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Pricing Contingent Claims under Interest Rate and Asset Price Risk

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ABSTRACT

This paper presents a general framework for pricing contingent claims under interest rate and asset price uncertainty. The framework extends Ho and Lee's (1986) valuation framework by allowing not only future interest rates but also future asset prices to depend on the current term structure of interest rates. The approach is shown to provide risk-neutral valuation relationships that are consistent with the initial term structure of interest rates and can be applied to valuation of a broad class of assets including stock options, convertible bonds, and junk bonds.

THIS PAPER PRESENTS A general framework for pricing contingent claims under interest rate and asset price uncertainty. Many examples of such claims exist. The price of an American stock option, for instance, depends not only on the underlying stock price but also on the interest rates, the level and pattern of which may induce early exercise. The dependence of contingent claim prices on interest rate risk is the most obvious in the case of interest rate options. In this case, option value is contingent solely on the market's expectation of future interest rates.

Theories of the term structure of interest rates suggest that the shape of the current term structure contains information about the market's expectation of future interest rates. Since interest rate contingent claims depend on interest rate expectations, using current term structure information to improve the precision of interest rate option pricing is crucial. The first authors to implement this important insight were Ho and Lee (1986) (H-L), who model the stochastic process for interest rates relative to the initial term structure and derive a valuation relationship for interest rate contingent claims.

This paper extends H-L's insight to assets whose risk is not solely determined by interest rate movements. Cox, Ingersoll, and Ross (1985) show that the equilibrium return of a risky asset over an infinitesimal period of time equals the prevailing risk-free rate plus risk premium. Given that all risky assets, including debt securities, are priced in a single market, the anticipation of risky asset price movements must be formed simultaneously with that of interest rate movements. This implies that the stochastic processes for risky assets must be in equilibrium with that for the risk-free rate, which, in turn, depends on the initial term

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structure. Specifically, this paper generalizes the H-L methodology by incorporating the market expectation of future interest rates embedded in the initial term structure into the stochastic processes for *both* interest rate and asset price movements.

Building the term structure into both stochastic processes has two important implications for pricing contingent claims. First, the valuation models based on this approach are consistent with the initial term structure. That is, if they are applied to the pure discount bonds, they provide the values which are consistent with the initial term structure. Since many contingent claims on interest rates and risky assets can be viewed as portfolios of pure discount bonds and embedded options, the consistency ensures that the pure discount bond components of the contingent claims are priced according to the initial term structure. Second, since the initial term structure contains information about the future movements of interest rates and asset prices, it follows from the consistency that this approach evaluates the embedded options of the contingent claims based on the market expectation of these uncertainties as implied by the initial term structure.[1]

Many studies address the valuation of contingent claims under interest rate and asset price uncertainty. For instance, Merton (1973) accounts for both interest rate and asset price risk in valuing European call options. Brennan and Schwartz (1980) extend their original valuation model for convertible bonds (Brennan and Schwartz (1977)) to allow interest rates to be stochastic. Grabbe (1983) and Cheng (1985), among others, apply Merton's approach to other types of contingent claims in the stochastic interest rate environment. Stapleton and Subrahmanyam (1984) develop a utility-based model to price multivariate contingent claims under interest rate uncertainty. Cox, Ingersoll, and Ross (1985) develop an intertemporal general equilibrium model which endogenizes interest rates. The approach used here differs from these models in that it prices contingent claims relative to the initial term structure of interest rates.

Methodologically, this study builds on the work of Cox, Ross, and Rubinstein (1979) (CRR) in modeling asset price risk and H-L in modeling interest rate risk. Combining the CRR and H-L models, however, presents a problem due to the multiplicity of uncertainties. To see this, consider an option on two risky assets under constant interest rates. If both risky asset prices follow binomial processes, the option value has four possible values at the end of each period. To obtain a valuation relationship based on a riskless hedge argument, therefore, requires four linearly independent assets other than the option whose values are specified by the two risky assets. However, only three assets meet this requirement—the two risky assets and the risk-free asset. The absence of a fourth asset implies that a hedge portfolio cannot be formed.[2]

A "sequential process" is proposed to avoid this problem. In a sequential process, the two uncertainties resolve only one at a time.[3] More specifically, each

[1] Ramaswamy and Sundaresan (1985) point out that the shape of the current term structure may have an important impact on futures option prices.

[2] This problem does not arise in continuous-time models. With diffusion processes, the stochastic change in option value is linearly dependent on the stochastic changes in the underlying state variables over infinitesimal periods of time.

[3] Subsequent to the development of the sequential process, it was discovered that Evnine (1983) proposed a similar, but not identical, process to price options on more than one stock.

period is divided into two subperiods. In the first subperiod, one source of uncertainty resolves, and then, in the second subperiod, the other source resolves. In such an environment, a hedge portfolio can be formed against each risk to yield a risk-neutral valuation relationship (RNVR). A model of options on two risky assets based on the sequential process is presented in Appendix A of Kishimoto (1987). It is shown that, with an appropriate definition of jump sizes, the sequential process converges to a two-dimensional Brownian motion, and the discrete-time RNVR converges to a continuous-time RNVR. Thus, the assumption about the resolution of uncertainty is not as restrictive as it seems.[4]

This paper is organized as follows. Section I models a stochastic process for the term structure and the risky asset price such that it reflects the initial term structure of interest rates. Section II develops a valuation model for contingent claims on the two uncertainties and discusses an estimation procedure for the model parameters. Section III concludes the paper with a summary.

I. The Stochastic Process for Term Structure and Asset Price

A. *Term Structure Movements*

First we list the basic assumptions of the model.

(A1) The time to expiration of the contingent claim is partitioned into N periods of equal length, and each period is further divided into two subperiods. The second subperiod is "short" relative to the first subperiod.[5] Trading takes place at the beginning of every subperiod.

The end of the nth period and the end of the first subperiod of the nth period are called time n and time $n - \frac{1}{2}$, respectively. Also, the initial point in time is referred to as time 0.

(A2) The market is frictionless. There are no taxes, no transaction costs, and no restrictions on short sales. All securities are perfectly divisible.

