The Expected Utility of the Doubling Strategy
Author(s): Edward Omberg
Source: *The Journal of Finance*, Vol. 44, No. 2 (Jun., 1989), pp. 515-524
Published by: Blackwell Publishing for the American Finance Association
Stable URL: http://www.jstor.org/stable/2328604
Accessed: 26/11/2009 07:26

THE JOURNAL OF FINANCE • VOL. XLIV, NO. 2 • JUNE 1989

The Expected Utility of the Doubling Strategy

EDWARD OMBERG*

ABSTRACT

It has been noted that a certain continuous-time trading strategy, termed the "doubling strategy", generates a positive net return on borrowed funds, with probability one and within a finite period of time. Since the doubling strategy seems to represent a "free lunch" or arbitrage opportunity, a variety of constraints to render it infeasible have been proposed. In this paper, we show that the doubling strategy generates infinite disutility for a large class of utility functions, and we can think of no utility function for a risk-averse agent which is a counterexample.

HARRISON AND KREPS [5] POINTED out that a certain continuous-time trading strategy in frictionless markets, since termed the "doubling strategy", generates a positive net return on borrowed funds, with probability one and within a finite period of time. Moreover, the returns can be made arbitrarily large by increasing the initial amount borrowed. Thus, the doubling strategy seems to represent a "free lunch" or arbitrage opportunity.

The theory of valuation of derivative assets also assumes continuous-time trading in frictionless markets, an assumption necessary to argue that the derivative asset can be perfectly hedged by the underlying asset and the risk-free asset, as in Black and Scholes [2], or equivalently that the returns on the derivative asset can be replicated by a self-financing trading strategy. (See Harrison and Kreps [5] or Harrision and Pliska [6].) In addition, it must be assumed that the economy is free of arbitrage opportunities so that the continuously revised hedge portfolio earns the risk-free rate or, again equivalently, that the value of the derivative asset is equal to the replicating strategy's initial portfolio. Consequently, Harrison-Kreps and others have viewed the existence of the doubling strategy and its apparent violation of the "arbitrage-free economy" condition as a serious problem in contingent-claims theory. More generally, the doubling strategy seems to suggest an inconsistency in standard sets of assumptions (e.g., diffusion-type asset prices, frictionless continuous-time markets, and market efficiency) when sufficiently complex trading strategies are permitted.

Since the doubling strategy was introduced by Harrison and Kreps, a number of constraints have been proposed to eliminate the doubling strategy, without at the same time ruling out the other continuous-time trading strategies essential to derivative-asset pricing and continuous-time portfolio theory: bounds on portfolio positions (Kreps [9]), a positive constraint on wealth (Dybvig [3]), and margin requirements (Heath and Jarrow [7]). More generally, Dybvig and Huang [4] show that a lower bound on wealth precludes other complex trading strategies

* Leavey School of Business, Santa Clara University. The author would like to thank Peter Carr, Tom Russell, and an anonymous referee for helpful comments and suggestions.

which result in a "free lunch". Finally, Ingersoll's [8] careful discussion of the doubling strategy, the theoretical problem it presents, and alternative resolutions available at the present time indicates a continuing concern.[1]

While restraints of the above types are not unrealistic, they are unappealing in two respects. First, they require an additional level of complexity in building models of frictionless continuous-time markets. A second objection is more theoretical: if all agents knew that the trading strategy would succeed with probability one, why would any agents be concerned with intermediate constraints?

In this paper, we take a different approach. Specifically, we calculate the expected utility of an agent pursuing a doubling strategy, to be sure that the strategy raises the agent's expected utility. At first glance this may appear to be naive since the doubling strategy generates a positive net return with probability one. However, both Ingersoll [8] and Omberg [10] have presented simple examples in other contexts showing that such "arbitrage" opportunities may not only decrease an agent's expected utility, but may even in some cases produce infinite disutility.[2] The reason is that a probability of zero for a set of events, in this case the events that produce negative returns, does not imply that those events cannot occur, merely that they do not occur with sufficient frequency to have a positive probability (just as a zero probability for a specific value of a continuous random variable does not rule out its occurrence). If the utility consequences of such a zero-probability set of events are infinite rather than finite, a direct expected utility calculation multiplies a zero probability by an infinite utility, yielding an indeterminate result. The actual expected utility can only be calculated by viewing the opportunity as a limit and, by taking the limit, comparing the rate at which the probability vanishes with the one at which the corresponding utility becomes infinite.

