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ABSTRACT

This paper investigates the term structure of interest rates in a multiperiod production and exchange economy with incomplete information. Unable to observe their stochastic investment opportunities, investors engage in dynamic Bayesian inference. This results in the endogenous identification of a more complex production function which generates a richer term structure, resembling the one that actual market prices imply. In addition, this paper introduces a characteristic function of the term structure and demonstrates that, in contrast with a fully observable economy, the widely investigated expectations hypothesis holds true only if interest rates are nonstochastic.

THIS PAPER INVESTIGATES THE term structure of interest rates in a dynamic production and exchange economy with an unobservable productivity factor. In order to estimate the unobservable stochastic investment opportunities, rational investors engage in dynamic Bayesian inference. Anticipated changes in investors' "quality of information" about the fluctuations of the economic state variable affect equilibrium demands and prices. This results in the endogenous identification of a more complex production function which induces a richer term structure. Term structures of interest rates in a partially observable economy resemble term structure curves that actual market prices imply. This paper also introduces a characteristic function of the term structure and demonstrates that, in contrast with a fully observable economy, the widely investigated expectations hypothesis holds true only if interest rates are nonstochastic.

The theory of the term structure of interest rates is a subject of renewed interest in the financial economics literature (see Section I). Thus far, the term structure of interest rates as implied by actual market prices has not been fully explained. In their seminal work, Cox, Ingersoll, and Ross (1985a, b) pursue a one-factor version of their model and, like Vasicek (1977), obtain explicit term structure curves that are monotonic or single humped.[1] Market prices, however, imply curves of considerable variation as well, e.g., multiple humped curves (see Levine (1975)).

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[1] Their model also includes closed form solutions for two factor economies, but they do not investigate the resulting higher variation yield curves. See this paper's later discussion of the relationship between the introduction of dynamic Bayesian inference of realization of state variables and the introduction of additional state variables.

Cox, Ingersoll, and Ross (1985a, b) present a dynamic equilibrium model in which realized outputs and all contingent claims are functions of fully observable state variables. In reality, demands, relative prices, and outputs are determined by both observable and unobservable factors. The latter category includes factors for which only proxies are observed (e.g., inflation rate and money supply) and factors that are difficult to quantify (e.g., labor input measured in units of efficiency and technological progress measured in aggregate terms).

This analysis investigates the informational effects on the term structure in a Cox, Ingersoll, and Ross (1985a, b) economy where an unobservable factor, or state varible, drives the production technologies. Investors draw inferences about the productivity factor from realized outputs and continuously update their estimates, thus engaging in a continuous Bayesian inference process.[2] The time changes in the precision of investors' estimates represents the changes in their "quality of information," or "learning ability," about the changes in the stochastic investment opportunity set. In this context, default-free bond prices reflect investors' expectations regarding their "quality of information" through the life of the bond.

For a certain class of information structures, the posterior distribution of the unobservable productivity factor enables the endogenous respecification of the state space and endogenously identifies a more complex production function which induces a richer term structure. The term structure of interest rates now reflects the changing precision of investors' estimates of their investment opportunities. The resulting term structure curve has a high degree of variation and resembles the term structure implied by actual market prices. For example, a productivity factor with a constant instantaneous variance coefficient will result in an interest rate with an instantaneous variance coefficient that changes over time. The nature of this time dependency will be affected primarily by the endogenously determined dynamic quality of information and by the covariance structure of the unobservable productivity factor and the observable realized returns.

In addition, if investors use the endogenously determined dynamic quality of information in their inference process, the expectations hypothesis of the term structure does not hold when interest rates are stochastic. This is in contrast to the findings of Cox, Ingersoll, and Ross (1981) in a fully observable economy.

All the preceding arguments apply to all smooth concave utility functions and to all information structures that yield solvable optimal control problems. This analysis further specifies preferences as logarithmic and obtains explicit expressions for the term structure and each of its biases.

Section I of the paper reviews and interprets recent research of the term structure; Section II describes the economy; and Section III analyzes the term structure of interest rates. Section IV examines the widely investigated expectations hypothesis; Section V discusses some of this model's assumptions and implications; and Section VI summarizes conclusions.

[2] Feldman (1983) and Dothan and Feldman (1986) explore such an economy, and similar economies are investigated in Detemple (1986) and Gennotte (1986).

I. Term Structure Models and Term Structure Hypotheses

The term structure of interest rates is the yield to maturity on default-free bonds as a function of the term to maturity. The theory of the term structure of interest rates began at the end of the 19th century with Fisher (1896); and Cox, Ingersoll, and Ross (1981) provide an excellent review of the theory's development. Recently, Vasicek (1977), Dothan (1978), Richard (1978), Brennan and Schwartz (1979), and Langetieg (1980) derived term structure models using arbitrage arguments. Beja (1979) developed an equilibrium term structure theory in a state preference economy, and Cox, Ingersoll, and Ross (1985a, b) developed a term structure theory within a dynamic general equilibrium framework. Cox, Ingersoll, and Ross (1981) examined the compatibility of the traditional hypotheses with modern asset valuation techniques (continuous time, rational expectations equilibrium) and went on to discuss how term structure theory is affected by modern portfolio theory and valuation of contingent claims under diffusion processes. Breeden (1986) presented additional analysis and synthesis, and Detemple and Kihlstrom (1987) examined equilibrium interest rates where investors can acquire information.

Existing term structure models contain a number of potentially limiting assumptions, such as: time separable preferences, homogeneous preferences, single factor, single good, linear production, and homogeneous information. Some of these assumptions are addressed in the works of Brennan and Schwartz (1979), Ulman and Wood (1982), Schaefer and Schwartz (1984), Sundaresan (1984), Cox, Ingersoll, and Ross (1985b), and Dunn and Singleton (1986). The present analysis, however, will focus on the assumption of complete information.

Before interpreting the monotonic or humped yield curves that have been theoretically explained thus far, let us examine the traditional hypotheses that are relevant to models such as Cox, Ingersoll, and Ross (1985a, b) and Vasicek (1977). The expectations hypothesis can be examined within these models since individuals have a probability distribution over realizations of the future spot rates of interest. The "market risk premium" hypothesis can also be investigated since risk-averse individuals might be willing to incur a premium for the ability to hedge the changes in the stochastic investment opportunity set and thereby reduce the variance of optimally invested wealth. The maturity preferences/market segmentation hypothesis, or, in its broader version, the habitat theory, could be examined if individuals' preferences are designed to reflect this specific structure. Cox, Ingersoll, and Ross (1981) provide a basis for proceeding in this direction by demonstrating the ways in which investors' risk aversion creates "habitats." Yet, none of the existing models have explored this avenue fully.

The monotonic or single humped yield curves obtained in current models can be interpreted as follows. Models like Vasicek (1977) and the main example in Cox, Ingersoll, and Ross (1985b) use a mean reverting autonomous diffusion process as the dynamics of the spot rate of interest. This spot rate is sufficient to determine bond prices of all maturities. Investors share a probability distribution over realizations of future spot rates; particularly, they believe that the spot rate will eventually attain its asymptotic rate. Indeed, the yield curve is flat for long terms to maturity (see Dybvig, Ingersoll, and Ross (1985) for discussion of uncertainty structures which result in asymptotically flat yield curves). Moreover, when the spot rate is low and individuals expect it to rise toward its asymptotic mean, the yield curve is upward sloping. When the spot rate is high and individuals expect it to fall, the yield curve is downward sloping. Up to this point, investors' behavior is consistent with the expectations hypothesis. The expectations hypothesis, however, fails to explain the humped term structure obtained for intermediate values of the spot rate.