Following H-L, interest rate uncertainty is represented by the movements in the term structure of interest rates. Let a pure discount bond with time to maturity T be a security which pays \$1 T periods later, with no cash flow at other times. Also, let the discount function express the equilibrium price of a pure discount bond as a function of its time to maturity.

(A3) The bond market is complete in the sense that there exists a pure discount bond for all time to maturity T, $T = 0, \frac{1}{2}, 1, 1 + \frac{1}{2}, 2, \cdots, N$.

[4] However, the discrete time aspect of the model should not be overemphasized. In fact, as shown by Heath, Jarrow, and Morton (1987), the new methodology proposed by H-L can be implemented by a continuous-time framework.

[5] The assumption of the second subperiod being "short" relative to the first is made to have a zero interest rate during the second subperiod. This simplifies the exposition of the model without changing its major results. See Appendix B of Kishimoto (1987) for a model which allows both subperiods to have equal length and bonds to earn a nonzero deterministic interest rate during the second subperiod.

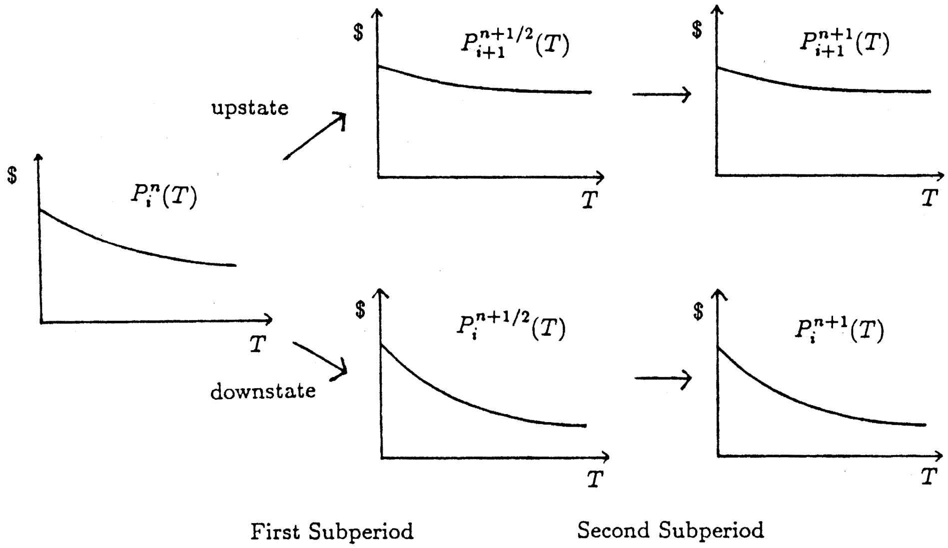

Figure 1. Movements of the discount function.

Therefore, the discount function is observable at time 0.[6] We assume the following movements of the discount function in the ensuing periods.

(A4) During the first subperiod, the interest rate uncertainty resolves, and, at the end of the first subperiod, the discount function takes one of two possible shapes. When the discount function "shifts up," we say that an "upstate of the term structure" is attained. When the discount function "shifts down," a "downstate of the term structure" is attained. During the second subperiod, the discount function remains unchanged.

The movements of the discount function for a generic period are depicted in Figure 1.[7] In addition, the following assumption is made about the term structure movements.

(A5) The shape of the discount function is completely determined by time and the number of upstates of the term structure.

Thus, if i upstates occur in the first n periods, the discount function is well defined for the pair (n, i), which is referred to as time-state (n, i), hereafter. The price at time-state (n, i) of a pure discount bond with T periods to maturity is denoted by $P_i^n(T)$. In case of time 0, however, time-state is suppressed

$$P(T) \equiv P_0^0(T).$$

[6] There are a number of techniques to estimate the discount function based on the market prices of bonds. See McCulloch (1975) and Litzenberger and Rolfo (1984).

[7] The assumption that the shape of the discount function is unchanged during the second subperiod corresponds to a zero interest rate during the second subperiod. Also, note that this assumption implies that the risk-free rate for the first subperiod is given by the one-period interest rate.

Also, time to maturity is suppressed when it is one.

$$P_i^n \equiv P_i^n(1).$$

The term structure prevailing at time-state (n, i) implies the forward prices of pure discount bonds deliverable at time $n + 1$. Thus, the discount function at time n is expressed as the product of the forward price $P_i^n(T + 1)/P_i^n$ and a deviation from it.

$$P_{i+1}^{n+1}(T) = \frac{P_i^n(T+1)}{P_i^n} h(T) \quad \text{for upstate;} \tag{1a}$$

$$P_i^{n+1}(T) = \frac{P_i^n(T+1)}{P_i^n} h^*(T) \quad \text{for downstate.} \tag{1b}$$

The deviations $h(T)$ and $h^*(T)$ are assumed to be functions of time to maturity but not time-state.[8]

B. Asset Price Movements

One of the key elements of this model is the way in which the asset price moves with the discount function. The asset price movements are decomposed into two components. The first component captures the co-movements of the asset price with the discount function during the first subperiod. It is referred to as the "interest-dependent component." The "asset-specific component" is residual noise and follows a binomial process during the second subperiod. Specifically:

(A6-1) During the first subperiod from time n to time $n + ½$, if an upstate of the term structure occurs, the asset price changes by a multiplicative factor of u_i^n/P_i^n, and, if a downstate of the term structure occurs, the asset price changes by a multiplicative factor of d_i^n/P_i^n.

These multiplicative factors are expressed relative to the one-period interest rate—the risk-free rate. Thus, u_i^n and d_i^n represent the returns the risky asset earns in excess of the then-prevailing risk-free rate in upstate and downstate of the term structure, respectively. Note that these multiplicative factors are intended to capture the interest-dependent component of the asset price. Thus, it is assumed that u_i^n and d_i^n are dependent upon the discount function prevailing at the beginning of the period and are functions of time-state (n, i).

During the second subperiod, the asset price either increases or decreases. We refer to one of the states as an "upstate of the asset price" and the other as a "downstate of the asset price."

(A6-2) During the second subperiod, if an upstate of the asset price occurs, the asset price changes by a multiplicative factor of $\bar{u}$ and, if a downstate of the asset price occurs, the asset price changes by a multiplicative factor of $\bar{d}$.

[8] It is possible to define $h(T)$ and $h^*(T)$ to be functions of time to maturity and time-state as shown in Ho and Lee (1986).