More precisely and generally, suppose we have an agent with utility function $u(W)$ and initial wealth W_o and a sequence of returns $\{Z_n\}$ such that $\lim_{n\to\infty} Z_n \to Z$ with probability one. If there are events with vanishing probabilities but infinite (positive or negative) utility, we cannot assume that $\lim_{n\to\infty} E[u(W_o + Z_n)] = u(W_0 + Z)$, i.e., that the limit of expected utility is the expected utility of the limit. Thus, given an investment opportunity which generates a positive net return with probability one, deciding whether or not it is an arbitrage opportunity is not the simple task it might appear to be at first glance. In order to prove that it is an arbitrage opportunity, we must show that there are at least some preferences in the class allowed to agents for which taking the opportunity increases expected utility; in order to prove that it is not an arbitrage opportunity, we must prove that there are no such preferences in the allowed class.

In Section I, we take the appropriate limits and find that the doubling strategy has both gains and losses which approach infinity, with probabilities which vanish but not as rapidly as the gains and losses increase. Thus, both the expected gains and the expected losses are infinite. The net result is an expected value of plus infinity. However, for a broad class of utility functions, the result of an

[1] See Ingersoll [8], pp. 385–86.

[2] See Ingersoll [8], pp. 171–72, and Omberg [10], pp. 24–25.

expected utility calculation is infinite disutility, and we know of no utility function for a risk-averse agent which would be a counterexample to this result. We therefore suggest that future research on the doubling strategy take the form of either generalizing this result or producing a counterexample and, only if the latter bears fruit, expending effort on identifying constraints to eliminate it.

I. The Expected Utility of the Doubling Strategy

The doubling strategy was described by Harrison and Kreps only by analogy to the game of betting on tosses of a fair coin, where the bet is doubled at each step until the coin toss results in a win. The essence of the strategy is that (a) the size of the bet can be altered at each step and (b) the game can be terminated when the desired amount is won. Ingersoll provides a precise description of a doubling strategy, where the bets are investments in a security, borrowing is employed to make the bets self-financing, and the investment horizon is successively reduced to ensure that the goal is achieved within a finite period of time.

Our description of the doubling strategy is similar to Ingersoll's. Consider an arbitrary time interval $[0, T]$. The goal of the strategy is to make a positive net return of at lesst a specified amount by time T, with probability one. Without loss of generality, we assume that $T = 1$ and the minimum amount which must be won is one dollar. The times at which the investment in the risky asset is altered are $t_n = 1 - (1/2)^n$ with $n = 0, 1, 2, 3, \cdots, \infty$. Denote the trading intervals by $\Delta_n = t_n - t_{n-1} = (1/2)^n$. The return on the risky asset over each interval is assumed to follow a geometric Brownian motion with drift rate α and variance rate σ^2, so its return over the interval $[t_{n-1}, t_n]$ is e^{y_n}, where

$$y_n = \mu\Delta_n + x_n\sigma\sqrt{\Delta_n}, \qquad \mu = \alpha - \sigma^2/2, \tag{1}$$

the x_n's represent independent draws from a normal distribution with mean zero and variance one, and α and μ are the mean and median logarithmic returns, respectively. We assume that

$$\alpha > \mu > r, \tag{2}$$

where r is the risk-free rate of return, and for simplicity take r to be zero. Since, given α and μ, borrowing costs increase with r, this favors the doubling strategy. The agent initiates the strategy at time zero by borrowing one dollar and setting it aside. His or her goal is then to pursue a self-financing trading strategy which pays off the one-dollar debt and generates a non-negative return by $T = 1$, leaving him or her with the initial one dollar pocketed plus the final non-negative return. Ignoring the initial one dollar gained, his or her position in the game at time t_n is Z_n, with $Z_0 = -1$ and the game ending when $Z_n \geq 0$ for the first time. At each date t_{n-1}, the agent borrows an amount B_n and invests $Z_{n-1} + B_n$ in the risky asset. His or her position at t_n is therefore

$$Z_n = (Z_{n-1} + B_n)e^{y_n} - B_n. \tag{3}$$

The agent chooses the amount borrowed B_n so that $Z_n \geq 0$ whenever $x_n \geq 0$;

