimator is less biased than the unadjusted estimator at all sample sizes. The correction factor simultaneously increases from 0.21 to 0.99, indicating the consistency of the corrected covariance matrix. By $T = 800$, the MSE of the corrected estimator closely approaches the MSE of the unadjusted estimator, for the typical covariance element.

It is not reported in Table III, but the simulation showed that the squared bias and the MSE are monotonic functions of the value of the correction factor, f, in the interval $0 < f \leqq \min\left[\frac{1}{|\min d_i|} - 0.01, 1\right]$. This implies that a value of f, between zero and the value used in obtaining $\mathbf{V}^*$, would produce a covariance estimator that has bias and MSE between their respective values for the unadjusted and corrected estimators. Because there is only a small difference in the MSE of $\mathbf{S}$ and $\mathbf{V}^*$, smaller values of the correction factor would increase the magnitude of the bias while achieving little decline in MSE. Therefore, the optimal value of the correction factor is in the region of the maximum value permissible in the theorem. It is possible that this result may differ across populations.

If bias and positive definiteness are important considerations in multivariate covariance estimation, the corrected covariance estimator, $\mathbf{V}^*$, appears to be a

Table III

Simulation Covariance Bias and Mean Squared Error

Results (in $\%^2$) for the average covariance element from 100 replications of T returns, where **S** is the unadjusted covariance matrix, $\mathbf{V} = \mathbf{S} + \mathbf{K}$ is the adjusted matrix, **K** is the Cohen et al. (1983) adjustment matrix, $\mathbf{V}^* = \mathbf{S} + f\mathbf{K}$ is the corrected matrix, and f is the positive definiteness correction factor with sampling mean, $\bar{f}$, and sampling standard deviation, s_f. The standard errors are computed as

$$\left[\sum_{\substack{i=1\\i\neq j}}^{n}\sum_{j=1}^{n}\sum_{\substack{k=1\\k\neq l}}^{n}\sum_{l=1}^{n}\operatorname{cov}(b_{ij}, b_{kl})/(nr \times nc^2)\right]^{1/2},$$

where $\operatorname{cov}(b_{ij}, b_{kl})$ is the simulation covariance between the biases on the ijth and klth elements of a covariance matrix, $n = 30$ is the number of assets, $nr = 100$ is the number of simulation replications, and $nc = 870$ is the number of covariance elements estimated. The implied z-statistic is asymptotically standard normal.

	Covariance Estimator		
Sample Size and Statistics	Unadjusted, S	Adjusted, V	Corrected, V*
$T = 2000$			$\bar{f} = 0.99$, $s_f = 0.03$
All Elements			
Squared Bias	0.11	0.00	0.00
MSE	0.11	0.07	0.07
Bias (Standard Error)	−0.26 (0.00)	−0.002 (0.0048)	−0.0002 (0.0048)
Positive Bias Elements			
Number	0	236	215
Bias	0.00	0.02	0.02
Negative Bias Elements			
Number	435	199	220
Bias	−0.26	−0.02	−0.02
$T = 1000$			$\bar{f} = 0.87$, $s_f = 0.11$
All Elements			
Squared Bias	0.11	0.00	0.00
MSE	0.12	0.15	0.12
Bias (Standard Error)	−0.259 (0.0018)	0.005 (0.0071)	−0.027 (0.0063)
Positive Bias Elements			
Number	1	251	93
Bias	0.02	0.03	0.02
Negative Bias Elements			
Number	434	184	342
Bias	−0.26	−0.03	−0.04
$T = 800$			$\bar{f} = 0.79$, $s_f = 0.12$
All Elements			
Squared Bias	0.11	0.00	0.01
MSE	0.12	0.18	0.13
Bias (Standard Error)	−0.259 (0.0018)	0.005 (0.0077)	−0.049 (0.0063)
Positive Bias Elements			
Number	2	255	60
Bias	0.01	0.03	0.02
Negative Bias Elements			
Number	433	180	375
Bias	−0.26	−0.03	−0.06