The factors $\bar{u}$ and $\bar{d}$ represent the asset-specific component of the asset price. Thus, the sizes of $\bar{u}$ and $\bar{d}$ are assumed to be independent of time-state (n, i).

The assumption that the discount function takes one of two possible shapes implies that all default-free bonds take on one of two possible prices at the end of the first subperiod depending on the shift of the discount function. Also note that the return on the risky asset for the first subperiod is determined by the shift of the discount function. Thus, we must ensure that there is no arbitrage opportunity among the risky asset and default-free bonds. H-L shows that no arbitrage condition among default-free bonds implies that there exists a constant π, $0 \leq \pi \leq 1$, such that

$$\pi h(T) + (1 - \pi)h^*(T) = 1, \tag{2}$$

for all T, where π is the risk-neutral probability of the upstate of the term structure. In addition, no-arbitrage condition must be ensured between the risky asset and default-free bonds, which implies the following restriction on u_i^n and d_i^n.

PROPOSITION 1: *u_i^n and d_i^n must be such that*

$$\pi u_i^n + (1 - \pi)d_i^n = 1, \tag{3}$$

for all time-states (n, i), where π is the risk-neutral probability of the upstate of the term structure.

Proof: See the Appendix.

This proposition implies that the process for the excess returns on the risky asset is martingale with respect to the risk-neutral probabilities.

Next, the multiplicative factors are examined in relation to the moments of the risky asset return distribution. Let p_1 and p_2 denote the true probabilities of the upstates of the term structure and asset price, respectively. Then the variance σ^2 of returns on the risky asset from time n to time $n + 1$ is given by

$$\sigma^2 = \sigma_1^2 + \sigma_2^2, \tag{4}$$

where

$$\sigma_1^2 = p_1(1 - p_1)\left\{\ln\left[\frac{u_i^n}{d_i^n}\right]\right\}^2,$$

$$\sigma_2^2 = p_2(1 - p_2)\left\{\ln\left[\frac{\bar{u}}{\bar{d}}\right]\right\}^2.$$

The correlation coefficient ρ between the return on the risky asset and the risk-free rate is given by

$$\rho = \frac{\pm\sigma_1}{\sigma}, \tag{5}$$

which takes on the positive (negative) sign when the risky asset is positively (negatively) correlated with the risk-free rate. Proposition 1 implies that there is a one-to-one relationship between u_i^n and d_i^n. Thus, for given values of p_1, p_2, $\bar{u}$, and $\bar{d}$, u_i^n determines the variance of the risky asset and its correlation with the risk-free rate. For example, if u_i^n is unity, the volatility of the risky asset stems from only the residual risk, and its correlation with interest rates is zero. If u_i^n is greater than unity, then d_i^n is smaller than unity, and the risky asset earns more in the upstate than in the downstate of the term structure. Since the interest rates decrease (increase) in the upstate (downstate) of the term structure, the risky asset is locally negatively correlated with interest rates. Furthermore, an increase in u_i^n results in a decrease in d_i^n, which induces a higher variance and stronger negative correlation with interest rates. By contrast, if u_i^n is smaller than unity, d_i^n is greater than unity and the interest-dependent component earns less in the upstate than in the downstate of the term structure. Thus, the risky asset is locally positively correlated with interest rates, and a smaller value of u_i^n yields a higher variance and higher correlation with interest rates. In short, u_i^n determines the interest rate sensitivity of the risky asset. Also, it is clear from the above equations that $\bar{u}$ and $\bar{d}$ partially determine the variance of the risky asset and its correlation with interest rates.

An additional assumption is made about the price of the risky asset.

(A7) The price of the risky asset at time n is completely determined by time, the number of upstates of the term structure, and the number of upstates of the asset price which have occurred through time n.

In this model, the numbers of upstates of the term structure and the asset price represent the underlying uncertainties and are analogous to standard Brownian motions in continuous-time models. Thus, (A7) specifies the price of the risky asset as a function of time and the current values of the underlying uncertainties as the stochastic differential equation for an asset price does in continuous-time models. It follows from assumptions (A5) and (A7) that the triplet (n, i, j) specifies uniquely the discount function and the risky asset price. This triplet is referred to as time-state (n, i, j) hereafter.

Since $\bar{u}$ and $\bar{d}$ are constant through time, restriction (A7) is always satisfied with respect to the asset-specific component. Thus, (A7) reduces to a condition on the interest-dependent component. Let S denote the asset price at time-state (n, i, j). If an upstate of the term structure is followed by a downstate of the term structure in the ensuing two periods, the interest-dependent component at time $n+2$ is

$$\frac{u_i^n d_{i+1}^{n+1}}{P_i^n P_{i+1}^{n+1}} S.$$

If a downstate is followed by an upstate, the interest-dependent component at time $n+2$ is

$$\frac{d_i^n u_i^{n+1}}{P_i^n P_i^{n+1}} S.$$

Assumption (A7) requires that the above two expressions are equal, which yields

$$\frac{u_i^n d_{i+1}^{n+1}}{P_{i+1}^{n+1}} = \frac{d_i^n u_i^{n+1}}{P_i^{n+1}}.$$

Substituting equation (3) into the above equation yields

$$u_i^{n+1}(1 - \pi u_i^n)P_{i+1}^{n+1} = u_i^n(1 - \pi u_{i+1}^{n+1})P_i^{n+1}.$$

Given the H-L model, this equation further reduces to

$$u_i^{n+1}(1 - \pi u_i^n) = u_i^n(1 - \pi u_{i+1}^{n+1})\delta, \tag{6}$$

where δ is a parameter of H-L's model and determines the spread between $h(T)$ and $h^*(T)$. This is the necessary and sufficient condition for restriction (A7) and is the basis for the following result.

Proposition 2: *Let U be a set of u_i^{N-1}, $i = 0, 1, 2, \cdots, N - 1$. Then a set of parameters π, δ, $\bar{u}$, $\bar{d}$, and U defines a unique stochastic process for both the term structure and the asset price.*

Proof: See the Appendix.

C. Special Case—State-Independent Model

A special case of the general model is obtained by imposing the following constraint.