In addition to investors' expectations regarding future spot rates, two other interacting phenomena determine the shape of the yield curve. (Ulman and Wood (1982) explicitly characterize the impact of these two phenomena for an Ornstein-Uhlenbeck process.) The first phenomenon is the Jensen's inequality bias, which is zero for instantaneous maturities and which monotonically assumes a negative asymptotic value for long maturities. The resulting yield curve is shifted downward from the curve of the average expected future spot rates in a monotonic fashion. The second phenomenon is the market risk premium bias, which is zero for instantaneous maturities and which monotonically attains an asymptotic value that can be negative or positive. Its sign is opposite that of the market risk premium. Assuming that the market risk premium is positive, the two biases on the yield to maturity would be negative.

The hump in the yield curve, for an intermediate range of spot rates, can be heuristically explained as follows. If one chooses a spot rate slightly lower than the asymptotic value of the expected future spot rate, expectations will drive the yield to maturity upward for short maturities when the biases are small. However, for longer maturities, when the biases become large, they "take over," i.e., stop the climb of the yield curve toward the average expected spot rates curve and drive it down to its asymptotic value.

Brennan and Schwartz (1979) analyze an economy where the term structure of interest rates is not solely determined by a single stochastic state variable, but instead is a function of two state variables which stand for the short and long rates of interest. Schaefer and Schwartz (1984) pursue a similar model and find an approximate analytical solution. Cox, Ingersoll, and Ross (1985b) attain closed form solutions for several two factor economies.

Sundaresan (1984) demonstrates that a common input across production technologies of different goods in a multigood production economy gives rise to a richer term structure. Dunn and Singleton (1986) explore the term structure where preferences are not time separable, causing interaction among utilities obtained from durable and nondurable goods. As a result of this interaction, shifts in the term structure may affect demands for consumption goods directly, and bonds with different terms to maturity may differ more widely in their risk characteristics. Benninga and Protopapadakis (1986) analyze the term premium, and Ahn and Thompson (1988) investigate the term structure where economic state variables follow jump-diffusion processes. Recent empirical studies of the term structure include Brennan and Schwartz (1980), Brown and Dybvig (1986), Fama (1984a, b), Gibbons and Ramaswamy (1986), and Stambaugh (1986).

The following analysis relaxes the assumption of complete observability and examines the effects of the resulting Bayesian inference on the term structure and its expectations hypothesis.

II. A Partially Observable Production Exchange Economy

The underlying economic structure is in the spirit of Cox, Ingersoll, and Ross (1985a, b). This model, however, allows the existence of an unobservable state variable. This economy is explored in Feldman (1983) and Dothan and Feldman (1986), and similar economies are investigated in Detemple (1986) and Gennotte (1986).[3] While these works concentrate on methodological issues, the following analysis focuses on the term structure.

In a competitive frictionless multiperiod economy, individuals observe an $(n \times 1)$ vector of realized outputs ξ_t of a single consumption/production numeraire good. These constant stochastic returns to scale outputs evolve as the Ito diffusions,

$$d\xi_t = I(\xi_t)(A_0 + A_1\theta_t)dt + I(\xi_t)B_1 dW_t,$$

with initial condition ξ_0. The underlying uncertainty in the economy is described by $W_t = (W_t, F_t)$, $W_0 = 0$, an $[(n + 1) \times 1]$ vector of independent Wiener processes defined over a complete probability space (Ω, F, P) with a nondecreasing right continuous family of sub-σ-algebras (F_t), $0 \le t < T$. A_0 and A_1 are known $(n \times 1)$ vectors of constants; B_1 is a known $[n \times (n + 1)]$ matrix of constants; and $B \equiv B_1 B_1'$—which is the instantaneous variance-covariance matrix of realized outputs—is positive definite. $I(\xi_t)$ is a diagonal matrix with ξ_{ti}'s as elements.

Individuals, however, do not observe the realization of an economic state variable (productivity factor) θ_t, which evolves as

$$d\theta_t = (a_0 + a_1\theta_t)dt + b_1' dW_t.$$

The initial condition θ_0 is normally distributed given ξ_0; b_1 is a known $[(n + 1) \times 1]$ vector of constants; $b_1' b_1 \equiv b$ is positive; and a_0 and a_1 are known constants.

Individuals with identical endowments and preferences maximize expected lifetime time additive utility of consumption, conditional on their observations,

$$E\left\{\int_t^T h[c(\tau), \tau]\, d\tau \mid \xi_s,\ 0 \le s \le t;\ w_t = w_0\right\}.$$

They continuously allocate their initial wealth w_t (henceforth denoted also as w) among consumption c, investment u, and instantaneous borrowing and lending at an endogenously determined market clearing rate r. The von Neumann-Morgenstern utility function h is concave and twice-differentiable.

[3] Contributing to the analysis of an equilibrium in a partially observable economy are: Merton (1973, 1980, 1981), Fleming and Rishel (1975), Klein and Bawa (1976, 1987), Liptser and Shiryayev (1977, 1978), Vasicek (1977), Williams (1977), Kreps (1979, 1982), Richard and Sundaresan (1980), Brock (1982), Duffie and Huang (1985a), Duffie (1986), Huang (1985), and Cox, Ingersoll, and Ross (1985a, b). Recent works that study problems of incomplete information are Detemple (1986), Feldman (1989), Duffie and Huang (1985b), Stulz (1986), Gennotte (1986), and Ghysels (1985).

Unfortunately, because the productivity factor is unobservable, this structure is non-Markovian and does not allow a state vector solution. In order to determine portfolio/consumption rules, investors identify an endogenous economy which has an identical information structure and which is compact and Markovian. Portfolio/consumption rules are optimal in the exogenous economy if and only if they are optimal in the endogenous economy. Investors in this endogenous economy represent information flow in terms of the posterior distribution (nonlinear filter) of θ_t given the observations ξ_s, $0 \leq s \leq t$ (see Lipster and Shiryayev (1978)):

$$dm_t = (a_0 + a_1 m_1)dt + (D + A_1\gamma_t)'B^{-1/2}d\bar{W}_t,$$

$$d\gamma_t = [b + 2a_1\gamma_t - (D + A_1\gamma_t)'B^{-1}(D + A_1\gamma_t)]dt,$$

$$d\bar{W}_t = B^{-1/2}[I^{-1}(\xi_t)d\xi_t - (A_0 + A_1 m_t)dt],$$

where $m_0 = E[\theta_0 \mid \xi_0]$, $\gamma_0 = E[(\theta_0 - m_0)^2 \mid \xi_0]$. The mean is m_t, and γ_t is the variance of the conditional distribution. $\bar{W}_t$ is the innovation process—a process that describes the deviations of the observations from their expected values—and it is a vector of independent Wiener processes. $D \equiv B_1 b_1$ is the instantaneous covariance between realized outputs and the unobservable factor.[4] It is important to note that the conditional mean, the estimate of the unobservable factor, is a sufficient statistic to the posterior distribution and is updated recursively.

