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ROBERT R. BLISS, JR. and EHUD I. RONN*

ABSTRACT

The purpose of this paper is to provide a test of a state-dependent multinomial model of intertemporal changes in the term structure of interest rates. The theoretical background for the model comes from Ho and Lee (1986). The current paper extends their model in several significant ways. First, we perform diagnostic tests on the data to demonstrate that the empirical results reject a binomial model in favor of a trinomial one. After theoretically deriving the appropriate trinomial model, the current paper extends their model to allow for state-dependent shifts which are determined by the set of ex ante observable state variables. The methodology for the study utilizes OLS regressions to identify the exogenous explanatory variables which drive the hypothesized trinomial process of term structure evolution. The empirical tests indicate that the set of state variables explains a significant portion of the variability in the shifts of the term structure over time. The model also identifies and quantifies a set of variables which impact on changes in the term structure of interest rates.

ARBITRAGE-BASED MODELS HAVE been a particularly appealing form of analysis in financial economics, relying as they do on a parsimonious set of assumptions. These arbitrage models have typically been applied to the valuation of equities (e.g., the APT) and their derivative products (e.g., the Black-Scholes option pricing model). More recent work has focused on the implication of such arbitrage-based models for the valuation of fixed-income securities and their derivative instruments. Cox, Ingersoll, and Ross (CIR) (1985) pioneered this approach by deriving an equilibrium model for the term structure of interest rates which implied that the short-term rate follows a mean-reverting process. Brennan and Schwartz (1979) extended the CIR model by incorporating both short- and long-term rates of interest. Conversely, Ho and Lee (1986) take the observed term structure of interest rates as given and derive the feasible, arbitrage-free subsequent movements in the term structure of interest rates. Their binomial process permits the entire term structure to move up or down by a maturity-dependent constant at each date. Ho and Lee's assumptions of a constant-parameter process

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for the movements of the term structure have the rather unfortunate consequence of leading, alternatively, to extremely high rates of interest along one "branch" (of the binomial tree) and negative interest rates along the other.

Our extension of the Ho and Lee approach entails a number of significant changes. Based upon diagnostic tests of the data, it can be shown convincingly that a trinomial, rather than binomial, model of the term structure fits the data in a much superior fashion. Further, the current model contains an explicit modeling of state-dependent shifts in the perturbation function of the trinomial process: the "up," "down," and "no change" states of the term structure are determined by the vector of state variables at time t. This paper consequently tests a parsimonious set of state variables, identifies the statistically significant ones, and quantifies their impact on the change in the riskless discount function, while simultaneously satisfying the cross-sectional constraint of no-arbitrage opportunities in the pricing of pure discount bonds. Further, the tests estimate the period-by-period trinomial probability of the "up," "down," and "no change" states in bond prices. Thus, using arbitrage arguments and a parsimonious set of assumptions, the current paper derives the trinomial time-varying shifts model of term structure evolution.

The application of factor analysis to term structure theory is contained in a substantial body of bond pricing literature, predominantly in the context of duration matching and immunization. Thus, Redington (1952) proposed the single-factor model extended and tested by Fisher and Weil (1971), Brennan and Schwartz (1983), Ingersoll (1983), and Nelson and Schaefer (1983).[1] Further, Ayers and Barry (1979, 1980) proposed a two-factor model: based on the short rate and the spread between the long and short rates of interest. Bierwag (1987) considered the relationship between multifactor models of bond returns and multiple duration measures corresponding to these factors. In contrast to these models, the current paper considers a vector of state variables which define the current "state of nature" and whose values determine the magnitude of the shifts (perturbations) in the term structure over time.

The state-dependent model derived here has clear implications for the pricing of all interest rate-dependent claims; one particularly attractive application may be found in the pricing of traded options on interest rate futures contracts. Thus, in addition to providing a quantitative description of the stochastic investment opportunity set, our use of a state-dependent formulation holds the promise of greater accuracy and explanatory power in the pricing of interest rate-dependent claims.

This paper is organized as follows. Section I briefly summarizes the Ho and Lee term structure model. Section II describes the data used in this study and presents the diagnostic tests which demonstrate the empirical appropriateness of the stochastic model postulated in the paper. Section III derives the trinomial model and its salient properties. Section IV presents the paper's empirical results. Section V contains a summary.

[1] See the excellent book edited by Bierwag, Kaufman, and Toevs (1983a) as well as their survey article (1983b) for other contributions to multifactor-based duration and immunization.

I. The Ho and Lee Fixed-Perturbation Model

Ho and Lee (1986) observe that, in an economy without uncertainty, the forward rates implied by the current term structure must be the interest rates prevailing in the future in order to avoid intertemporal arbitrage. A simple way to introduce uncertainty is to postulate a binomial process, where the spot rate term structure at time $t + 1$ is equal to the implied forward term structure at time t, multiplied by a perturbation function that distinguishes the "upstate" from the "downstate." In the Ho and Lee model,

1) markets are frictionless,
2) riskless pure discount bonds are available for all maturities,
3) term structure movements are path independent (an up-move followed by a down-move results in the same term structure as a down-move followed by an up-move),
   and
4) the perturbation functions are time and state invariant.

Under these four postulates, Ho and Lee solve for the functional form of the perturbations. Their model can be completely specified by two parameters: π and δ, where π corresponds to the binomial, or risk-neutral, probability of an up-move in discount prices (declining interest rates) and where $0 < \delta < 1$ is essentially a volatility parameter.[2]

Formally, following Ho and Lee (1986),[3] consider the term structure at time n and state i in terms of the implied prices of discount bonds, $P_i^n(m)$, where i denotes the number of up-moves which have occurred up to time n, and m is the time to maturity of the pure discount bond. The next-period price is equal to the current forward price times a perturbation function. The perturbation function depends on the state realized next period—either "up" or "down."

Thus, the discount functions at time $n + 1$ are given by

$$P_{i+1}^{n+1}(m) = \frac{P_i^n(m + 1)}{P_i^n(1)}\, h(m) \tag{1}$$

for the upstate and

$$P_i^{n+1}(m) = \frac{P_i^n(m + 1)}{P_i^n(1)}\, h^*(m) \tag{2}$$

for the downstate. By convention, asterisked variables refer to downstates.

Then the perturbation function for the upstate and downstate, respectively, will be

$$h(m) = \frac{1}{\pi + (1 - \pi)\delta^m} > 1 \tag{3}$$

[2] Under $\delta = 1$, the certainty case results.

[3] We have made minor changes in notation to conform with the usage in this paper. We retain the Ho and Lee designations of $h(\cdot)$ and $h^*(\cdot)$ when referring to the perturbations in their model.

and

$$h^*(m) = \frac{\delta^m}{\pi + (1 - \pi)\delta^m} < 1 \tag{4}$$

for $m = 0, 1, 2, \cdots$.