(A8) $u_i^{N-1} = \gamma$ for $i = 0, 1, 2, \cdots, N - 1$, where $N - 1$ represents the beginning of the last period and γ is a constant number.

In other words, the interest rate sensitivity of the risky asset is constant across all states of the term structure in the last period.[9] This assumption has two important implications. First, the interest rate sensitivity of the risky asset is uncorrelated with the state of the term structure in all periods. Second, the stochastic process for both the term structure and the risky asset is characterized completely by a set of five parameters: π, δ, $\bar{u}$, $\bar{d}$, and γ.

These implications can be seen by substituting assumption (A8) into equation (6):

$$u_i^{N-2} = \frac{\gamma}{\delta + \pi\gamma - \pi\delta\gamma}. \tag{7}$$

The right-hand side of equation (7) does not depend on i. Thus, u_i^{N-2} is constant across all states. We can repeat this procedure to determine u's at time n, $n = N - 3, N - 4, \cdots, 0$ and verify that u's are constant across all states in the same period. Solving equation (3) for d_i^n yields

$$d_i^n = \frac{1 - \pi u_i^n}{1 - \pi}. \tag{8}$$

Thus, d's in the same period are constant across all states. In short, assumption (A8) implies that the interest rate sensitivity of the risky asset is uncorrelated with interest rates and that all the multiplicative factors are specified by five

[9] This assumption is made to examine the model in its simplest form and can be relaxed to allow for negative or positive correlation of the interest rate sensitivity of the risky asset with the state of the term structure.

parameters: π, δ, $\bar{u}$, $\bar{d}$, and γ. We refer to this special case of the general model as the state-independent model. It can be shown that u_i^n is greater (smaller) than untiy for all n, if γ is greater (smaller) than unity.[10] Thus, $\gamma > 1$ ($\gamma < 1$) implies negative (positive) correlation of the risky asset with interest rates through time N. Also, the further γ deviates from unity, the more sensitive the asset is to the shifts of the discount function. The general expression for the asset price is given below.

PROPOSITION 3: *Let S and $S_{i,j}^n$ be the prices of the risky asset at time 0 and time-state (n, i, j), respectively. Then,*

$$S_{i,j}^n = \frac{F_1(n, i; \pi, \delta, \gamma) F_2(n, j; \bar{u}, \bar{d})}{P(n)} S, \tag{9}$$

where

$$F_1(n, i; \pi, \delta, \gamma) = \frac{\prod_{k=0}^{i-1} u^{(k)} \prod_{m=i}^{n-1} \dfrac{1 - \pi u^{(m)}}{1 - \pi} \prod_{h=0}^{n-1} \pi + (1 - \pi)\delta^h}{\delta^{(n-i)(n-i-1)/2}},$$

$$F_2(n, j; \bar{u}, \bar{d}) = \bar{u}^j \bar{d}^{n-j},$$

$$u^{(n-1)} = \frac{u^{(n)}}{\delta + \pi u^{(n)} - \pi\delta u^{(n)}}, \qquad n = N-1, N-2, \cdots, 1, \tag{10}$$

with a boundary condition $u^{(N-1)} = \gamma$.

Proof: See the Appendix.

$F_1(n, i; \pi, \delta, \gamma)/P(n)$ represents the returns earned on the interest-dependent component through time n, while $F_2(n, j; \bar{u}, \bar{d})$ represents the returns earned on the asset-specific component for the same period. $F_1(n, i; \pi, \delta, \gamma)$ and $F_2(n, j; \bar{u}, \bar{d})$ depend only on time-state (n, i, j) and the model parameters π, δ, $\bar{u}$, $\bar{d}$, γ.

Next, we examine the effects of the initial term structure on the stochastic process for the risky asset. Suppose that an upstate of the term structure and an upstate of the asset price occur from time-state (n, i, j) to $(n+1, i+1, j+1)$. Then, using the expression for P_i^n in H-L, the return earned on the risky asset during this period is given by

$$\frac{u_i^n}{P_i^n} \bar{u} = u_i^n \frac{P(n)}{P(n+1)} G(n, i; \pi, \delta)\bar{u}, \tag{11}$$

where

$$G(n, i; \pi, \delta) = \frac{\pi + (1-\pi)\delta^n}{\delta^{n-i}}.$$

$G(n, i; \pi, \delta)$ depends on n, i, π, and δ, while u_i^n is a function of n, i, π, δ, and γ in case of the state-independent model. $P(n)/P(n+1)$ represents one plus one-

[10] For this property to hold, the value of γ must be such that either of the following two conditions hold: (a) $1 < \gamma < 1/\pi$ and $u^{(0)} < 1/\pi$ or (b) $0 < \gamma < 1$ and $u^{(0)} < 1$, where $u^{(0)}$ is defined by the equation (10).

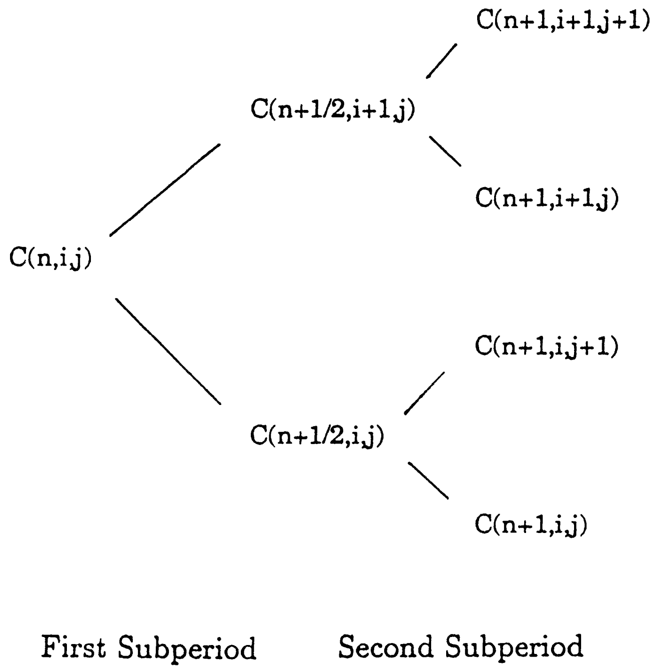

Figure 2. Movements of the price of the contingent claim.

period forward rate implied by the initial term structure. Therefore, the one-period return on the risky asset is composed of two parts. The first depends on the initial term structure, and the other depends on time-state and model parameters. Similarly, it can be shown that the n-period return on the risky asset is composed of two parts—one dependent on the n-period forward rate implied by the initial term structure and the other on time-state and model parameters. It is in this sense that the stochastic process for the risky asset price reflects the initial term structure of interest rates.[11]

II. Valuation of Contingent Claims

A. One-Period Valuation Relationship

This section considers the valuation of contingent claims that depend only on interest rates and an asset price. Let $C(n, i, j)$ be the price of a contingent claim at time-state (n, i, j). Then the price of the contingent claim would follow the paths depicted in Figure 2.