The filter equations define the instantaneous variance of m_t, the estimate of the unobservable factor, as

$$\sigma_m(\gamma_t) = B^{-1/2}(D + A_1\gamma_t),$$

and define the instantaneous covariance G between the rates of return on investment in ξ_t and the realizations of m_t, the estimate of the unobservable factor, as

$$G = B^{1/2}\sigma_m(\gamma_t) = D + A_1\gamma_t.$$

In equilibrium, G is set to be an average of D and A_1, dynamically weighted by the filtering error. D, the instantaneous covariance, represents the diffusion source of covariance, and A_1 represents the drift source of covariance between the unobservable factor and realized outputs.

Using the filter equations, individuals reformulate the problem as a Markovian one,[5]

$$\max_{c,u} E\left\{\int_t^T h[c(\tau), \tau]\, d\tau \mid w_t = w_0\right\},$$

[4] Liptser and Shiryayev ((1978), Theorems 12.5 and 12.7) prove that if the conditional distribution of θ_0 given ξ_0 is Gaussian, $N(m_0, \gamma_0)$, then $\bar{W}_t$, m_t, and γ_t are the unique, continuous, F_t^{ξ} measurable (strong) solutions of the filter equations. Moreover, the innovation process $\bar{W}_t$ generates the economy, i.e., the σ-algebras $F_t^{\xi_0, \bar{W}}$ and F_t^{ξ} are equivalent.

[5] Compare with the investors' original problem

$$\max_{c,u} E\left\{\int_t^T h[c(\tau), \tau]\, d\tau \mid \xi_s,\ 0 \leq s \leq t;\ w_t = w_0\right\},$$

where

$$dm_t = (a_0 + a_1 m_t)dt + (D + A_1\gamma_t)'B^{-1/2}d\bar{W}_t,$$

$$d\gamma_t = [b + 2a_1\gamma_t - (D + A_1\gamma_t)'B^{-1}(D + A_1\gamma_t)]dt,$$

$$I^{-1}(\xi_t)d\xi_t = (A_0 + A_1 m_t)dt + B^{1/2}d\bar{W}_t,$$

s.t.

$$dw = w[u'I^{-1}(\xi_t)d\xi_t + (1 - u'\mathbf{1})r] - cdt,$$

$$c \geq 0, \quad u \geq 0, \quad w_0 > 0, \quad w_T \geq 0.$$

Assuming all production processes are active, they determine the optimal portfolio/consumption rules as found in Cox, Ingersoll, and Ross (1985a):

$$r = u'(A_0 + A_1 m_t) + \frac{wJ_{ww}}{J_w} u'Bu + \frac{J_{wm}}{wJ_w} u'G,$$

or

$$r = (\mathbf{1}'B^{-1}\mathbf{1})^{-1}\left[\mathbf{1}'B^{-1}(A_0 + A_1 m_t) + \frac{wJ_{ww}}{J_w} + \frac{J_{wm}}{J_w}\mathbf{1}'B^{-1}G\right],$$

and

$$u = \frac{-J_w}{wJ_{ww}} B^{-1}[(A_0 + A_1 m_t) - r\mathbf{1}] + \frac{-J_{wm}}{wJ_{ww}} B^{-1}G,$$

where J, $J = J(w, m_t, t)$ is the indirect utility function of the optimization problem, $w^2u'Bu$ is the variance of optimally invested wealth, and $wu'G$ is the covariance between the estimate of the unobservable factor m_t and optimally invested wealth. Subscripts denote partial derivatives, and $\mathbf{1}$ is an $(n \times 1)$ vector of ones.

In order to obtain explicit solutions, preferences are specified as Bernoulli-logarithmic,

$$h[c(t), t] = e^{-\zeta t}\log c(t).$$

The resulting equilibrium, market-clearing, instantaneous rate of interest r is a deterministic linear function of the estimate of the unobservable factor m_t:

$$r = \frac{\mathbf{1}'B^{-1}A_0 - 1}{\mathbf{1}'B^{-1}\mathbf{1}} + \frac{\mathbf{1}'B^{-1}A_1}{\mathbf{1}'B^{-1}\mathbf{1}} m_t.$$

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where

$$d\theta_t = (a_0 + a_1\theta_t)dt + b_1'dW_t,$$

$$I^{-1}(\xi_t)d\xi_t = (A_0 + A_1\theta_t)dt + B_1 dW_t,$$

s.t.

$$dw = w[u'I^{-1}(\xi_t)d\xi_t + (1 - u'\mathbf{1})r] - cdt,$$

$$c \geq 0, \quad u \geq 0, \quad w_0 > 0, \quad w_T \geq 0.$$

r evolves as

$$dr = (\bar{a}_0 + \bar{a}_1 r)dt + \sigma_r'(\gamma_t)d\bar{W}_t,$$

where

$$\bar{a}_0 = \frac{\mathbf{1}'B^{-1}A_1}{\mathbf{1}'B^{-1}\mathbf{1}}\, a_0 + \frac{1 - \mathbf{1}'B^{-1}A_0}{\mathbf{1}'B^{-1}\mathbf{1}}\, a_1, \quad \bar{a}_1 \equiv a_1.$$

The instantaneous variance of the spot rate of interest $\sigma_r^2(\gamma_t)$ is

$$\sigma_r(\gamma_t) = \frac{\mathbf{1}'B^{-1}A_1}{\mathbf{1}'B^{-1}\mathbf{1}}\, B^{-1/2}(D + A_1\gamma_t).$$

Next, the model introduces markets for claims contingent on the interest rate. Determined by arbitrage, their prices obey the general valuation equation

$$\tfrac{1}{2}\sigma_r^2(\gamma_t)f_{rr} + [\bar{a}_0 + \bar{a}_1 r - u'B^{1/2}\sigma_r(\gamma_t)]f_r - rf + f_t + \delta(r, t) = 0,$$

for any boundary conditions γ_0, $f(r, \bar{t}) = \bar{f}$, and with payout function δ. The market risk parameter is $u'B^{1/2}\sigma_r(\gamma_t)$.

In a partially observable economy, individuals observe realized outputs and continuously draw inferences about the realizations of the unobservable factor. At each point in time, they possess a posterior distribution of the realization of the factor, which they continuously update. This conditional distribution is normal. Its mean m_t, a sufficient statistic for the conditional distribution, is the best estimate of the unobservable factor. Its variance γ_t, the filtering error, defines the precision of the estimate. In fact, at each point in time individuals have a measure of the accuracy of their estimate or the quality of their information. Since the second moment of the posterior distribution changes over time, so does the estimate's precision. In other words, individuals' quality of information about the realizations of the unobservable factor changes over time.

In equilibrium, the stochastic spot rate of interest is a one-to-one function of the best estimate of the factor. Additionally, prices of multiperiod claims on the spot rate (i.e., bonds) reflect the path of the estimate's precision—or investors' quality of information—through the lives of the claims. Since this quality of information changes over time, so does investors' ability to infer the changes of the stochastic investment opportunity set.

In a fully observable economy, the changes of the stochastic investment opportunity set are observable, and equilibrium interest rates are set to perfectly imitate these changes (see Cox, Ingersoll, and Ross (1985b)). In a partially observable economy, however, it is impossible to perfectly duplicate these unobservable changes. Rather, investors' quality of information determines the extent to which these changes can be followed.

Changes in the quality of information, and therefore in individuals' perception of investment opportunities, induce two effects. First, the volatility of the interest rate changes as a function of individuals' ability to infer the fluctuation in their investment opportunities. In other words, the variance of the spot rate changes as a function of individuals' dynamic quality of information. Secondly, the covariance between the return on optimally invested wealth and the perceived changes in the investment opportunities varies as a function of the (changing) quality of information. This covariance is crucial in determining the market risk parameter, which, in turn, determines the market risk premium. These two effects combine to influence bond prices and give rise to term structure curves with a high degree of variation.