Ho and Lee's binomial model gives rise to an important cross-sectional constraint: in order to prevent arbitrage profits, the maturity-independent parameter π must satisfy

$$h(m)\pi + h^*(m)(1 - \pi) = 1 \quad \forall m. \tag{5}$$

The intuitive rationale for equation (5) follows from the requirement that any portfolio (of pure discount bonds) whose one-period return is riskless must earn the one-period rate of interest.

The Ho and Lee perturbation functions, equations (3) and (4), have two serious limitations. First, the parameters, δ and π, are time and state invariant. Therefore, regardless of the current state of the world, the "up" and "down" perturbations will be the same. The second problem with the model is that with fixed perturbations the interest rates at the extreme branches assume unreasonable values if we allow the number of periods to increase without bound.[4]

The objective of our paper is to generalize the binomial fixed-perturbation Ho and Lee framework. To avoid the potential degeneracy of the fixed-perturbation model, we use a time-varying state-dependent framework, which permits the magnitude of the perturbations at time t to depend on the vector of observed state variables at time $t - 1$. This permits the perturbations and risk-neutral probabilities to change, allowing for mean reversion.[5] Section III below will describe our model.

[4] Taking the limit as the number of periods goes to infinity will demonstrate the problem. Ho and Lee show that the value of the discount functions in any future time and state can be expressed in terms of a given *initial* discount function, $P(m) \equiv P^0(m)$, and the parameters π and δ. For a future period n and $m = 1$, that expression is

$$P_i^n(1) = \frac{P(n+1)}{P(n)} \frac{\delta^{n-i}}{[\pi - (1 - \pi)\delta^n]}.$$

Consider, for example, an infinite sequence of down-moves:

$$\lim_{n\to\infty} P_0^n(1) = \lim_{n\to\infty} \frac{P(n+1)}{P(n)} \frac{\delta^n}{[\pi + (1 - \pi)\delta^n]} = 0 \tag{6}$$

for any $0 < \delta < 1$, assuming only that the one-period forward rate implied by the initial term structure is bounded from above for all maturities. Equation 6 implies that the short-term interest rate will tend to infinity if we have a sufficiently long run of down-moves.

Similarly, consider the limit of the one-period discount function if an infinite number of up-moves occur:

$$\lim_{n\to\infty} P_n^n(1) = \lim_{n\to\infty} \frac{P(n+1)}{P(n)} \frac{\delta^0}{[\pi + (1 - \pi)\delta^n]} = \frac{1}{\pi(1 + f_\infty)},$$

where $1 + f_\infty \equiv \lim_{n\to\infty} P(n)/P(n+1)$. Negative interest rates will result if $\pi < 1/(1 + f_\infty)$.

[5] Empirically, we do not impose mean reversion, so it is still possible for interest rates to explode. However, since we estimate the model from data that do not exhibit this behavior, this is unlikely to happen. In contrast, the Ho and Lee model is not mean reverting.

II. Data

In order that our model present as accurate a representation of reality as possible, we will find it useful to consider the data in the construction of the theoretical model and its subsequent empirical tests. Consequently, we first describe the underlying data set for the current model and tests.

A. Data Description

The data required for this project may be classified into two categories. The first category is a time series of the term structure of interest rates or, equivalently, the pure discount prices P_{tm} for a series of times t and maturities m. We begin with the Fama-Bliss (1987) set of pure discount bond prices. The Fama-Bliss methodology for extracting the prices of pseudo-discount bonds from the available coupon bonds results in a term structure which exactly fits the data. These data may be subject to quotation errors and indeterminacy of the true price within the quoted spread. The resulting spot rate structure is therefore "jagged." The current paper smooths the Fama-Bliss spot rate curve,[6] before constructing the discount bond price series used in the empirical tests. The resulting monthly series of pseudo-discount bond prices covers the period 1962:2–1987:12.

The second set of data contains the state variables $\mathbf{x}_t$. The following set of state variables is used:

1) the three-month T-bill rate,
2) the term premium (defined as the thirty-year government bond yield less the three-month T-bill yield),
3) the risk premium (defined as the yield on BBB corporate debt less the yield on AAA-rated debt),
4) the current dividend yield in the CRSP value-weighted market portfolio,
5) a measure of the "hump" in the term structure.[7]

The set of state variables was chosen with a view to including variables of a forward-looking nature, i.e., the values of which incorporate market participants' information concerning the future evolution of the prices of real and financial

[6] Based on a modification of the Nelson and Siegel (1987) term structure estimation technique, the current paper uses the following approximating function for the Treasury bill yield curve:

$$r(t) = \beta_0 + \beta_1 \frac{1 - \exp(-t/\tau_1)}{t/\tau_1} + \beta_2 \exp\left(-\frac{t}{\tau_2}\right),$$

where the parameters $\beta_0, \beta_1, \beta_2, \tau_1, \tau_2$ were estimated anew for each month. This estimation technique has several advantages: the resulting yield curve can adopt various shapes (monotonic, "humped," or S-shaped) while remaining asymptotically flat and avoiding the unbounded forward rates which plague other functional forms; the model is parsimonious, requiring the estimation of only four parameters; and experimentation indicated that the function constituted a good fit for the yield curve over the maturity range out to five years.

Details of the methodology may be found in Bliss (1989).

[7] The "hump" is empirically defined as the maximum excess, if any, of the intermediate-term interest rates (of maturities one, two, three, five, seven, and ten years) over the maximum of the three-month T-bill rate and the thirty-year yield.

assets. Hence, the three-month T-bill rate is designed to capture the near-term inflation expectations and the near-term level of the real interest rates. The inclusion of the dividend yield, the term premium, and the default premium as significant state variables is suggested by Fama and French (1987), who note, "The default spread is a clear business cycle variable, high when business is poor and low when the economy is strong. We argue that dividend yields and the term spread also have strong business cycle components."

Thus, to the extent that term structure perturbations are functions of the business cycles, these variables will capture that effect. Finally, the "hump" is designed to capture the information, if any, inherent in the curvature of the term structure of interest rates.

The three-month T-bill rate was obtained from the Fama T-bill Files, whereas the remaining interest rate state variables were available from the CITIBANK data base.

B. Analysis of Intertemporal Nonstationarity

The first order of business in our analysis is to determine the degree of intertemporal nonstationarity in the data. Due to the lack of suitable long-maturity issues in the 1960's, we focus our analysis on the term structure movements out to a five-year maturity. Let y_{tm} denote the empirical analogue to the Ho-Lee perturbation functions $h(\cdot)$ and $h^*(\cdot)$:

$$y_{tm} \equiv \frac{P_{tm}P_{t-1,1}}{P_{t-1,m+1}}, \tag{7}$$

where P_{tm} is the price at the time t of an m-period pure discount bond.