First we examine the valuation problem for the second subperiod. During this subperiod, the uncertainty stems only from the risky asset. Thus, the valuation problem reduces to that of CRR, and a hedge argument is used to derive the RNVR. Consider a portfolio of the risky asset and the risk-free asset that replicates the value of the contingent claim at the end of the period. Here any

[11] This argument holds true for the general model of asset price movements in Subsection I.*B*.

bonds can serve as the risk-free asset, for their values do not change during the second subperiod. No-arbitrage condition between the replicating portfolio and the contingent claim yields

$$C(n+\tfrac{1}{2}, i+1, j) = qC(n+1, i+1, j+1) + (1-q)C(n+1, i+1, j), \tag{12a}$$

$$C(n+\tfrac{1}{2}, i, j) = qC(n+1, i, j+1) + (1-q)C(n+1, i, j), \tag{12b}$$

where

$$q = \frac{1-\bar{d}}{\bar{u}-\bar{d}}.$$

The definition of q is identical to the risk-neutral probability of CRR except that its numerator is $(1 - \bar{d})$, not $(r - \bar{d})$ of CRR. This is because r in CRR's expression is one plus the risk-free rate, which reduces to unity when the risk-free rate is zero. Thus, q is the risk-neutral probability of CRR, which would be the probability of the upstate of the term structure if all investors were risk neutral.

During the first subperiod, the uncertainty arises only from a shift of the discount function. Thus, a hedge portfolio is formed by the risk-free asset and either a bond or the risky asset. The no-arbitrage condition between the replicating portfolio and the contingent claim implies

$$C(n, i, j) = P_i^n[\pi(C(n + \tfrac{1}{2}, i + 1, j) + (1 - \pi)C(n + \tfrac{1}{2}, i, j)], \tag{13}$$

where π is the risk-neutral probability of the upstate of the term structure. Equations (12) and (13) are combined to obtain the following relationship:

$$\begin{aligned} C(n, i, j) = P_i^n[&\pi qC(n + 1, i + 1, j + 1) + \pi(1 - q)C(n + 1, i + 1, j) \\ &+ (1 - \pi)qC(n + 1, i, j + 1) + (1 - \pi)(1 - q)C(n + 1, i, j)]. \end{aligned} \tag{14}$$

πq, $\pi(1 - q)$, $(1 - \pi)q$, and $(1 - \pi)(1 - q)$ are the risk-neutral probabilities of the four events at the end of the period. Thus, equation (14) represents the expected value of the contingent claim at time $n + 1$, discounted by the one-period interest rate prevailing at time n. Needless to say, the expectation is computed with respect to the risk-neutral probabilities. Thus, an amazingly simple RNVR is obtained for a generic period.[12]

B. Pricing of Contingent Claims

Next we price contingent claims by applying the one-period RNVR recursively. Assume that the contingent claim expires at time N with a payoff function $\phi(i, j)$, $i, j = 0, 1, 2, \cdots, N$. Since the price of the contingent claim at time N must be identical to the terminal payoff, we have

$$C(N, i, j) = \phi(i, j), \qquad i, j = 0, 1, 2, \cdots, N. \tag{15}$$

[12] The covariance between the two uncertainties does not appear in the RNVR. This is because in this model the covariance is fully captured by the interest-dependent component and asset-specific component is independent of the events in the first subperiod.

The contingent claim may have upper and lower bounds to its price. Let $U(n, i, j)$ and $L(n, i, j)$ be the upper and lower bounds for time-state (n, i, j), respectively. Then we have

$$L(n, i, j) \leq C(n, i, j) \leq U(n, i, j) \quad \text{for all } (n, i, j). \tag{16}$$

Given this specification of the contingent claim, we can proceed as follows to determine its initial price. First, the payoff function $\phi(i, j)$ determines the price of the contingent claim at time N in all states. Then the one-period RNVR is used to derive the arbitrage-free price of the contingent claim at time $N - 1$. Let such a price be denoted by $C^*(N - 1, i, j)$. The price $C(N - 1, i, j)$ of the contingent claim at time-state $(N - 1, i, j)$ is determined by comparing it with equation (16):

$$C(N - 1, i, j) = \max[L(N - 1, i, j), \min(C^*(N - 1, i, j), U(N - 1, i, j))].$$

Once the prices of the contingent claim are given for all states at time $N - 1$, we can determine the prices for all states at time $N - 2$, which, in turn, determine the prices at all states at time $N - 3$. We continue this procedure recursively up to time 0 to obtain the initial price of the contingent claim.

Alternatively, valuation formulas may be derived based on the following result.

PROPOSITION 4: *The price $A(n, i, j)$ of the Arrow-Debreu security which pays* \$1 *at time n if and only if state (i, j) occurs at time n is given by*

$$A(n, i, j) = P(N)H_1(n, i; \pi, \delta)H_2(n, i, j; \pi, q). \tag{17}$$

where

$$H_1(n, i; \pi, \delta) = \frac{g(n - i, i)}{\binom{n}{i} \prod_{m=1}^{n-1} \pi + (1 - \pi)\delta^m},$$

$$H_2(n, i, j; \pi, q) = \binom{n}{i}\binom{n}{j} \pi^i(1 - \pi)^{n-i}q^j(1 - q)^{n-j},$$

$$g(k, m) = \delta^k g(k, m - 1) + \delta^{k-1}g(k - 1, m),$$

with boundary conditions $g(k, 0) = \delta^{k(k-1)/2}$ and $g(0, m) = 1$ for all non-negative integers k and m.

Proof: See the Appendix.