The following sections analyze the term structure. Propositions 1 and 2 characterize the possible paths of the quality of information. Propositions 3 and 4 identify the consequent behavior of the variance of interest rates, and the final series of propositions describes the resulting term structure.

III. The Term Structure of Interest Rates

To simplify the notation and the discussion, assume that the factor is a "good factor," i.e., $A_1 > 0$. As discussed earlier, this assumption implies that the "drift source" of covariance is positive. Despite the applicability of this paper's results to the multitechnology case, assume for simplicity that there is a single realized output, i.e., that $n = 1$. In addition, assume that $a_1 < 0$ so that θ_t follows a mean reverting sample path.

PROPOSITION 1: *Prices of multiperiod default-free bonds are a function of the path of the estimation error through the life of the bond. In partiuclar, the price of a unit default-free discount bond with term to maturity τ, $f(r, \tau)$ is*

$$f(r,\tau) = \exp\left\{\frac{\bar{a}_0}{\bar{a}_1}\tau + \left(-\frac{\bar{a}_0}{\bar{a}_1} - r\right)\frac{1}{(-\bar{a}_1)}(1 - e^{\bar{a}_1\tau}) - \frac{B^{1/2}}{\bar{a}_1}\int_0^\tau \sigma_r(\gamma_t)(1 - e^{\bar{a}_1 t})\,dt + \frac{1}{2\bar{a}_1^2}\int_0^\tau \sigma_r^2(\gamma_t)(1 - e^{\bar{a}_1 t})^2\,dt\right\}.$$

Proof: See Appendix.

PROPOSITION 2:

(*a*) *The estimation error γ_t evolves monotonically from its initial value γ_0, to a non-negative asymptotic steady state γ_{ss}, according to the rule*

$$\gamma_t = \gamma_u \frac{\dfrac{\gamma_s}{\gamma_u} - \dfrac{\gamma_0 - \gamma_s}{\gamma_0 - \gamma_u}\exp\left[\left(-\dfrac{A_1^2}{B}\right)(\gamma_s - \gamma_u)t\right]}{1 - \dfrac{\gamma_0 - \gamma_s}{\gamma_0 - \gamma_u}\exp\left[\left(-\dfrac{A_1^2}{B}\right)(\gamma_s - \gamma_u)t\right]},$$

$$\gamma_s = k_2\{(k_1 - \rho) + [(k_1 - \rho)^2 + (1 - \rho^2)]^{1/2}\},$$

$$\gamma_u = k_2\{(k_1 - \rho) - [(k_1 - \rho)^2 + (1 - \rho^2)]^{1/2}\},$$

$$k_1 \equiv \frac{a_1/b^{1/2}}{A_1/B^{1/2}},\quad k_2 \equiv \frac{(bB)^{1/2}}{A_1},$$

and

$$\rho \equiv \frac{D}{(bB)^{1/2}}.$$

(*b*) *This steady state value is a bounded concave function of the instantaneous correlation* $D/(bB)^{1/2}$. *A sufficient condition for a zero steady state value of the estimation error is perfect local correlation between* θ_t *and* ξ_t, *and similar signs of the two covariance sources. That is,* $D^2/bB = 1$ *and* $sign[D] = sign[A_1]$. *Under local perfect correlation but with covariance sources of different signs, the steady state estimation error vanishes for an identifiable, non-trivial parameter set.*

Proof: See Appendix.

The behavior of γ_t over time is as follows. γ_t is decreasing (stationary, increasing) if its initial value γ_0 is lower than (equal to, higher than) its steady state value γ_{ss}. Proposition 3 further characterizes the best case of the steady state hedging ability; this occurs when there is perfect local correlation between the factor and realized outputs with both sources of covariance having the same sign. In this case, the precision of the estimate continuously increases (assuming a finite initial precision), and the estimation error vanishes asymptotically. The rate of convergence of the estimation error is an increasing function of the distance from its steady state value; hence, most of the improvement in the precision is achieved early. In fact, under perfect local correlation and similar signs of covariance sources, investors learn as fast as they would learn if the factor were constant over time. In both cases, an infinitely long time interval is needed to reduce the uncertainty about the factor's realizations to zero. (Williams (1977) demonstrates the case of the unknown constant parameter.)

A. The Path of the Instantaneous Variance of the Spot Rate

PROPOSITION 3: *The instantaneous variance of the interest rate* σ_r^2 *has the following characteristics*:

(*a*) *It is a quadratic convex function of the estimation error* γ_t.

(*b*) *It depends on time, only through its dependence on the estimation error.*

(*c*) *It is an increasing, decreasing, or decreasing-increasing, function of time. In particular,*

$$\sigma_r^2 = \frac{A_1^4}{B}(\gamma_t - \gamma_m)^2, \quad \gamma_m \equiv -D/A_1.$$

Proof: See Appendix.

PROPOSITION 4:

(*a*) *Except for the degenerate case where* $\gamma_0 = \gamma_{ss}$ *and, therefore,* σ_r^2 *is constant over time, a sufficient condition for a monotonic (in time) instantaneous variance of the interest rate* σ_r^2 *is a non-negative instantaneous covariance coefficient D.*

(*b*) *If* $D \geq 0$ *and if* γ_t *is increasing (decreasing), then* σ_r^2 *is monotonic increasing (decreasing).*

(*c*) *A negative instantaneous correlation coefficient D is a necessary condition for the instantaneous variance of the interest rate to be a decreasing-increasing function of time.*

(*d*) *If* $D < 0$, *then the value of the estimation error that gives rise to the minimum variance is* $\gamma_m = -D/A_1$ *and the value of this minimum variance is zero, i.e.,* $\sigma_r^2(\gamma_m) = 0$.

(*e*) *Except for the degenerate case where* $\gamma_0 = \gamma_{ss}$ *and* σ_r^2 *is constant over time, a necessary and sufficient condition for a decreasing-increasing instantaneous variance of the spot rate is* $D < 0$ *and* γ_m *is between the initial value and the steady state value of* γ_t: $\gamma_m \epsilon [min(\gamma_0, \gamma_{ss}), max(\gamma_0, \gamma_{ss})]$.

(*f*) *If* $D < 0$ *but the asymptotic total covariance between the unobservable factor and the rates of return on realized outputs is positive,* σ_r^2 *is a decreasing function of time if* $\gamma_0 > \gamma_{ss}$, *an increasing function if* $\gamma_m < \gamma_0 < \gamma_{ss}$, *or decreasing-increasing if* $\gamma_0 < \gamma_m$.

(*g*) *If D and the asymptotic total covariance are negative (i.e., the effect of the local covariance D is strong enough to dictate the sign of the asymptotic total covariance) then* σ_r^2 *is a decreasing, increasing, or decreasing-increasing function of time if* $\gamma_0 < \gamma_{ss}$, $\gamma_{ss} < \gamma_0 < \gamma_m$, *or* $\gamma_m < \gamma_0$, *respectively.*

Proof: See Appendix.

The behavior of the instantaneous variance of the spot rate can be explained as follows. Changes in realizations of the productivity factor cause changes in both the spot rate and the investment opportunity set. In a fully observable economy, the variance of the interest rate is a sure linear function of the factor's variance. This would have been the result here if θ_t were observable, and it is the result in Cox, Ingersoll, and Ross (1985a, b). By contrast, in a partially observable economy where the factor is unobservable, investors do not have the privilege of setting the equilibrium interest rate to duplicate the changes in the investment opportunity set. Instead, they use realized outputs to make inferences. Now the stochastic changes of the spot rate are based on changes in the unobservable productivity factor as seen through the changes in realized outputs.