Figure 1A presents the time series y_{tm} for $m = 60$ months over t from 1962:02–1987:12. Three distinct time periods are discernible: the low-variability period 1962:02–1965:11, the moderate-variability period 1965:12–1979:09, and the post-October 1979 high-variability period. A substantially similar pattern obtains for $m = 30$; moreover, the estimated (unconditional) $\text{Corr}(y_{t,30}, y_{t,60}) = 0.94$, thus establishing consistency across different maturities and reliability of the subperiod classification used herein.

C. Empirical Appropriateness of a Trinomial Model

Figure 1B re-examines Figure 1A, revealing a significant number of observations on or about the $y_{t,60} = 1$ line. Further, consider Figure 2, which plots y_{tm} for $m = 1, \cdots, 60$ and at several alternative t in the last subperiod, 1979:10–1987:12. These two graphs are strongly suggestive of an alternative modeling of term structure movements: a trinomial model, with an additional state and an additional perturbation function h^n_{tm} which will represent the "no change" state.[8]

The time-varying parameter trinomial model includes the Ho and Lee model as a special case. In the Ho and Lee model the "no change" state never occurs;

[8] The term "no change" does not require that $y_{tm} \equiv 1\ \forall t$ and m; rather, it refers to the state where the perturbations are "close to" unity. Section IV includes a discussion of the classification scheme for determining which of the three states materializes.

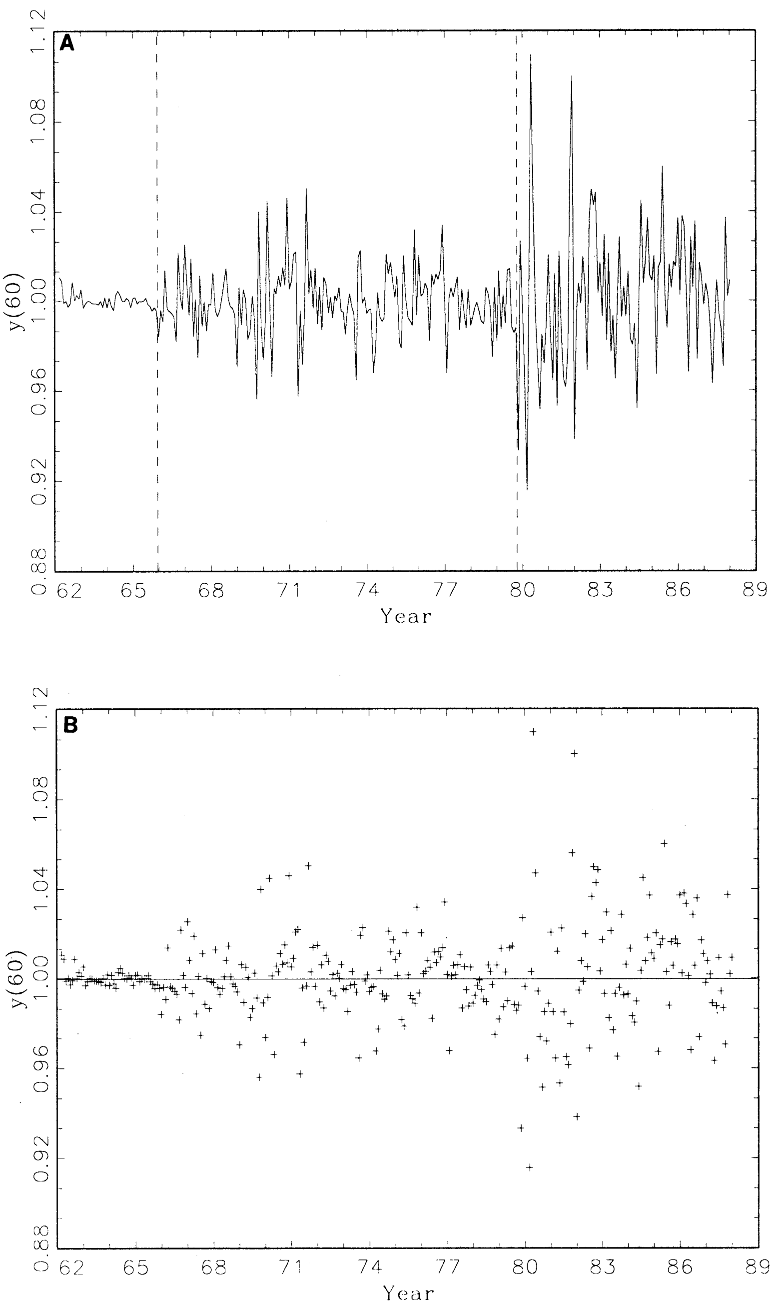

Figure 1. Realizations of five year maturity perturbations.

only the constant terms in the "up" and "down" regressions are significant; and the classification of observations uses unity to distinguish "up" and "down."

To obtain an estimate of the impact of the restrictions implied in the Ho and Lee model, and in other models which are special cases of the model we propose,

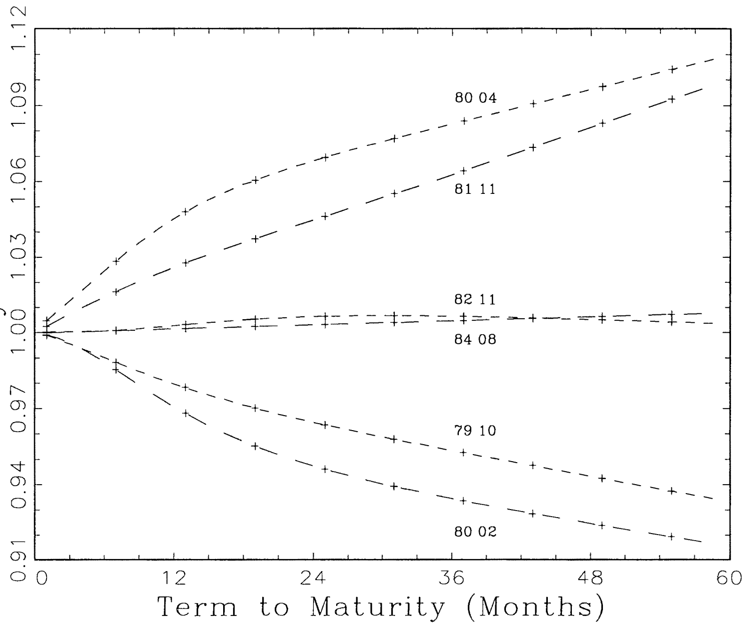

Figure 2. Term stuctures of perturbations for selected months.

we use a series of unconstrained regressions. Cross-sectional constraints, such as (5), are not imposed, and each maturity is treated separately in estimating the parameters. The criterion function for comparing models is the sum of the squared deviations of the realized perturbations from the fitted ones, summed across all maturities.[9]

The simplest possible model is a constant-parameter uninomial one, wherein the perturbation is the same regardless of the current state of the world and is not conditional on the state of the world which materializes next period. The variable-parameter uninomial model posits that the expected perturbation is a function of the current value of the state variables but is not conditional on the state of the world realized next period.