Table III—*Continued*

Sample Size and Statistics	Covariance Estimator: Unadjusted, S		Adjusted, V	Corrected, V*	
$T = 400$				$\bar{f} = 0.53$, $s_f = 0.06$	
All Elements					
Squared Bias	0.11		0.00	0.03	
MSE	0.14		0.36	0.16	
Bias (Standard Error)	−0.26	(0.0032)	−0.013 (0.011)	−0.13	(0.0066)
Positive Bias Elements					
Number	2		177	10	
Bias	0.01		0.04	0.02	
Negative Bias Elements					
Number	433		258	425	
Bias	−0.26		−0.05	−0.13	
$T = 200$				$\bar{f} = 0.35$, $s_f = 0.05$	
All Elements					
Squared Bias	0.11		0.01	0.06	
MSE	0.17		0.70	0.21	
Bias (Standard Error)	−0.26	(0.0045)	−0.036 (0.015)	−0.19	(0.0071)
Positive Bias Elements					
Number	1		146	1	
Bias	0.01		0.05	0.01	
Negative Bias Elements					
Number	434		289	434	
Bias	−0.26		−0.08	−0.19	
$T = 100$				$\bar{f} = 0.21$, $s_f = 0.03$	
All Elements					
Squared Bias	0.11		0.03	0.08	
MSE	0.22		1.32	0.28	
Bias (Standard Error)	−0.26	(0.0061)	−0.089 (0.021)	−0.23	(0.0082)
Positive Bias Elements					
Number	5		95	5	
Bias	0.02		0.06	0.02	
Negative Bias Elements					
Number	430		340	430	
Bias	−0.26		−0.13	−0.23	

preferable estimator, possessing less bias than the unadjusted estimator and lower MSE than the adjusted estimator for the typical element of the covariance matrix. The corrected estimator dominates the CHMSW adjusted estimator, and perhaps other additive estimators, because of their propensity to be nonpositive definite with finite return samples.

IV. Estimation of Functions of Daily Covariance Matrices

There are many analytical and statistical methods employed in finance that involve estimated covariance matrices or functions thereof. Many methods also require the inverse of the estimated covariance matrix *viz.* regression analysis.

Table IV

Estimated Efficient Set Constants

Efficient set constants, $a = \mu' \Sigma^{-1} \mu$, $b = e' \Sigma^{-1} \mu$, and $c = e' \Sigma^{-1} e$, for three groups of securities (each comprised of 29 NYSE stocks and the S&P 500 index) estimated using 252 returns from Jan. 2, 1985 to Dec. 31, 1985 which are used to compute the sample mean, $\bar{r}$, and three covariance estimators **S**, **V**, and **V***, where **S** is the unadjusted covariance matrix, $\mathbf{V} = \mathbf{S} + \mathbf{K}$ is the adjusted matrix, **K** is the Cohen et al. (1983) adjustment matrix, $\mathbf{V}^* = \mathbf{S} + f\mathbf{K}$ is the corrected matrix, and f is the correction factor for positive definiteness.

Group No. and Covariance Estimator	No. of Negative Eigenvalues in Covariance Matrix	Efficient Set Constant Estimates $\hat{a}$	$\hat{b}$	$\hat{c}$
#1				
S, Unadjusted ($f = 0.0$)	0	0.16	0.57	5.04
V*, Corrected ($f = 0.2$)	0	0.15	0.53	4.73
V, Adjusted ($f = 1.0$)	3	0.87	2.31	9.52
#2				
S, Unadjusted ($f = 0.0$)	0	0.13	0.71	6.46
V*, Corrected ($f = 0.2$)	0	0.12	0.65	6.04
V, Adjusted ($f = 1.0$)	5	0.00	0.37	6.68
#3				
S, Unadjusted ($f = 0.0$)	0	0.12	0.54	6.83
V*, Corrected ($f = 0.2$)	0	0.12	0.55	7.56
V, Adjusted ($f = 1.0$)	5	0.03	0.56	4.32

Functions of covariance matrices, adjusted for trading frictions, are exposed to the nonpositive definite property generated by the CHMSW adjustment or other additive adjustments. Depending on the function, the effect on estimated values of the function is potentially severe.[7] The efficient set constants of Merton (1972) and Roll (1977) are functions of the inverse of the covariance matrix, as well as the mean vector, and therefore seem appropriate candidates for illustrating the effects of a multivariate nonsynchronous trading adjustment. Because the purpose here is to illustrate the severe estimation problems that can arise, small return samples are used.