Moreover, the covariance between realized outputs and that factor takes over the role played by the factor's variance in a fully observable economy. This covariance is an average of two sources of covariance, the systematic and nonsystematic, which are dynamically weighted by the quality of information. The systematic source of covariance is that of the unobservable factor and the instantaneous expected return on realized outputs. (The sign of this covariance is the sign of A_1.) The nonsystematic source of covariance is the instantaneous covariance of the unobservable factor and realized outputs (D). A necessary condition for a nonmonotonic variance of the spot rate is that the sources of covariance are of different signs, i.e., $\text{sign}[D/A_1] = -1$.

The variance of the spot rate reflects the weight that consumers put on the new information contained in the observed realized outputs. If the estimation error decreases (increases) the covariance between the factor and realized outputs, then the variance of the spot rate is set to be lower (higher). Therefore, this variance vanishes in two different equilibria. The first equilibrium occurs when all future realizations of the factor are known with certainty. Future equilibrium spot rates can be determined in advance as a nonstochastic function of time. In this case, no weight is put on future information since all is known in advance. The second equilibrium, the relevant one, is a partially observable economy with a "good" ("bad") factor and a negative (positive) instantaneous covariance coefficient. In this case, the two sources of covariance—the systematic and the nonsystematic—counteract. Changes in the estimation error might weigh the sources of the covariance in a way that makes the overall covariance vanish temporarily or, in special cases, permanently. When this covariance vanishes, changes in the observed realized outputs do not contain information about changes in the unobservable factor. Therefore, individuals do not use these stochastic changes to update their estimates. The weight they put on new information is zero, and they set the variance of the spot rate to zero as well. Typically, the variance vanishes for an instant; it vanishes for a finite period only when $\gamma_{ss} = \gamma_m$.

The interplay between the two sources of covariance determines the behavior of the variance of interest rates. An interesting complex behavior arises when the two sources have opposite impact and the changes in the estimation error shift the weight from one source of covariance to another. In addition, if $\gamma_0 < \gamma_m < \gamma_{ss}$, the instantaneous variance will decrease from its initial value of $\sigma_r^2(\gamma_0)$ to $\sigma_r^2(\gamma_m) = 0$ and then increase to $\sigma_r^2(\gamma_{ss})$. In this case, the estimation error increases monotonically from γ_0 to γ_{ss}, and the variance is a decreasing-increasing function of the estimation error and a decreasing-increasing function of time. On the other hand, if $\gamma_{ss} < \gamma_m < \gamma_0$, the estimation error decreases monotonically, and the variance is an increasing-decreasing function of the estimation error but a decreasing-increasing function of time. The first case occurs if the "systematic source" of covariance, positive by assumption, dominates the sign of the asymptotic total covariance and hence $\gamma_m < \gamma_{ss}$. The second case occurs when the "diffusion source of covariance," negative by assumption, dominates the total asymptotic covariance and hence $\gamma_{ss} < \gamma_m$.

In a fully observable economy, the term structure of interest rates is determined by the expectations regarding future spot rates, the market risk premium, and the Jensen's inequality bias. In a partially observable economy, an additional component is involved: the path of investors' quality of information. As established earlier, the variance coefficient of the spot rate $\sigma_r^2(\gamma_t)$ is a quadratic function of the estimation error of the unobservable factor. The market risk parameter $u'B^{1/2}\sigma_r(\gamma_t)$ is also a function of the filtering error. The market risk parameter and the variance of the spot rate can be increasing, decreasing, increasing-decreasing, or decreasing-increasing functions of the estimation error, depending on the covariance coefficients between the outputs and the factor. In addition, depending on the initial value of the estimation error, the market risk parameter and the variance of the spot rate will be increasing, decreasing, or decreasing-increasing functions of time.

B. The Term Structure Function

A transformation of the equation describing the value of a default-free discount bond (given in Proposition 1) yields the term structure function,

$$\begin{aligned} R(r, \tau) &= -\frac{1}{\tau} \ln f(r, \tau) \\ &= -\frac{\bar{a}_0}{\bar{a}_1} + \left(r + \frac{\bar{a}_0}{\bar{a}_1}\right)\left(\frac{1}{-\bar{a}_1 \tau}\right)(1 - e^{\bar{a}_1 \tau}) \\ &\quad + \frac{B^{1/2}}{\bar{a}_1 \tau} \int_0^\tau \sigma_r(\gamma_t)(1 - e^{\bar{a}_1 t})\, dt \\ &\quad - \frac{1}{2\bar{a}_1^2 \tau} \int_0^\tau \sigma_r^2(\gamma_t)(1 - e^{\bar{a}_1 t})^2\, dt. \end{aligned}$$

This function can be expressed as

$$R(r, \tau) = AER(r, \tau) + MRPB(\tau) + JEB(\tau),$$

where

- $R(r, \tau)$ is the yield to maturity of a default-free discount bond expressed as a function of the bond's term to maturity τ, and a function of the instantaneous spot rate of interest.
- $AER(r, t)$ is the expected value of the average instantaneous future spot rate of interest, averaged over τ time units,

$$AER(r, \tau) = E\left[\frac{1}{\tau} \int_0^\tau r(t)\, dt\right] = -\frac{\bar{a}_0}{\bar{a}_1} + \left(r + \frac{\bar{a}_0}{\bar{a}_1}\right)\left(\frac{1}{-\bar{a}_1 \tau}\right)(1 - e^{\bar{a}_1 \tau}).$$

$AER(r, t)$ is determined from the distribution of r (see, for example, Vasicek (1977)). In addition,

$$\lim_{\tau \to 0} AER(r, \tau) = r, \qquad \lim_{\tau \to \infty} AER(r, \tau) = -\frac{\bar{a}_0}{\bar{a}_1},$$

and

$$\operatorname{sign}\left(\frac{\partial}{\partial \tau} [AER(r, \tau)]\right) = \operatorname{sign}\left(-\frac{\bar{a}_0}{\bar{a}_1} - r\right).$$

That is, the average expected rates curve starts at the spot rate and approaches monotonically the asymptotic value of the spot rate.
- $MRPB(\tau)$ is the market risk premium bias on the yield to maturity, for maturity of τ time units,

$$MRPB(\tau) = \frac{B^{1/2}}{\bar{a}_1 \tau} \int_0^\tau \sigma_r(\gamma_t)(1 - e^{a_1 t})\, dt.$$

Hence,

$$\lim_{\tau \to 0} MRPB(\tau) = 0, \quad \text{and} \quad \lim_{\tau \to \infty} MRPB(\tau) = \frac{B^{1/2}}{\bar{a}_1} \sigma_{ss}.$$

Generally *MRPB* will be a function of r as well as of τ. Here there is a single output, so *MRPB* is a function of τ only. This does not qualitatively affect the behavior of the yield curve because the dependency on r would have changed the size of the $MRPB(\tau)$ by the same proportion for different terms to maturity.