We test both the Ho and Lee constant-parameter binomial model and a variant where the break point used to classify outcomes may be different from unity. In the latter case the value used was that which produced the lowest value of the criterion function. The variable-parameter binomial model classified outcomes based on their proximity to the fitted values. Details are given in Section IV.*A*. Again, the classification used was the one which produced the lowest value of the criterion function.

The trinomial models results were obtained by an analogous procedure.

[9] For these tests five maturities of one through five years were used. The time-varying models were of the form:

$$h^s_{tm} = \alpha^s_{1m} + \sum_{k=2}^{K} \alpha^s_{km} x_{kt},$$

where s denotes u, n, or d as appropriate.

Table I

SSR for Various Models

y_{60} is used as the classification variable.

Type of Model		SSR
Uninomial	Constant (Mean)	0.246
	Variable (Regression)	0.218
Binomial	Ho and Lee $y > 1 \rightarrow up$ $1 > y \rightarrow dn$	0.112
	Constant ϵ $y > 1 + \epsilon \rightarrow up$ $1 + \epsilon > y \rightarrow dn$	0.113
	Variable ϵ_t	0.077
Trinomial	Constant ϵ^u, ϵ^d $y > 1 + \epsilon^u \rightarrow up$ $1 + \epsilon^u > y > 1 - \epsilon^d \rightarrow nc$ $1 - \epsilon^d > y \rightarrow dn$	0.110
	Variable $\epsilon^u{}_t$, $\epsilon^d{}_t$	0.039

Table I presents the sum of squared residuals (SSR) for these models. For the binomial and trinomial models, the classification scheme by which observations have been classified (into "up," "down," or "no change" states) is included in the table. These schemes consist of both constant and time-varying limits for the inclusion of observations into the various states.[10] The time-varying trinomial model exhibits a much reduced SSR relative to the other models.[11] Thus, our modeling of a trinomial process constitutes a compromise between the greater generality of a multinomial process and the analytical complexity of such higher order processes; whereas the binomial process is found empirically to be overly constrained, the trinomial process is more appealing empirically while remaining analytically tractable.

III. The Trinomial Lattice

A. Theoretical Derivation

Theorem 1 provides the theoretical derivation of the trinomial lattice.

THEOREM 1: *Consider an economy in which the term structure of interest rates can adopt one of three alternative forms in the next period (period t). These forms*

[10] For example, the Ho and Lee classification uses the following rule: if $y_{t,60} > 1$, that observation is classified as an "up" state; conversely, if $y_{t,60} < 1$, that period is considered a "down" state.

[11] The optimized break point for the constant-parameter binomial model was arrived at by optimizing with respect to the classification maturity ($m = 60$ months) only. This solution was not optimal for all maturities, as evidenced by the fact that the SSR over all maturities for the Ho-Lee classification scheme was slightly lower than that for the $m = 60$ optimization shceme utilized.

are determined by the current $(t-1)$ *term structure and the three possible perturbations* h^u_{tm}, h^n_{tm}, *and* h^d_{tm}. *The future prices are determined as follows*:

$$P_{tm} = \frac{P_{t-1,m+1}}{P_{t-1,1}} h^s_{tm}, \tag{8}$$

where "s" denotes the state realized (u, n, or d). We assume that state realizations are the same across all maturities, m, for any given period t.

Then, if riskless arbitrage opportunities are to be precluded, the perturbation functions h^u_{tm}, h^n_{tm}, *and* h^d_{tm} *must satisfy the conditions*:

$$h^u_{tm}\pi^u_t + h^n_{tm}\pi^n_t + h^d_{tm}\pi^d_t = 1 \quad \forall m, \tag{9}$$

$$\pi^u_t + \pi^n_t + \pi^d_t = 1. \tag{10}$$

Proof: Appendix A. The demonstration consists of constructing a portfolio of three distinct issues which has the same payoff in each of the three states and is therefore riskless.

Appendix B provides a set of intuitively appealing sufficiency conditions for the existence of a finite-weights portfolio (of pure discount bonds) whose one-period return is riskless.

B. State-Dependent Shifts in the Term Structure of Interest Rates

To introduce state dependency into the Ho and Lee model, assume the following notation: $P_{tm} \equiv$ price at time t of an m-period pure discount bond; $\mathbf{x}_t \equiv K \times 1$ vector of state variables observed at time t.

Consider the deviations of realized spot prices from their respective forward prices in the previous period. Let $h^u(\mathbf{x}_{t-1}, m)$ be the perturbation in the forward function for maturity m conditional on the current vector of state variables $\mathbf{x}_{t-1}$ and conditional on an "up" state materializing the next period; $h^d(\mathbf{x}_{t-1}, m)$ and $h^n(\mathbf{x}_{t-1}, m)$ are the analogous function for the "down" and "no change" states. The explicit time-varying dependency of $h^u(\cdot)$, $h^n(\cdot)$, and $h^d(\cdot)$ on a vector of state variables is designed to remedy the Ho-Lee degeneracy result described in Section I: the explicit introduction of time-varying state variables is designed to allow for a dependency of the perturbations to, among others, the short-term interest rate, thus potentially alleviating a situation in which these perturbations give rise to an unreasonably low or high level of interest rates.

Finally, $\pi^u(\mathbf{x}_{t-1})$, $\pi^n(\mathbf{x}_{t-1})$, and $\pi^d(\mathbf{x}_{t-1})$ are the probabilities, conditional on $\mathbf{x}_{t-1}$, such that, under this state-dependent model, equation (11) obtains:

$$h^u(\mathbf{x}_{t-1}, m)\pi^u(\mathbf{x}_{t-1}) + h^n(\mathbf{x}_{t-1}, m)\pi^n(\mathbf{x}_{t-1}) + h^d(\mathbf{x}_{t-1}, m)\pi^d(\mathbf{x}_{t-1}) = 1, \tag{11}$$

where $\pi^u(\cdot)$, $\pi^n(\cdot)$, and $\pi^d(\cdot)$ are as defined in equations (9) and (10).[12]

Thus, the generalization of the Ho and Lee framework has allowed a state dependency to enter into the $\{h^u(\cdot)$, $h^n(\cdot)$, and $h^d(\cdot)\}$ and $\{\pi^u(\cdot)$, $\pi^n(\cdot)$, and $\pi^d(\cdot)\}$ functions, while retaining the no-arbitrage condition implied in equation (9).

[12] Henceforth, a reference to the π_t will imply $\pi(\mathbf{x}_{t-1})$, even when that dependency on the state vector $\mathbf{x}_{t-1}$ is not explicitly included.