In total, 87 stocks with no missing observations over the period January 2, 1985 to December 31, 1985 (252 returns) were selected from the CRSP daily file. The stocks were arranged alphabetically and placed in three groups of 29 securities each. To each group was added the S&P index, making a total of 30 securities in each group. The unadjusted, CHMSW adjusted, and corrected estimates, **S**, **V**, and **V***, respectively, of the daily covariance matrix were determined. The numbers of negative eigenvalues in the adjusted matrix were determined. The correction factor, f, was computed according to the theorem of Section II. The inverses of these matrix estimates and the sample mean vector, $\bar{r}$, were then employed to estimate the parameter inverse covariance, Σ^{-1}, and mean vector, μ, contained in the expressions for the efficient set constants, given

[7] For example, it is possible for F-statistics computed from restricted multivariate regressions to be negative.

by $a = \mu' \Sigma^{-1}\mu$, $b = e' \Sigma^{-1}\mu$, and $c = e' \Sigma^{-1}e$, where $e(n \times 1)$ is the unit vector. The results are tabulated in Table IV.

The results are illustrative of the fact that estimates of efficient set constants can be affected drastically by the adjustment for trading frictions. For example, in group 1, the estimate of a ranges from 0.15 to 0.87, the estimate of b ranges from 0.53 to 2.31, and the estimate of c ranges from 4.73 to 9.52. In these samples of 252 daily returns, the correction factor, f, is quite small (~0.20), which makes the corrected estimates close to the unadjusted estimates. However, the adjusted estimates are often very different from the unadjusted and corrected estimates. With $T = 252$ daily returns, the bias in the unadjusted and corrected covariances is quite large, given the combined results of Shanken (1987) and Table III. Therefore, both these estimates of the efficient set constants are biased. On the other hand, uncorrected CHMSW trading adjustments are inadmissible and may not be used in multivariate applications unless the sample size is sufficiently large. The adjusted estimates, from these three groups, are to be discounted because the estimated covariances are not positive definite covariance matrices, as evidenced by the presence of negative eigenvalues in the adjusted matrix.

While the analysis of Table III suggests that the corrected estimator is dominant at all sample sizes investigated, there does not seem to be a satisfactory estimator at small sample sizes. This is problematic in that many studies involving daily stock returns have relatively small samples of daily returns. A recent example is Brennan and Copeland (1988), which uses 252 days to compute standard deviations and 40 days to compute benchmark mean returns. One future possibility is a Bayes-Stein *individualized* shrinking of elements of the adjustment matrix. In the absence of a superior estimator in the presence of trading frictions, one must use the unadjusted (but biased) estimator or an adjusted estimator such as CHMSW corrected for nonpositive definiteness.

V. Concluding Comments

The additive, nonsynchronous trading adjustment of CHMSW induces nonpositive definiteness in the adjusted covariance and correlation matrices. It occurs with seemingly large samples of daily returns (i.e., $T \leqq 2000$), with occurrences diminishing as daily return sample sizes grow to infinity. The problem may occur with any nonsynchronous trading adjustment that does not restrict the resulting adjusted covariance and correlation matrices to be positive definite. In many multivariate applications in finance, covariance estimators adjusted for nonsynchronous trading are required. For example, asset opportunity set estimates and restricted multivariate regressions, associated with asset pricing, are functions of the covariance matrix and its inverse. Nonpositive definite covariance matrices can produce very unreliable inference results and are inadmissible.