- $JEB(t)$ is the Jensen's inequality bias on the yield to maturity, for maturity of τ time units,

$$JEB(\tau) = -\frac{1}{2\bar{a}_1^2 \tau} \int_0^{\tau} \sigma_r^2(\gamma_t)(1 - e^{\bar{a}_1 t})^2 \, dt.$$

The $JEB(\tau)$ is the difference between the yield to maturity $R(r, \tau)$ and the average expected rates $AER(r, \tau)$ when the market risk premium is zero.

$$\lim_{\tau \to 0} JEB(\tau) = 0, \quad \text{and} \quad \lim_{\tau \to \infty} JEB(\tau) = -\frac{1}{2\bar{a}_1^2} \sigma_{ss}^2.$$

Like $MRPB(\tau)$, $JEB(\tau)$ evolves monotonically from its initial value to its asymptotic steady state when the common factor is observable. Moreover, because of the convexity of the negative logarithm function, Jensen's inequality says that *JEB* is nonpositive. (See the analysis and discussion of this phenomenon in Cox, Ingersoll, and Ross (1981).)

If the common factor is observable, the instantaneous variance of the interest rate $\sigma_r^2(t)$ does not change in time. Vasicek (1977) and Ulman and Wood (1982) analyze this case. The following two propositions, developed from their work, are cited for comparison with the results of this more generalized model. Proposition 5F summarizes the behavior of the components of the term structure in a fully observable economy.

PROPOSITION 5F (Fully observable economy): *If the common factor is observable, then the average expected rates curve $AER(\tau)$ is monotonic and bounded, and both the market risk premium bias $MRPB(\tau)$ and the Jensen's inequality bias $JEB(\tau)$ are increasing and bounded.*

A qualitatively similar behavior will be displayed when the common factor is unobservable and the initial value of the filtering error is identical to its steady state value. In this case, γ_t does not change in time, and therefore $\sigma_r^2(\gamma_t)$ is constant. Combining the effects of the term structure's components when the common factor is observable leads to the following proposition.

PROPOSITION 6F (Fully observable economy): *If the common factor is observable, then the term structure of interest rates can be monotonic increasing, monotonic decreasing, or single humped.*

Vasicek (1977) identified two spot rates, r_L and r_H, $r_L < r_H$. In this case, the yield curve rises for spot rates lower than r_L; the yield curve falls for spot rates higher than r_H; and the yield curve is humped for spot rates between r_L and r_H. Cox, Ingersoll, and Ross (1985b) demonstrate an equivalent phenomenon for different interest rate dynamics. In their model, the instantaneous variance of the interest rate is proportional to its value.

The monotonic or humped yield curve results from the behavior of the market risk premium bias, the Jensen's inequality bias, and the average expected rates curve. The term structure of interest rates, when the common factor is observable, was explained in Section I. Now, however, one can see that the slope of the yield curve is determined by the relative sizes and signs of the *AER* adjustment, the *MRPB* adjustment, and the *JEB* adjustment. Clearly, a sufficient condition for a monotonic yield curve is that all adjustments are of the same sign; a necessary condition for a nonmonotonic yield curve is that the adjustments are not all of the same sign; a sufficient condition for a nonmonotonic yield curve is that the adjustments are of different signs and that their relative sizes change.

When the common factor is unobservable, different term structure curves arise as a function of the different paths of the filtering error. The fully observable economy, representing an observable common factor, is equivalent now to only one particular path, the path where the value of the filtering error is zero at all times. In addition, any particular path of the estimation error will characterize a family of yield curves parameterized by different spot rates. Anticipating the complexity of reporting the different qualitative behaviors of the various term structure families, the following proposition introduces the characteristic function of the term structure and describes it for a fully observable economy.

C. The Characteristic Function of the Term Structure

PROPOSITION 7F (Fully observable economy): *There exists a (characteristic) function $r = CFTS(\tau)$ such that the term structure is increasing (flat, decreasing) at a point (r, τ) if and only if r is less than (equal to, greater than) $CFTS(\tau)$. If the common factor is observable, then the characteristic function is bounded and decreasing. This function will be referred to as the characteristic function of the term structure CFTS.*

Because of the additive separability of the term structure function and the monotonicity of its components, one can see that for humped yield curves, the term to maturity at which the peak of the hump is reached is a decreasing function of the spot rate. That is, if $r_L < r_1 < r_2 < r_H$ then $\tau^*(r_1) > \tau^*(r_2)$, where $\tau^*(r)$ is the term to maturity at which a humped yield curve, starting at r, attains its peak.

Therefore, for all $\tau > 0$, all points (r, τ) $r > r_H$ lie above the characteristic function, and all points (r, τ), $r < r_L$ lie below the characteristic function. In addition, all points (r, τ), $r_L < r < r_H$ lie below the characteristic function for all τ, $\tau < \tau^*(r)$—representing the ascending part of the hump. They cross the characteristic function at $\tau = \tau^*(r)$—representing the peak of the hump, and they lie above the characteristic function for all τ, $\tau > \tau^*(r)$—representing the descending part of the hump. Since $\tau^*(r)$ is decreasing in r, so is the characteristic function. The resulting characteristic function, therefore, exhibits the following properties: $CFTS(0) = r_H$, $CFTS'(\tau) < 0$, and $\lim_{\tau\to\infty} CFTS(\tau) = r_L$. In fact, $CFTS^{-1}(r) = \tau^*(r)$.

PROPOSITION 5P (Partially observable economy): *If the common factor is unobservable, then the average expected rates curve $AER(\tau)$ is monotonic and bounded. The market risk premium bias $MRPB(\tau)$ and the Jensen's inequality bias $JEB(\tau)$ are bounded also. However, these biases can be increasing or increasing-decreasing-increasing, depending on the path of the filtering error and on the relative signs and sizes of the drift source of covariance and the diffusion source of covariance. If the instantaneous variance of the interest rate is increasing (decreasing), then the biases increase at a higher (lower) rate. If this variance is decreasing-increasing, then the biases are increasing-decreasing-increasing. In addition, the rate of change of the instantaneous variance of the interest rate is a function of the filtering error's rate of change.*

The average expected rates curve is qualitatively unaffected by the estimation process. The market risk premium bias and the Jensen's inequality bias are monotonic increasing functions of the instantaneous variance of the interest rates. This variance in turn is an increasing, or decreasing, or decreasing-increasing function of the estimation error.

PROPOSITION 7P (Partially observable economy): *If the common factor is unobservable, then the characteristic function of the term structure is bounded and either decreasing or humped, depending on the path of the filtering error and on the relative signs and sizes of the drift source of covariance and the diffusion source of covariance.*

With an unobservable factor, the shape of the characteristic function is determined primarily by the relationship between the initial value of the estimation error and the covariances of the factor with the outputs.

IV. The Expectations Hypothesis of the Term Structure of Interest Rates

This section examines the expectations hypothesis of the term structure of interest rates and compares it with the analysis of Cox, Ingersoll, and Ross (1981) in a fully observable economy. (See also Leroy (1982).) Their analysis demonstrates that only one version of the expectations hypothesis is consistent with general equilibrium in continuous time. This version states that bonds of all maturities have identical expected instantaneous return. It further demonstrates that (a) the logarithmic utility function, because of its myopic nature, seems to be most strongly related to the expectations hypothesis, and (b) under logarithmic preferences, the uncorrelatedness of the returns on wealth with changes in the state variables is a sufficient condition for the expectations hypothesis.