Consider the price of a security paying \$1 if and only if a single state, say "up," materializes. Using a procedure identical to the derivation of equations (20) in Appendix A, it can be shown that the price of such a security is $\pi_t^u P_1 \equiv \pi_t^u/(1 + r)$, where r is the current one-period interest rate. Analogous prices for the "no change" and "down" states are $\pi_t^n/(1 + r)$ and $\pi_t^d/(1 + r)$, respectively. Thus, since prices of state-contingent securities having non-negative payoffs in all states must be non-negative, we require as a property of the trinomial lattice that

$$(\pi_t^u, \pi_t^n, \pi_t^d) \geq 0 \quad \forall t. \tag{12}$$

IV. Empirical Tests

A. Classification of Time Periods

In order to test our model it is necessary to classify each realization as either an "up," a "down," or a "no change" state.

Our first decision is the choice of the variable to use to classify each outcome. We observe that the y_{tm} observed in the data are typically monotone in m (a condition implied by the Ho and Lee model) and are highly correlated across maturities. We therefore select the longest maturity, $y_{t,60}$, as the variable on which to base our classification.[13]

Our hypothesized (and estimated) perturbations are changing through time. Thus, a fixed-classification scheme would be inappropriate in a model where the three possible outcomes in each period are state dependent.

If it were the case that the only source of uncertainty was which of the three outcomes was realized, then we would classify each observation according to whether y matched h^u, h^n, or h^d. Unfortunately, we can only estimate the expected perturbation functions. Furthermore, we cannot expect the model to hold exactly. We therefore use a maximum-likelihood approach, based on the predicted outcomes, to assign each observation to one of the three outcomes. Since the parameter estimation needed to forecast possible perturbations and standard errors itself depends on a prior classification of observations into "up," "down," or "no change" groups, it is necessary to use an iterative approach.

A.1. Step 0: Initial Classification of State Realizations

Consider the fixed parameters ϵ^u and ϵ^d: if $1 - \epsilon^d < y_{t,60} < 1 + \epsilon^u$, then "no change" will be said to have occurred at t; otherwise, either an "up" or "down" state is said to have occurred. The ϵ^u, ϵ^d are arbitrarily chosen as one standard deviation above and below the mean value of $y_{t,60}$.

[13] The assumption of a single state realization across all maturities is implicit in the development of our model. The empirical results are somewhat dependent on the maturity—in this case, the longest available maturity of sixty months—by which the classification is determined. Tests were run using other maturities. Classifying on the basis of the longest maturity resulted in the best overall classification based on the SSR calculated across all maturities. Using other maturities as the classification variable reduced the SSR for that maturity's particular regression but increased the overall SSR.

A.2. Step 1: Estimate Fitted Values

Using the outcome assignments from the preceding step, calculated the fitted values for $\hat{h}^u_{t,60}$, $\hat{h}^n_{t,60}$, and $\hat{h}^d_{t,60}$ from the unconstrained regressions. Also compute the standard deviations of $(y - \hat{h})$ for each of the realizations. Denote these as $\hat{\sigma}^u$, $\hat{\sigma}^n$, $\hat{\sigma}^d$.

A.3. Step 2: Reclassification of State Realizations

If $y_{t,60} > \hat{h}^u_{t,60}$, assign t to the "up" state. If $y_{t,60} < \hat{h}^d_{t,60}$, assign t to the "down" state. If $\hat{h}^n_{t,60} < y_{t,60} < \hat{h}^u_{t,60}$ and

$$\frac{\hat{h}^u_{t,60} - y_{t,60}}{\hat{\sigma}^u} < \frac{y_{t,60} - \hat{h}^n_{t,60}}{\hat{\sigma}^n}, \tag{13}$$

assign t to the "up" state; otherwise, assign t to the "no change" state. If $\hat{h}^d_{t,60} < y_{t,60} < \hat{h}^n_{t,60}$ and

$$\frac{y_{t,60} - \hat{h}^d_{t,60}}{\hat{\sigma}^d} < \frac{\hat{h}^n_{t,60} - y_{t,60}}{\hat{\sigma}^n}, \tag{14}$$

assign t to the "down" state; otherwise, assign t to the "no change" state.

Repeat steps one and two until no further changes in outcome classifications occur. Convergence is achieved in nine iterations.

Figure 3 plots the fitted $\hat{h}^u_{t,60}$, $\hat{h}^n_{t,60}$, and $\hat{h}^d_{t,60}$ following the final iteration together with the $y_{t,60}$ and their classification for the interval 82/01 through 84/12. The time variation in the perturbation functions is readily apparent. A classification scheme based on fixed break points (such as Step 0 in our procedure)

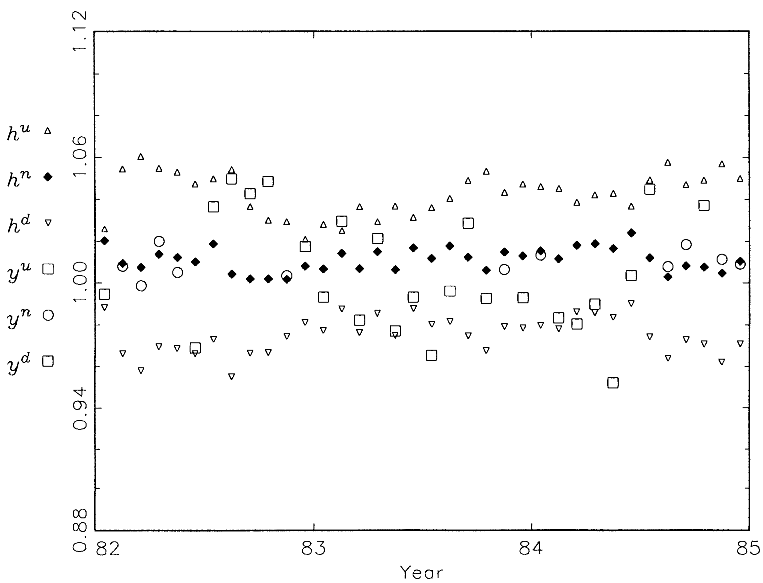

Figure 3. Predicted perturbations and classified realizations, 1982–1984.

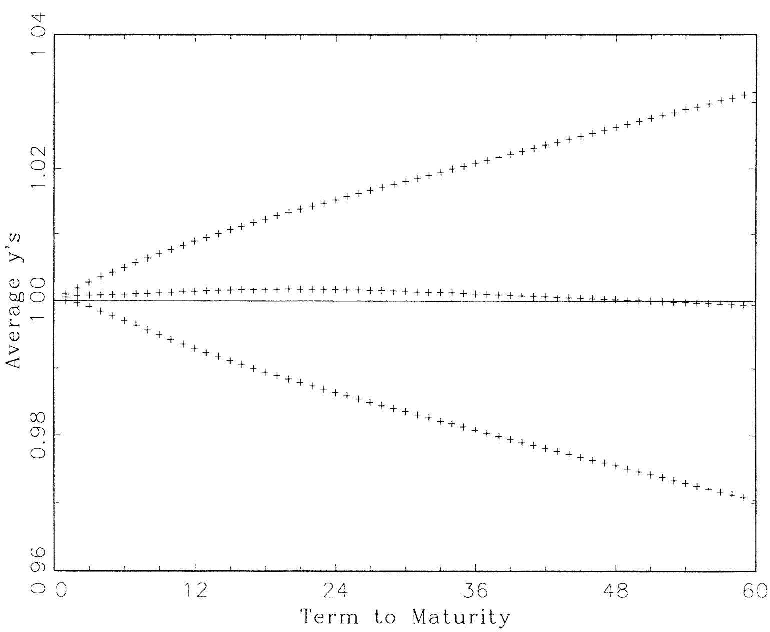

Figure 4. Average "up," "down," and "no change" perturbation functions by maturity.

would have classified the observations on 82/04 and 82/12 identically. We can see, however, that the state-dependent perturbations changed in such a way as to make the different classifications justified.