$P(n)H_1(n, i; \pi, \delta)$ represents the "discount factor" for time n, conditional on the event that i upstates of the term structure occur in the first n periods. In fact, it can be shown that the reciprocal of $P(n)H_1(n, i; \pi, \delta)$ represents one plus the average cumulative return on investing in the risk-free asset from the initial period to time-state (n, i).[13] Thus, if i is small relative to n, then there are few upward and many downward shifts of the discount function, which implies that the risk-free rate tends to be high through time n and that $1/P(n)H_1(n, i; \pi, \delta)$

[13] See Chapter 4 of Kishimoto (1987) for the proof of this claim.

is greater than $1/P(n)$. $H_2(n, i, j; \pi, q)$ represents the probability that a bivariate binomial distribution takes the pair of values (i, j) in n trials, where π and q represent the probabilities of the "up" events for each of the two variables. The price C of a contingent claim which has no interim cash flow and no boundary condition except for the terminal condition can be determined by summing up all the products of the terminal payoffs and Arrow-Debreu security prices:

$$C = \sum_{i=0}^{N} \sum_{j=0}^{N} A(N, i, j)\phi(i, j). \tag{18}$$

The approach proposed in this paper predicts the relationship between the initial term structure and the prices of options on risky assets. For instance, equation (18) can be used to examine the effects of the initial term structure on the prices of European call options. Let the exercise price of a European call option be K. Then its terminal payoff is given by

$$\phi(i, j) = \max[S(N, i, j) - K, 0].$$

Thus, the price C of the option is:

$$\begin{aligned} C &= \textstyle\sum_{\theta} A(N,i,j)S(N,i,j) - K\sum_{\theta} A(N,i,j) \\ &= S\textstyle\sum_{\theta} F_1(N,i;\pi,\delta,\gamma)F_2(N,j;\bar{u},\bar{d})H_1(N,i;\pi,\delta)H_2(N,i,j;\pi,q) \\ &\quad - KP(N)\textstyle\sum_{\theta} H_1(N,i;\pi,\delta)H_2(N,i,j;\pi,q), \end{aligned} \tag{19}$$

where

$$\theta = \{(i, j) \mid S(n, i, j) \geq K\}.$$

The expressions summed over θ in the first and second terms do not depend explicitly on the initial term structure. However, the second term contains the N-period discount factor $P(N)$. Thus, the price of the European call option is explicitly dependent on only a single point of the term structure—the N-period discount factor.[14]

By contrast, the prices of American options on risky assets may depend explicitly upon more than one point of the term structure. To clarify this point, we consider an American put option with N periods to expiration in two hypothetical economies. The two economies have identical values for the model parameters but differ in the shape of the initial term structure. Economy 1 has an upward sloping yield curve up to maturity N, while economy 2 has a downward sloping yield curve up to time N. Also, the two economies have the identical value for the N-period discount factor. In this paper, as well as H-L, the stochastic process for interest rates is constructed relative to the initial term structure. Furthermore, as shown in Subsection I.C., the stochastic process for the risky asset is modeled relative to the initial term structure. Thus, at each time-state (n, i, j), both the risky asset price and the discount function differ between the two economies, which implies that the optimal exercise decisions differ between them, and so do the prices of the put option.

[14] This property holds for the general model for asset price movements, as long as the risky asset and the Arrow-Debreu security prices at time N depend only on the model parameters and the N-period discount factor.

C. Consistency with the Initial Term Structure

In general, every model for contingent claims on both interest rates and a risky asset can be used to price pure discount bonds. This is because pure discount bonds can be viewed as contingent claims on the *two* uncertainties whose payoffs at their maturities are \$1 in all states. The following proposition states that the models based on our approach price pure discount bonds relative to the initial term structure.[15]

PROPOSITION 5: *If the RNVR presented in this paper is applied to the pure discount bonds, it yields values identical to the ones implies by the initial discount function.*

Proof: See the Appendix.

This has important implications to contingent claims on both interest rates and a risky asset. First, note that many of them can be viewed as portfolios of pure discount bonds and embedded options. Thus, the above proposition implies that the models based on this approach price the pure discount bond components of contingent claims relative to the initial term structure. For example, consider a coupon bond which gives its holders an option to exchange it for a fixed number of shares of common stock at a fixed point in time. This option is a European call option on both interest rates and a risky asset. Thus, the value of the coupon bond can be decomposed into pure discount bonds and the call option, and our approach can be shown to price the pure discount bond component of the coupon bond relative to the initial term structure.

In addition, since the initial term structure contains information on the future movements of interest rates and risky assets, the consistency of the RNVR with the initial term structure implies that this RNVR prices contingent claims with the market expectations embedded in the initial term structure. For instance, for the above coupon bond, our approach prices not only the pure discount bond component, but also the embedded call option relative to the initial term structure.

D. Estimation of the Model Parameters

This subsection provides a procedure to estimate the model parameters π, δ, $\bar{u}$, $\bar{d}$, and γ.[16] It involves three steps. The first step is to estimate the discount function based on the market prices of default-free bonds. There are a number of methods for this, including the cubic spline method in McCulloch (1975) and Litzenberger and Rolfo (1984). The second step is to estimate the model parameters related to the term structure uncertainty—π and δ. First, a theoretical model for a subclass of interest rate contingent claims is developed based on H-

[15] This approach differs from previous models such as Brennan and Schwartz (1980) in the sense that we take the initial term structure as given while Brennan and Schwartz endogenize it. As a result, the consistency of the RNVR with the initial term structure holds exactly in our approach while only approximately in Brennan and Schwartz.

[16] Alternative estimation procedures can be implemented for the state-independent model as well as other specifications of the general model.

L. Then, π and δ are estimated so that the model has the best fit with the observed prices of the interest rate contingent claims. Ho and Lee (1985), for instance, apply this procedure to call and sinking fund provisions of corporate bonds. The third step begins with the development of a pricing model for options on the underlying risky asset or contingent claims on both interest rates and the risky asset. Then, the model parameters $\bar{u}$, $\bar{d}$, and γ are estimated so that the model has the best fit with the observed prices.