To examine the expectations hypothesis in a partially observable economy, one can rearrange the valuation equation to define the instantaneous expected return on bonds:

$$E\,\frac{df}{f} = r + u'B^{1/2}\sigma_r(\gamma_t)\,\frac{f_r}{f},$$

and substitute for $\sigma_r(\gamma_t)$:

$$E\,\frac{df}{f} = r + \frac{\mathbf{1}'B^{-1}A_1}{\mathbf{1}'B^{-1}\mathbf{1}}\,u'(D + A_1\gamma_t)\,\frac{f_r}{f}.$$

The term $\dfrac{\mathbf{1}B^{-1}A_1}{\mathbf{1}'B^{-1}\mathbf{1}}\,u'(D + A_1\gamma_t)$ is the covariance of changes between returns on optimally invested wealth and the interest rate. One can assume, without loss of generality, that all production processes are active.

The expectations hypothesis holds if $E\,\dfrac{df}{f} = r$, hence, if $D + A_1\gamma_t = 0$. The case where $D = 0$ is equivalent to the case that Cox, Ingersoll, and Ross (1981) found as sufficient for the expectations hypothesis in a fully observable economy. However, in a partially observable economy, it is clearly not necessary or sufficient. Now the sufficient condition becomes $\gamma_t = -D/A_1$. Thus, necessary conditions for the expectations hypothesis are that the "sources" of covariance are of opposite signs (since D/A_1 should be positive) and that the economy is in a steady state (since the filtering error must be constant in time).

Moreover, this condition implies an additional quality: the spot rate of interest should become nonstochastic. When $D + A_1\gamma_t = 0$, the two sources of covariance cancel each other; realized outputs do not contain information about changes in the investment opportunity set; and, therefore, the equilibrium spot rate is not updated. Quantitatively one can see that if $D + A_1\gamma_t = 0$, then $\sigma_r^2(t) = 0$. Therefore, a nonstochastic instantaneous spot rate of interest is necessary for the expectations hypothesis to hold when the common factor is unobservable.

The association of the expectations hypothesis with nonstochastic interest rates in a partially observable economy has a straightforward interpretation. Recall that, in a fully observable economy, one of the necessary conditions for the expectations hypothesis is no correlation between returns on optimally invested wealth and changes in the investment opportunity set. However, in a partially observable economy, where the changes in the investment opportunity set are unobservable, the absence of a similar covariance causes realized returns to be ineffective in conveying information about unexpected changes in the investment opportunity set. Consequently, the equilibrium interest rate is not stochastically updated.

V. The Model's Assumptions and Some Implications

Acknowledgement of the existence of unobservable state variables generalizes the fully observable economy to a partially observable economy. This generalization, however, induces a specification of the information structure of the economy. The processes that describe the information flow should give rise to a compact Markovian nonlinear filter. Only in this case can the induced optimal control problem be solved. A sufficient, but not necessary, condition for this is a system (θ_t, ξ_t) which is conditionally Gaussian. That is, the probability distribution of θ_t is Gaussian conditional on the observations ξ_s, $0 \leq s \leq t$.

Since the square variation of an Ito diffusion process is observable, a situation where the instantaneous variance of a state variable is an invertible function of it would enable investors to make a perfect and immediate inference of its realizations. (See, for example, Williams (1977).) This, in fact, switches the economy from a partially observable to a fully observable economy.

Optimally, it would be desirable to analyze diffusion coefficients, which are functions of observable state variables (see Cox, Ingersoll, and Ross (1985b)), and drifts, which are functions of both observable and unobservable state variables. Unfortunately, explicit solutions for problems with more than one state variable are tractable only in special cases. Therefore, in order to avoid a perfect and instantaneous inference of the state variable and to enjoy the full range of analysis available for a single state variable, the model uses a state variable with a nonstochastic diffusion coefficient. However, the term structures generated by processes with a nonstochastic diffusion coefficient are similar to those generated by processes with a stochastic diffusion coefficient. (See Vasicek (1977) and Cox, Ingersoll, and Ross (1985b).) The choice of a single state variable is merely for notational convenience. The model's structure holds for any finite number of unobservable state variables. In this case, the equilibrium rate of interest will be a deterministic function of the weighted average of the investors' estimates of the unobservable factors.

For a conditionally Gaussian information structure with parameters that are not functions of the observations, the filtering error is nonstochastic though dynamic, and it converges to a steady state. Therefore, the deviation of a bond's return from that predicted by the expectations hypothesis changes over time as a function of the quality of information; then it asymptotically converges to a constant. A more plausible and realistic description of the quality of information is one that allows stochastic fluctuations. In this case, the market risk premium, or the deviation from the expectations hypothesis, will continuously fluctuate as well. The current structure makes this possible by allowing parameters that are functions of the observations. The identification of a compact Markovian system that describes the posterior distribution of the unobservable factor and has stochastic higher moments would also result in a stochastic quality of information. These extensions, however, would probably not generate qualitatively different term structures.

The closed form solutions for the term structure make it possible to test this model using methods like those in Brown and Dybvig (1986), Gibbons and Ramaswamy (1986), or Stambaugh (1986). When considering the empirical implications of this analysis, one should carefully distinguish between four economies, similar in their endowments and preferences. The first economy is a fully observable one, where the previously specified conditionally Gaussian system (θ_t, ξ_t) describes the information flow. The system (θ_t, ξ_t) describes the information flow in the second economy as well. However, θ_t is unobservable here, so it is a partially observable economy. The third economy is endogenously derived from the second one. The system (m_t, ξ_t, γ_t) describes the information flow; and the nonlinear filter specifies the evolution of m_t and γ_t, the posterior moments of the distribution of θ_t given ξ_s, $0 \leq s \leq t$. The fourth economy is a fully observable one. The system $(\bar{m}_t, \bar{\xi}_t, \bar{\gamma}_t)$ describes the information flow, where $\bar{m}_t$, $\bar{\xi}_t$, and $\bar{\gamma}_t$ are exogenously assumed to be identical to m_t, ξ_t, and γ_t.

While all these economies are qualitatively different, the last three give rise to identical demands and prices and therefore are not distinguishable empirically. The previously presented analysis of equilibrium in a partially observable economy is necessary and sufficient for the endogenous identification of the third economy, which, by theorems from Liptser and Shiryayev (1978), is informationally equivalent to the second one. In fact, the identification of this third economy is necessary and sufficient for determining equilibrium demands and prices in the second economy as well as for identification of the fourth economy. The fourth economy is an exogenous fully observable economy. By construction, it is informationally equivalent to the second partially observable economy.

In other words, the analysis of partially observable economies is essential for the endogenous selection of the family of economies to be tested empirically; it provides a method for endogenously identifying a particular family of fully observable economies. For example, one may assume a simple structure, say, a productivity factor with a constant instantaneous variance coefficient. This will endogenously identify a more complex production function that will induce an interest rate with an instantaneous variance coefficient that changes over time according to an endogenously specified rule. By contrast, in a fully observable economy, one needs to exogenously assume a time-varying instantaneous variance coefficient of the productivity factor in order to achieve a similarly varying instantaneous variance coefficient of the interest rate.

Since an equivalent fully observable economy exists for every partially observable economy, it is quantitatively irrelevant whether one calls the selected economy fully observable or partially observable. The equivalent fully observable economy, however, is not always identifiable;[6] and, as stated earlier, where the information structure is conditionally Gaussian, this analysis is necessary and sufficient for the identification. Qualitatively, however, fully and partially observable economies are easy to distinguish: if investors do not directly observe their investment opportunities, they perform within a partially observable economy.