B. Properties of the Perturbation Functions h_m^u, h_m^n, and h_m^d

As a means of presenting an overview of the empirical results, Figure 4 plots the average (unconditional) values of h_m^u, h_m^n, and h_m^d over the period 1979:10–1987:12. It is instructive to note that, on average, h_t^u and h_m^d display monotonicity of value in m, with h_m^u increasing and h_m^d decreasing as m increases. Recall that the binomial model (equations (3) and (4)) also shares this monotonicity property, in that the analytical expressions for $h_m^u(\cdot)$ and $h_m^d(\cdot)$ satisfy $\partial h_m^u/\partial m > 0$ and $\partial h_m^d/\partial m < 0$.[14]

C. Analysis of the Empirical Results

Consider the following set of three regressions:

$$h_t^u(\mathbf{x}_{t-1}, m) = \alpha_{1m}^u + \textstyle\sum_{k=2}^K \alpha_{km}^u \mathbf{x}_{k,t-1}, \tag{15}$$

$$h_t^n(\mathbf{x}_{t-1}, m) = \alpha_{1m}^n + \textstyle\sum_{k=2}^K \alpha_{km}^n \mathbf{x}_{k,t-1}, \tag{16}$$

$$h_t^d(\mathbf{x}_{t-1}, m) = \alpha_{1m}^d + \textstyle\sum_{k=2}^K \alpha_{km}^d \mathbf{x}_{k,t-1}, \tag{17}$$

[14] The monotonicity properties of $h_m^u(\cdot)$ and $h_m^d(\cdot)$ will be exploited in deriving both the theoretical and empirical properties of the trinomial model. See Appendix B for a discussion of the theoretical aspects.

Table II

Regression Results for the Unconstrained Model

Maturity	Variable	Up		No Change		Down	
$m = 12$	Constant	1.071	(10.5)	1.069	(11.1)	1.112	(8.27)
	3-Month T-Bill	7.37e-4	(0.70)	5.14e-5	(0.11)	1.09e-3	(1.16)
	Term Premium	−4.86e-3	(−3.57)	5.15e-4	(0.67)	3.82e-3	(2.75)
	Risk Premium	−1.28e-3	(−0.30)	−7.64e-5	(−0.04)	7.01e-4	(0.20)
	Hump	0.229	(0.66)	0.099	(0.70)	−0.442	(−1.69)
	Div. Yield	−3.80e-3	(−0.76)	3.53e-3	(0.84)	5.97e-3	(1.22)
	Lagged y_m	−6.93e-2	(−0.68)	−7.12e-2	(−0.73)	−0.114	(−0.85)
	R^2	0.853		0.134		0.349	
$m = 24$	Constant	0.965	(7.93)	1.093	(12.9)	1.166	(9.23)
	3-Month T-Bill	1.67e-3	(0.90)	3.07e-4	(0.45)	1.43e-3	(0.91)
	Term Premium	−5.65e-3	(−2.30)	1.84e-3	(1.57)	6.78e-3	(2.92)
	Risk Premium	−3.47e-3	(−0.45)	−1.04e-3	(−0.35)	6.01e-3	(1.01)
	Hump	0.151	(0.25)	8.45e-2	(0.40)	−0.649	(−1.49)
	Div. Yield	−4.18e-3	(−0.46)	9.83e-3	(1.57)	1.37e-2	(1.68)
	Lagged y_m	4.38e-2	(0.36)	−9.47e-2	(−1.11)	−0.179	(−1.42)
	R^2	0.771		0.234		0.351	
$m = 36$	Constant	0.857	(6.82)	1.137	(16.8)	1.191	(10.4)
	3-Month T-Bill	2.17e-3	(0.91)	7.82e-4	(1.05)	1.71e-3	(0.88)
	Term Premium	−4.90e-3	(−1.54)	3.46e-3	(2.67)	8.78e-3	(3.07)
	Risk Premium	−9.39e-3	(−0.94)	−1.51e-3	(−0.45)	9.34e-3	(1.28)
	Hump	0.266	(0.33)	1.41e-2	(0.06)	−0.744	(−1.38)
	Div. Yield	−3.81e-4	(−0.32)	1.44e-2	(2.13)	1.86e-2	(1.87)
	Lagged y_m	0.158	(1.27)	−0.140	(−2.07)	−0.215	(−1.90)
	R^2	0.725		0.413		0.372	
$m = 48$	Constant	0.804	(6.28)	1.169	(21.8)	1.197	(11.5)
	3-Month T-Bill	2.70e-3	(0.92)	1.21e-3	(1.62)	2.18e-3	(0.99)
	Term Premium	−3.82e-3	(−0.98)	4.97e-3	(3.83)	1.07e-2	(3.30)
	Risk Premium	−1.55e-2	(−1.25)	−1.30e-3	(−0.38)	1.08e-2	(1.31)
	Hump	0.442	(0.45)	−4.55e-2	(−0.20)	−0.846	(−1.38)
	Div. Yield	−3.23e-3	(−0.22)	1.87e-2	(2.77)	2.21e-2	(1.96)
	Lagged y_m	0.214	(1.69)	−0.173	(−3.25)	−0.230	(−2.25)
	R^2	0.687		0.580		0.401	
$m = 60$	Constant	0.789	(5.87)	1.187	(25.6)	1.199	(12.4)
	3-Month T-Bill	3.43e-3	(0.94)	1.56e-3	(2.03)	2.69e-3	(1.10)
	Term Premium	−2.42e-3	(−0.50)	6.34e-3	(4.68)	1.26e-2	(3.50)
	Risk Premium	−2.09e-2	(−1.34)	−7.57e-4	(−0.21)	1.18e-2	(1.28)
	Hump	0.617	(0.51)	−7.37e-2	(−0.31)	−0.949	(−1.40)
	Div. Yield	−2.57e-3	(−0.14)	2.25e-2	(3.22)	2.52e-2	(2.02)
	Lagged y_m	0.230	(1.72)	−0.195	(−4.23)	−0.241	(−2.54)
	R^2	0.633		0.671		0.428	
All	# obs.	22		37		40	

for all t and m. Table II reports the coefficients (with t-statistics in parentheses) for the state variables from these OLS regressions performed over the latest subperiod, 1979:10–1987:12. Several observations are noteworthy here.