The first and second steps can be implemented based on the market prices of actively traded Treasury securities and interest rate futures options, respectively. The feasibility of the third step depends on individual risky assets. For example, if the risky asset is a common stock whose options are traded on an organized exchange, then the procedure proposed above can be implemented. Since calls and puts are listed in various strike prices as well as in many expiration dates, more than thirty options may be traded on the same stock on a single day. If only a few contingent claims on the risky asset are traded in a day, it may be possible to use their prices over a short period of time, during which the model parameters are assumed to be constant.

III. Summary and Conclusions

This paper presents a general framework for pricing contingent claims under interest rate and asset price uncertainty. First, a stochastic process for interest rates and asset price is modeled. This process recognizes explicitly the initial term structure of interest rates. A valuation relationship is then derived by virtue of a riskless hedge argument. The RNVR is shown to price pure discount bonds and contingent claims relative to the initial term structure. Thus, valuation models based on this approach are more powerful than previous models.

It is shown in discrete time that stochastic processes for risky assets cannot be specified arbitrarily without reference to the stochastic process for the risk-free rate. In other words, under stochastic interest rates, the stochastic processes for risky asset prices must be modeled such that they are arbitrage free with the then-prevailing risk-free rate. This insight is incorporated in the return-generating processes for risky assets.

The method proposed in this paper can be viewed as an integration of the Cox, Ross, and Rubinstein (1979) and Ho and Lee (1986) approaches. Cox, Ross, and Rubinstein (1979) is used to model asset price movements and Ho and Lee (1986) to model term structure movements. However, the multiplicity of risk makes it nontrivial to derive a risk-neutral valuation relationship (RNVR) in a discrete-time framework. A sequential process is proposed to avoid this problem. If the multiplicative factors are defined appropriately, the discrete-time RNVR converges to a continuous-time RNVR. Thus, the discrete-time approach based on the sequential process is consistent with the continuous-time approach. Needless to say, this technique can be used to price other types of multifactor contingent claims.

Finally, a method for estimating the model parameters based on the market prices is provided. First, the initial discount function is estimated based on the market prices of Treasury securities. The second step is to estimate π and δ using the prices of interest rate contingent claims. Finally, $\bar{u}$, $\bar{d}$, and γ are estimated based on the prices of contingent claims on the risky asset in question.

Appendix

Proof of Proposition 1: We form a portfolio of a T-period discount bond and ξ units of the risky asset at time-state (n, i). At the end of the first subperiod, if an upstate of the term structure occurs, the value V_u of the portfolio is[17]

$$V_u = \frac{P_i^n(T)}{P_i^n} h(T-1) + \xi \frac{u_i^n}{P_i^n} S.$$

If a downstate occurs, the value V_d of the portfolio is

$$V_d = \frac{P_i^n(T)}{P_i^n} h^*(T-1) + \xi \frac{d_i^n}{P_i^n} S.$$

We choose ξ such that $V_u = V_d$:

$$\xi = -\frac{h(T-1) - h^*(T-1)}{u_i^n - d_i^n} \frac{P_i^n(T)}{S}.$$

No-arbitrage condition implies that this portfolio earns the risk-free rate:

$$\frac{P_i^n(T) + \xi S}{P_i^n} = \frac{P_i^n(T)}{P_i^n} h^*(T-1) + \xi \frac{d_i^n}{P_i^n} S.$$

Substituting ξ into the above equation yields

$$\frac{1 - h^*(T-1)}{h(T-1) - h^*(T-1)} = \frac{1 - d_i^n}{u_i^n - d_i^n}.$$

The left-hand side of this equation is defined to be π. Thus,

$$\frac{1 - d_i^n}{u_i^n - d_i^n} = \pi,$$

or

$$\pi u_i^n + (1 - \pi) d_i^n = 1.$$

Proof of Proposition 2: Since it is shown by H-L that π and δ define the entire process for the term structure, it suffices to show that this set of parameters provides a unique specification of the asset price movements. First we solve equation (6) for u_i^n and let $n = N - 2$:

$$u_i^{N-2} = \frac{u_i^{N-1}}{\delta + \pi u_i^{N-1} - \pi\delta u_{i+1}^{N-1}}.$$

This implies that u_i^{N-2} is uniquely determined by u_i^{N-1} and u_{i+1}^{N-1} for all i, $i = 0$,

[17] Since there is no time value of money in the second subperiod, $P_i^{n+1/2}(T - 1/2)$ is equal to $P_i^{n+1}(T - 1)$ for all T.

1, 2, $\cdots$, $N - 2$. Thus, U specifies all u's at time $N - 2$. However, once u's at time $N - 2$ are given, one can repeat this procedure to determine u's at time $N - 3$, which, in turn, determine u's at time $N - 4$. Thus, the repetition of this procedure through time 0 yields the values of u_i^n for all time-states. Also, note that u_i^n determines d_i^n through equation (8). Thus, π, δ, and U specify the entire movements of the interest-dependent component. In addition, $\bar{u}$ and $\bar{d}$ are sufficient to determine the movements of the asset-specific component. Therefore, the stochastic process for the risky asset is completely specified by π, δ, $\bar{u}$, $\bar{d}$, and U.

Proof of Proposition 3: The price $S_{i,j}^n$ of the risky asset at time-state (n, i, j) is given by multiplying the initial price S by the multiplicative factors accumulated over n periods, each of which is the product of the multiplicative factor for the interest-dependent component and that for the asset-specific component. Thus, $S_{i,j}^n$ can be obtained by multiplying three parts: the initial price, the product of all multiplicative factors for the interest-dependent component, and the product of all multiplicative factors for the asset-specific component. The first and third parts are S and $\bar{u}^j \bar{d}^{n-j}$, respectively. The second part is obtained as follows. The path-independent condition implies that any path from time 0 to time-state (n, i, j) yields the identical price for the risky asset. Thus, we can choose an arbitrary path to represent the second part. Let such a path be the one on which upstates of the term structure occur in the first i periods and downstates of the term structure occur in the ensuing $(n - i)$ periods. Then the product of all multiplicative factors for the interest-dependent component is represented by $F_1(n, i; \pi, \delta, \gamma)/P(n)$.