Any incomplete information learning model will be observationally and informationally equivalent to a similar complete information model without learning where investors are supplied with the inferences outcome. This analysis demonstrates that an n factor partially observable economy has an n factor observable economy which is observationally and informationally equivalent to it. While the introduction of dynamic Bayesian inference clearly does not increase the number of state variables (in fact, it decreases the Martingale multiplicity of the economy since the estimates and the precision of the unobservable variables are measurable with respect to the observations), it does endogenously specify dynamic patterns of parameters in the economy, such as the instantaneous variance of the interest rate and the market risk parameter. These dynamic patterns could not have been otherwise motivated in a fully observable economy.

[6] Frequently, the posterior distribution of the unobservable state variables is characterized by an infinite number of moments. In this case, the economic problem has an infinite number of state equations. It is possible, though, to identify an approximately equivalent full information economy by considering only a finite number of the posterior moments.

In other words, an n factor partially observable economy is observationally and informationally equivalent to a n factor fully observable economy where the economic parameters change over time. Intuitively, the equilibrium of an n factor partially observable economy travels over time across equilibria of n factor fully observable economies of different parameter sets.

Finally, for preferences other than logarithmic, the informational effects of the term structure will be even more distinguished. This is true because the expressions for the spot rate of interest and for optimal demands will both have an additional term which will explicitly depend on the dynamic quality of information. Under logarithmic preferences, this term vanishes because the cross partials of the indirect utility function are zero.

VI. Conclusion

In a production and exchange economy with an unobservable source of nondiversifiable risk, consumer-investors, while determining equilibrium demands and prices, replace unobservable means of asset returns with their estimates, the conditional means. The precision path of these estimates is determined by the estimation error, and individuals take the current and anticipated changes in this path into account while updating their demands. That is, they know that their estimates are imperfect, and they modify their policies according to the relative level of accuracy of their new estimates.

The path of the precision with which investors can infer the changes in the stochastic investment opportunity set is called the path of investors' quality of information. Although the model initially assumes a simple production function, the fluctuating quality of information endogenously induces a more complex one which, in turn, induces changes in the variance of the equilibrium rate of interest and changes in the market risk parameter. These changes are reflected in bond prices and give rise to term structure curves that actual market prices imply.

This paper has introduced and explored the characteristic function of the term structure of interest rates. Compared to a fully observable economy, the characteristic function here exhibits added complexity and generates yield curves resembling those implied by market prices. In addition, except when there is a unique combination of parameter values that results in a nonstochastic equilibrium rate of interest, the widely investigated expectations hypothesis of the term structure does not hold in partially observable economies.

Appendix

Proof of Proposition 1: This price is the unique solution of the general valuation equation when $\delta = 0$ and $f(r, 0) = 1$.

Proof of Proposition 2: The filter equations define γ_t as

$$\frac{d\gamma_t}{dt} = \left(b - \frac{D^2}{B}\right) + 2\left(a_1 - \frac{A_1 D}{B}\right)\gamma_t - \frac{A_1^2}{B}\gamma_t^2,$$

rearranged to get

$$\frac{d\gamma_t}{dt} = \left(-\frac{A_1^2}{B}\right)(\gamma_t - \gamma_s)(\gamma_t - \gamma_u),$$

where

$$\gamma_s = k_2\{(k_1 - \rho) + [(k_1 - \rho)^2 + (1 - \rho^2)]^{1/2}\},$$

$$\gamma_u = k_2\{(k_1 - \rho) - [(k_1 - \rho)^2 + (1 - \rho^2)]^{1/2}\},$$

$$k_1 \equiv \frac{a_1/b^{1/2}}{A_1/B^{1/2}}, \quad k_2 \equiv \frac{(bB)^{1/2}}{A_1}, \quad \text{and} \quad \rho \equiv \frac{D}{(bB)^{1/2}}.$$

Observe that $\text{sign}\left[\frac{d\gamma_t}{dt}\right] = \text{sign}[-(\gamma_t - \gamma_s)(\gamma_t - \gamma_u)]$. Note that ρ is the instantaneous correlation coefficient, $\gamma_s \geq \gamma_u$, $k_2 \geq 0$, and that $k_1 \leq 0$ (since $A_1 > 0$ and $a_1 < 0$). Recall that, since $1 - \rho^2 \geq 0$, γ_s is non-negative. The signs of the time derivative of γ_t reveal that γ_u is an unstable steady state of γ_t, while γ_s is a stable steady state of γ_t. For an initial condition γ_0, the solution of the Riccati equation that describes the evolution of γ_t is

$$\gamma_t = \gamma_u \frac{\dfrac{\gamma_s}{\gamma_u} - \dfrac{\gamma_0 - \gamma_s}{\gamma_0 - \gamma_u}\exp\left[\left(-\dfrac{A_1^2}{B}\right)(\gamma_s - \gamma_u)t\right]}{1 - \dfrac{\gamma_0 - \gamma_s}{\gamma_0 - \gamma_u}\exp\left[\left(-\dfrac{A_1^2}{B}\right)(\gamma_s - \gamma_u)t\right]}.$$

In addition, define

$$\gamma_{ss} \equiv \lim_{t\to\infty} \gamma_t = \gamma_s,$$

and

$$\sigma_{ss} \equiv \lim_{t\to\infty} \sigma_r(t) = \lim_{t\to\infty} \sigma_r(\gamma_t) = \sigma_r(\gamma_s).$$

These demonstrate (a).

The functional form of γ_s implies the concavity in ρ (distribute the arguments of the square root). If $\rho^2 = 1$, then $\gamma_s = k_2[(k_1 - \rho) + |k_1 - \rho|]$. For $\rho = 1$, $\gamma_s = 0$ since $k_1 < 0$. For $\rho = -1$, $\gamma_1 = 0$ only if $k_1 < -1$, i.e., when $\left|\frac{a_1}{b^{1/2}}\right| > \frac{A_1}{B^{1/2}}$. (Again, remember $A_1 > 0$, $a_1 < 0$.)

Proof of Proposition 3: From the filter equations,

$$\sigma_r^2 = \frac{A_1^2}{B}(D^2 + 2DA_1\gamma_t + A_1^2\gamma_t^2),$$

or

$$\sigma_r^2 = \frac{A_1^4}{B}(\gamma_t - \gamma_m)^2, \quad \gamma_m \equiv -D/A_1. \tag{1A}$$

This demonstrates (a) and (b). Together with the monotonicity of γ_t, established in Proposition 2, it demonstrates (c) as well.

Proof of Proposition 4: Sections (a) through (e) follow directly from (1A). Remember that the estimation error assumes the value γ_m if γ_m is between its initial value and its steady state value. From the proof of Proposition 3,

$$\gamma_s = \frac{a_1 B}{A_1^2} + \gamma_m + \left[\left(\frac{a_1 B}{A_1^2} + \gamma_m\right)^2 + \frac{bB - D^2}{A_1^2}\right]^{1/2}.$$

Therefore, $\gamma_{ss} > \gamma_m$ ($\gamma_{ss} < \gamma_m$) if and only if $\frac{bA_1}{2a_1} + D > 0\left(\frac{bA_1}{2a_1} + D < 0\right)$. Again, remember $a_1 < 0$, $A_1 > 0$. Feldman (1983) demonstrates that the asymptotic total covariance between θ_t and ξ_t is positive (negative) if and only if $\left|\frac{bA_1}{2a_1}\right| + D > 0\left(\left|\frac{bA_1}{2a_1}\right| + D < 0\right)$.

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