First, the explanatory power (in terms of R^2's) of the model is substantial: the maximum R^2's are 0.853 for "up," 0.725 for "no change," and 0.428 for "down." Given the restricted region in which "no change" is defined to occur, the relatively high R^2 is substantial.

In terms of statistically significant impact, the term premium is the only variable impacting on the h^u functions; the short-term interest rate, term premium, dividend yield, and lagged y significantly influence the h^n perturbations; and the term premium, "hump," dividend yield, and lagged y affect the h^d functions. These state variables constitute a statistically relevant set for conditioning the magnitude of the term-structure perturbations.

Finally, note that virtually all statistically significant state variables (excluding only the term premium in the h^u regressions) exhibit the "monotonicity property": the coefficients are monotonic (increasing or decreasing) with respect to the maturity m.

V. Summary

This paper devlops a no-arbitrage trinomial model of state-dependent shifts in the term structure of interest rates. Empirical tests using a prespecified set of economic state variables display statistically significant explanatory power for the "up," "down," and "no change" states of nature and thus constitute important evidence on intertemporal changes in riskless rates of interest.

Among other implications, we believe that this analysis is suggestive of future work along the following two lines of inquiry. First, the cross-sectional constraint[15] can be explicitly imposed, resulting in parameter estimates which are consistent with that no-arbitrage condition. Second, the model we have proposed gives rise to a natural application in the valuation of interest rate-dependent securities: for example, exchange-traded options on Treasury Note and Treasury Bond futures contracts.

Appendix A

Derivation of Trinomial Model

We begin by showing that—under the trinomial model and given the availability of pure discount of three distinct maturities—it is possible to form a riskless portfolio. The results of this analysis imply the cross-sectional constraint, equation (11).

A.1. Derivation Using Riskless Portfolio

The necessary no-arbitrage condition is that a portfolio which has the same value in each possible next-period state must earn the riskless rate over the next period.

Consider a portfolio of three pure discount bonds of distinct maturities $m + 1$, $m_1 + 1$, $m_2 + 1$ with proportions 1, ω_1, and ω_2. The current value of the portfolio is

$$V_0 = P_{m+1} + \omega_1 P_{m_1+1} + \omega_2 P_{m_2+1},$$

where prices are at $t = 0$. The next-period value of a pure discount bond, given

[15] See equation (11).

an "up" realization for the perturbation function, is

$$P_{t+1,m} = \frac{P_{t,m+1}}{P_{t1}} h_{tm}^{u}.$$

This follows from the definition of h_{tm}^{u}. Dropping the time subscript for convenience, the values in each of the three possible next-period states are

$$V_1^u = (P_{m+1}h_m^u + \omega_1 P_{m_1+1}h_{m_1}^u + \omega_2 P_{m_2+1}h_{m_2}^u)/P_1,$$

$$V_1^d = (P_{m+1}h_m^d + \omega_1 P_{m_1+1}h_{m_1}^d + \omega_2 P_{m_2+1}h_{m_2}^d)/P_1,$$

$$V_1^n = (P_{m+1}h_m^n + \omega_1 P_{m_1+1}h_{m_1}^n + \omega_2 P_{m_2+1}h_{m_2}^n)/P_1,$$

If the portfolio is to earn the riskless rate in each of the three outcomes, it follows that $V_1^u P_1 = V_1^d P_1 = V_1^n P_1$. Hence,

$$P_{m+1}h_m^n + \omega_1 P_{m_1+1}h_{m_1}^n + \omega_2 P_{m_2+1}h_{m_2}^n = P_{m+1}h_m^u + \omega_1 P_{m_1+1}h_{m_1}^u + \omega_2 P_{m_2+1}h_{m_2}^u$$

and

$$P_{m+1}h_m^n + \omega_1 P_{m_1+1}h_{m_1}^n + \omega_2 P_{m_{2e}+1}h_{m_2}^n = P_{m+1}h_m^d + \omega_1 P_{m_1+1}h_{m_1}^d + \omega_2 P_{m_2+1}h_{m_2}^d.$$

Rearranging the terms and defining $dh_x^u \equiv h_x^u - h_x^n$ and $dh_x^d \equiv h_x^d - h_x^n$, we have

$$\omega_1 P_{m_1+1}dh_{m_1}^u + \omega_2 P_{m_2+1}dh_{m_2}^u = -P_{m+1}dh_m^u,$$

$$\omega_1 P_{m_1+1}dh_{m_1}^d + \omega_2 P_{m_2+1}dh_{m_2}^d = -P_{m+1}dh_m^d. \tag{18}$$

Define

$$A' \equiv dh_m^d dh_{m_2}^u - dh_m^u dh_{m_2}^d \quad \text{and} \quad A \equiv P_{m+1}P_{m_2+1}A',$$

$$B' \equiv dh_m^u dh_{m_1}^d - dh_m^d dh_{m_1}^u \quad \text{and} \quad B \equiv P_{m+1}P_{m_1+1}B',$$

$$C' \equiv dh_{m_1}^u dh_{m_2}^d - dh_{m_1}^d dh_{m_2}^u \quad \text{and} \quad C \equiv P_{m_1+1}P_{m_2+1}C'.$$

Solving the system of equations (18) yields $\omega_1^* = A/C$ and $\omega_2^* = B/C$ for $C \neq 0$. An important condition for the no-arbitrage argument to proceed is that ω_1 and ω_2 should be finite in order to ensure the feasibility of the reference portfolio. As ω_1^*, ω_2^* is the solution to a system of two linear equations, this condition needs to be verified.[16] The denominator of the solution ω_1^*, ω_2^* is proportional to

$$(h_{m_1}^n - h_{m_1}^u)(h_{m_2}^n - h_{m_2}^d) - (h_{m_2}^n - h_{m_2}^u)(h_{m_1}^n - h_{m_1}^d). \tag{19}$$

The above expression must differ from zero in order that the weights ω_1^*, ω_2^* be finite. Appendix B provides a set of economically reasonable assumptions which are sufficient to guarantee this result.

Because the portfolio with ω_1^* and ω_2^* so defined must earn the riskless rate, it follows that

$$V_0 = V_1^u P_1 = V_1^d P_1 = V_1^n P_1.$$

[16] Ho and Lee do not explicitly verify the finiteness of ω in their binomial model. Nevertheless, a sufficient condition for ω to be finite in their model is that $h_m^u \neq h_m^d \ \forall m$; this condition is, in fact, satisfied by their model.