Proof of Proposition 4: This is proved by mathematical induction. First we show that the formula holds when n is one. Consider time-state (1, 1, 1). In this case, the formula reduces to $P(1)\pi q$. However, by the one-period RNVR, the price of the Arrow-Debreu (A-D) security which pays \$1 at time 1 in state (1, 1) is $P(1)\pi q$. Thus, the formula holds for this time-state. For the other time-states at time 1, the formula gives $P(1)\pi(1 - q)$, $P(1)(1 - \pi)q$, and $P(1)(1 - \pi)(1 - q)$ to (1, 1, 0), (1, 0, 1), and (1, 0, 0), respectively, and all of them can be shown to be valid by the same argument.

Next, we assume that the formula holds for all periods up to time $n - 1$ and consider the valuation of the A-D security which pays \$1 at time-state (n, i, j). By definition, this security pays \$1 if and only if the state (i, j) occurs at time n. Then, at time $n - 1$, there are only four states in which this security has a nonzero price. They are $(n - 1, i - 1, j - 1)$, $(n - 1, i - 1, j)$, $(n - 1, i, j - 1)$, and $(n - 1, i, j)$. At time-state $(n - 1, i - 1, j - 1)$, the price $V(n - 1, i - 1, j - 1)$ of this A-D security is given by the one-period RNVR:

$$V(n - 1, i - 1, j - 1) = P_{i-1}^{n-1}\pi q.$$

Similarly for the other time-states,

$$V(n - 1, i - 1, j) = P_{i-1}^{n-1}\pi(1 - q),$$

$$V(n - 1, i, j - 1) = P_i^{n-1}(1 - \pi)q,$$

$$V(n - 1, i, j) = P_i^{n-1}(1 - \pi)(1 - q).$$

Then the price $V(0, 0, 0)$ at time 0 of this A-D security is given by

$$\begin{aligned} V(n, i, j) = {} & P_{i-1}^{n-1}\pi q A(n-1, i-1, j-1) \\ & + P_{i-1}^{n-1}\pi(1-q)A(n-1, i-1, j) \\ & + P_{i}^{n-1}(1-\pi)qA(n-1, i, j-1) \\ & + P_{i}^{n-1}(1-\pi)(1-q)A(n-1, i, j). \end{aligned}$$

Substituting the formula for $A(n-1, \cdot, \cdot)$ into the above equation yields the desired result.

Proof of Proposition 5: Consider a pure discount bond which matures at time n. Since it pays \$1 at time n in all states, we have $\phi(i, j) = 1$ for $i, j = 0, 1, \cdots, n$. Then its model price C is given by equation (18):

$$\begin{aligned} C &= \textstyle\sum_{i=0}^{n}\sum_{j=0}^{n} A(n,i,j) \\ &= \textstyle\sum_{i=0}^{n}\sum_{j=0}^{n} \dfrac{P(n)g(n-i,i)}{\prod_{m=1}^{n-1}\pi + (1-\pi)\delta^m}\dbinom{n}{j}\pi^i(1-\pi)^{n-i}q^j(1-q)^{n-j} \\ &= \textstyle\sum_{i=0}^{n} \dfrac{P(n)g(n-i,i)}{\prod_{m=1}^{n-1}\pi + (1-\pi)\delta^m}\pi^i(1-\pi)^{n-i}\textstyle\sum_{j=0}^{n}\dbinom{n}{j}q^j(1-q)^{n-j}. \end{aligned}$$

The summation over j can be viewed as the sum of the probabilities of the binomial distribution, which is equal to one. Thus, we have

$$C = \sum_{i=0}^{n} \frac{P(n)g(n-i, i)}{\prod_{m=1}^{n-1}\pi + (1-\pi)\delta^m}\pi^i(1-\pi)^{n-i}.$$

It follows from Appendix C of H-L that this expression is equal to $P(n)$.

REFERENCES

Brennan, Michael J. and Eduardo S. Schwartz, 1977, Convertible bonds: Valuation and optimal strategies for call and conversion, *Journal of Finance* 32, 1699–1715.

———, 1980, Analyzing convertible bonds, *Journal of Financial and Quantitative Analysis* 15, 907–929.

Cheng, Susan T., 1985, Multiple factor contingent claims in a stochastic interest rate environment, Unpublished manuscript, Stanford University.

Cox, John C., Stephen A. Ross, and Mark Rubinstein, 1979, Option pricing: A simplified approach, *Journal of Financial Economics* 7, 229–263.

Cox, John C., Jonathan E. Ingersoll, Jr., and Stephen A. Ross, 1985, An intertemporal general equilibrium model of asset prices, *Econometrica* 53, 363–384.

———, 1985, A theory of the term structure of interest rates, *Econometrica* 53, 385–407.

Evnine, Jeremy J., 1983, Three essays in the use of option pricing theory, Unpublished Ph.D. dissertation, University of California, Berkeley.

Grabbe, J. Orlin, 1983, The pricing of call and put options on foreign exchange, *Journal of International Money and Finance* 2, 239–253.

Heath, David, Robert Jarrow, and Andrew Morton, 1987, Bond pricing and the term structure of interest rates: A new methodology, Unpublished manuscript, Cornell University.

Ho, Thomas S. Y. and Sang-bin Lee, 1985, Pricing of corporate bond provisions: An empirical evidence, Unpublished manuscript, New York University.

———, 1986, Term structure movements and interest rate contingent claims pricing, Unpublished manuscript, New York University.

———, 1986, Term structure movements and interest rate contingent claims pricing, *Journal of Finance* 41, 1011–1029.

Kishimoto, Naoki, 1987, Pricing of asset-linked bonds under interest rate risk, Unpublished Ph.D. dissertation, New York University.

Litzenberger, Robert H. and Jacques Rolfo, 1984, An international study of tax effects on government bonds, *Journal of Finance* 39, 1–22.

McCulloch, J. H., 1975, The tax-adjusted yield curve, *Journal of Finance* 30, 811–830.

Merton, Robert C, 1973, Theory of rational option pricing, *Bell Journal of Economics and Management Science* 4, 141–183.

Ramaswamy, Krishna and Suresh M. Sundaresan, 1985, The valuation of options on futures contracts, *Journal of Finance* 40, 1319–1340.

Stapleton, R. C. and M. G. Subrahmanyam, 1984, The valuation of multivariate contingent claims in discrete time models, *Journal of Finance* 39, 207–228.