A theorem was developed which provides sufficient restrictions on the eigenvalues of a function of the adjusted and adjustment matrices. The restrictions led to a tractable correction factor which is applied only to the elements of the adjustment matrix. The correction is applicable to any nonsynchronous trading adjustment, including the CHMSW adjustment. The resulting estimator has the desirable property of consistency if the adjustment is consistent.

The small sample properties of the corrected version of the CHMSW adjusted covariance matrix were investigated in a CHMSW market, where observed returns are generated with frictions in the trading process. In general, the corrected estimator is less biased than the unadjusted estimator and has lower mean squared error than the adjusted estimator, while possessing the required property of positive definiteness that is lacking in the adjusted estimator.

Appendix: Theorem Proof

a) The unadjusted matrix is real, symmetric, and positive definite. The adjustment matrix, **K**, is real and symmetric. If its diagonal elements are zero, then the trace of this matrix is zero and the sum of its eigenvalues is also zero. Therefore, the eigenvalue set (spectrum) must contain both positive and negative values and the adjustment matrix is neither positive semi-definite or negative semi-definite in this special case.

There exists a nonsingular matrix, **C**, such that $\mathbf{K} = \mathbf{CDC}^T$ and $\mathbf{S} = \mathbf{CIC}^T$, where **D** is a diagonal matrix with elements d_i, $i = 1, 2, \cdots, n$, and **I** is the identity matrix. (See Theorems 7.6.4 and 7.6.5 in Horn and Johnson (1985).) The adjusted matrix is therefore $\mathbf{V} = \mathbf{S} + \mathbf{K} = \mathbf{C}[\mathbf{I} + \mathbf{D}]\mathbf{C}^T$. **V** is positive definite if and only if $\mathbf{xC}[\mathbf{I} + \mathbf{D}]\mathbf{C}^T\mathbf{x}^T > 0$ (**x** an arbitrary vector) or, equivalently, $\Sigma_i y_i^2(1 + d_i) > 0$, where $\mathbf{y} = \mathbf{xC}$. The eigenvalues of $\mathbf{I} + \mathbf{D}$ are $1 + d_i$, $i = 1, 2, \cdots, n$. Therefore, $(1 + d_i) > 0$ or $d_i > -1$, $i = 1, 2, \cdots, n$, is required to keep $\mathbf{y}[\mathbf{I} + \mathbf{D}]\mathbf{y}'$ positive definite. Because $\mathbf{KS}^{-1} = \mathbf{CDC}^{-1}$, the eigenvalues of $\mathbf{KS}^{-1}$ are the d_i.

Let $0 < f < \min\left[\dfrac{1}{|\min d_i|}, 1\right]$ when $\min d_i \leqq -1$. This implies that $(1 + fd_i) > 0$, $i = 1, 2, \cdots, n$, and $\mathbf{I} + f\mathbf{D}$, a diagonal matrix, is therefore positive definite and so is $\mathbf{C}[\mathbf{I} + f\mathbf{D}]\mathbf{C}^T = \mathbf{V}^*$, from the definition of $\mathbf{V}^*$ and the expressions for **S** and **K** in the theorem proof. When $\min d_i > -1$, $\mathbf{V}^*$ is always positive definite and f is restricted in the theorem to $f = 1$, ensuring that the multivariate adjustment never exceeds the univariate adjustment.

The correlation matrix, derived from a positive definite covariance matrix, $\mathbf{V}^*$, is positive definite if the absolute value of any correlation is less than one. See Theorems 8.7.2 and 8.7.3 in Graybill (1969). If correlations violating the bound are found, the value of f is further reduced until the violation disappears. Q.E.D.

b) From Theorem 7.6.3 in Horn and Johnson (1985) and the properties of the adjustment matrix, the d_i contain both positive and negative values when no variance adjustment is present in **K**. Therefore, $\min d_i < 0$ and the limit applies. Q.E.D.

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