In particular,

$$P_{m+1} + \omega_1 P_{m_1+1} + \omega_2 P_{m_2+1} = P_{m+1} h_m^u + \omega_1 P_{m_1+1} h_{m_1}^u + \omega_2 P_{m_2+1} h_{m_2}^u.$$

Expanding and rearranging the latter equation yields

$$h_m^u X + h_m^n Y + h_m^d Z = X + Y + Z,$$

where

$$X \equiv h_{m_2}^d h_{m_1}^n - h_{m_2}^n h_{m_1}^d + h_{m_1}^d - h_{m_1}^n - h_{m_2}^d + h_{m_2}^n,$$

$$Y \equiv h_{m_1}^d h_{m_2}^u - h_{m_2}^d h_{m_1}^u + h_{m_1}^u - h_{m_1}^d + h_{m_2}^d - h_{m_2}^u,$$

$$Z \equiv h_{m_2}^n h_{m_1}^u - h_{m_1}^n h_{m_2}^u - h_{m_1}^u + h_{m_1}^n + h_{m_2}^u - h_{m_2}^n.$$

Dividing through by $X + Y + Z$ yields

$$h_m^u \pi^u + h_m^n \pi^n + h_m^d \pi^d = 1,$$

where

$$\pi^u \equiv \frac{X}{X + Y + Z},$$

$$\pi^n \equiv \frac{Y}{X + Y + Z},$$

$$\pi^d \equiv \frac{Z}{X + Y + Z}. \tag{20}$$

By construction, $\pi^u + \pi^n + \pi^d = 1$. Note that the π's are defined in terms of $\{h^u, h^d, h^n\}$, which are known at time $t = 0$. The π's, which are not free, are used as a notational convenience in defining the constraints.

Appendix B

Sufficient Conditions for the Trinomial Model

In Appendix A it was shown that, for the no-arbitrage condition to hold in the trinomial model, equation (20) must obtain. This appendix develops an economically appealing set of sufficiency conditions on the h_{tm}^s which provide for finite values of the weights ω_1 and ω_2 of Appendix A.

B.1. Path Independence

In the context of the Ho and Lee constant-parameter binomial model, path independence implies that an "up"-"down" perturbation sequence results in a term structure identical to that resulting from a "down"-"up" sequence. The trinomial analogue would be that the following sequences are identical:

$$\{u_{t-1}, n_t\} \equiv \{n_{t-1}, u_t\},$$

$$\{n_{t-1}, n_t\} \equiv \{u_{t-1}, d_t\} \equiv \{d_{t-1}, u_t\},$$

$$\{d_{t-1}, n_t\} \equiv \{n_{t-1}, d_t\},$$

where "u" denotes "up," "n" denotes "no change," and "d" denotes "down."

The assumption of path independence is intuitive in a model wherein all parameters are constant. In contrast, the current model seeks to permit state-dependent nonstationarity in intertemporal term structure movements. We may define *state-dependent* path independence—which we denote "quasi"-path independence—as path independence which obtains only when the set of state variables $\mathbf{x}$ is constant across two time periods.

Taking the two sequences "up"-"no change" and "no change"-"up" as an example, assume that path independence holds. Then, for the first sequence,

$$P_{t+1,1} = \frac{P_{t,2}}{P_{t,1}} h_1^u,$$

$$P_{t+1,m+1} = \frac{P_{t,m+2}}{P_{t,1}} h_{m+1}^u,$$

and

$$P_{t+2,m} = \frac{P_{t+1,m+1}}{P_{t+1,1}} h_m^n$$

$$= \frac{P_{t,m+2}}{P_{t,2}} \frac{h_{m+1}^u h_m^n}{h_1^u}.$$

If the "no change"-"up" sequence occurs, then

$$P_{t+2,m} = \frac{P_{t,m+2}}{P_{t,2}} \frac{h_{m+1}^n h_m^u}{h_1^n}.$$

Notice that time subscripts t are omitted from the perturbations under the assumption that $\mathbf{x}_t = \mathbf{x}_{t+1}$. Since, by assumption, $P_{t+2,m}$ must have the same value regardless of which of these two paths was taken, it follows that

$$\frac{h_{m+1}^n h_m^u}{h_1^n} = \frac{h_{m+1}^u h_m^n}{h_1^u}. \tag{21}$$

By similar arguments we may show that path independence implies that

$$\frac{h_{m+1}^n h_m^n}{h_1^n} = \frac{h_{m+1}^u h_m^d}{h_1^u}, \tag{22}$$

$$\frac{h_{m+1}^n h_m^d}{h_1^n} = \frac{h_{m+1}^d h_m^n}{h_1^d}. \tag{23}$$

Dividing equation (21) by equation (22) (or equivalently equation (22) by equation (23)) yields

$$\frac{h_m^n}{h_m^d} = \frac{h_m^u}{h_m^n} \Leftrightarrow h_m^n = \sqrt{h_m^u h_m^d}. \tag{24}$$

B.2. Finiteness of Portfolio Weights

Sufficient conditions for obtaining finite sets of α_1, α_2 in the no-arbitrage portfolios are

1) state-dependent "quasi"-path independence:

$$h_m^n = \sqrt{h_m^u h_m^d} \quad \forall m;$$

2) monotonicity of h_m^u and h_m^d in m:

$$m_1 < m_2 \Rightarrow h_{m_1}^u < h_{m_2}^u \quad \text{and} \quad h_{m_1}^d > h_{m_2}^d;$$

3) ordering of perturbations:

$$h_m^u > h_m^n > h_m^d \quad \forall m.$$

Substituting (24) into (19) yields

$$\begin{aligned}(\sqrt{h_{m_1}^u h^{dm_1}} - h_{m_1}^u)(\sqrt{h_{m_2}^u h^{dm_2}} - h^{dm_2}) - (\sqrt{h_{m_2}^u h^{dm_2}} - h_{m_2}^u)(\sqrt{h_{m_1}^u h^{dm_1}} - h^{dm_1}) \\ = (\sqrt{h_{m_1}^u} - \sqrt{h^{dm_1}})(\sqrt{h_{m_2}^u} - \sqrt{h^{dm_2}})(\sqrt{h_{m_2}^u h^{dm_1}} - \sqrt{h_{m_1}^u h^{dm_2}}).\end{aligned}$$

The first two expressions in parentheses are positive by virtue of assumption 3. The last expression is nonzero by virtue of assumption 2, the monotonicity of h_m^u and h^{dm} in m:[17]

$$\sqrt{h_{m_2}^u h^{dm_1}} - \sqrt{h_{m_1}^u h^{dm_2}} = \sqrt{h_{m_1}^u h^{dm_1}}\left(\frac{\sqrt{h_{m_2}^u}}{\sqrt{h_{m_1}^u}} - \frac{\sqrt{h^{dm_2}}}{\sqrt{h^{dm_1}}}\right) > 0,$$

since

$$\frac{h_{m_2}^u}{h_{m_1}^u} > 1 > \frac{h^{dm_2}}{h^{dm_1}}.$$

[17] It is instructive to note that equation (25) cannot be signed without (some) additional assumptions, e.g., the three conditions of Section *B.2.* Absent additional assumptions, numerical examples can be constructed to yield positive, negative, and zero values for (25).

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