i.e., the probability of winning the game at each step is one half. Then

$$B_n = \frac{-Z_{n-1}e^{\mu\Delta_n}}{e^{\mu\Delta_n} - 1}, \tag{4}$$

and the position of the agent is given recursively by

$$Z_n = b_n Z_{n-1}, \tag{5}$$

$$b_n = \frac{e^{\mu\Delta_n} - e^{y_n}}{e^{\mu\Delta_n} - 1}, \tag{6}$$

so that

$$Z_n = b_n b_{n-1} \cdots b_1(Zo = -1). \tag{7}$$

Note from equations (1) and (6) that

$$b_n > 0 \quad \text{for } x_n < 0 \Rightarrow y_n < \mu\Delta_n, \tag{8a}$$

$$b_n = 0 \quad \text{for } x_n = 0 \Rightarrow y_n = \mu\Delta_n, \tag{8b}$$

and

$$b_n < 0 \quad \text{for } x_n > 0 \Rightarrow y_n > \mu\Delta_n. \tag{8c}$$

Since the game terminates as soon as a non-negative value for x_i occurs, we can assume that after n trials of the game the result is either

$$b_i > 0 \quad \text{for } i = 1, n, \text{ i.e., "failure so far"}, \tag{9a}$$

or

$$b_i > 0 \quad \text{for } i = 1, n - 1$$

and

$$b_n < 0, \text{ i.e., "success at step } n\text{"}. \tag{9b}$$

Define $E[b_n]$ to be the expected value of b_n, and $\beta_{n,s}$ and $\beta_{n,f}$ to be the expected values of b_n conditional on success or failure at step n, i.e., over the time interval $[t_{n-1}, t_n]$:

$$E[b_n] = \int_{-\infty}^{\infty} b_n\phi(x_n)\, dx_n$$

$$= \frac{e^{\mu\Delta_n} - e^{\alpha\Delta_n}}{e^{\mu\Delta_n} - 1} < 0, \tag{10}$$

$$\beta_{n,s} = E[b_n\colon x_n > 0] < 0$$

$$= \frac{\int_0^\infty b_n\phi(x_n)\, dx_n}{1/2}, \tag{11a}$$

$$\beta_{n,f} = E[b_n\colon x_n < 0] > 0$$

$$= \frac{\int_{-\infty}^0 b_n\phi(x_n)\, dx_n}{1/2}, \tag{11b}$$

where $\phi(x)$ is the standard normal-density function. Note that, since the unconditional expectation $E(b_n) = (1/2)\beta_{n,s} + (1/2)\beta_{n,f}$ is negative and $\beta_{n,s} < 0$, $\beta_{n,f} > 0$, the inequality

$$|\beta_{n,s}| > \beta_{n,f} \tag{12}$$

must hold. Further, let $E[Z_{n,s}]$ and $E[Z_{n,f}]$ be the corresponding expected positions conditional on success and failure at trial n:

$$E[Z_{n,s}] = E[Z_n\colon x_n > 0 \text{ and } x_i < 0 \text{ for } i = 1, n-1]$$

$$= [\textstyle\prod_{i=1}^{n-1} \beta_{i,f}]\beta_{n,s}(Zo = -1) > 0, \tag{13a}$$

$$E[Z_{n,f}] = E[Z_n\colon x_i < 0 \text{ for } i = 1, n]$$

$$= \textstyle\prod_{i=1}^{n} \beta_{i,f}(Zo = -1) < 0, \tag{13b}$$

with, given (12),

$$E[Z_{n,s}] > |E[Z_{n,f}]|. \tag{14}$$

The expected value of the strategy, accounting for the initial one dollar borrowed and set aside, is therefore

$$EV_{DS} = 1 + \lim_{N\to\infty} \textstyle\sum_{n=1}^{N} (1/2)^n E[Z_{n,s}] + (1/2)^N E[Z_{n,f}]$$

$$= 1 + \lim_{N\to\infty} [1 - (1/2)^N]E[Z_s] + (1/2)^N E[Z_{N,f}], \tag{15}$$

where $1 + E[Z_s]$ is the expected gain from the strategy given success.

The variables $|\beta_{n,s}|$ and $\beta_{n,f}$ have upper and lower bounds of the form[3]

$$k[2^{n/2}] < \beta_{n,f} < |\beta_{n,s}| < K[2^{n/2}], \qquad K > k > 0. \tag{16}$$

It follows from equations (13a) and (13b) that $E[Z_{n,s}]$ and $|E[Z_{n,f}]|$ have bounds of the form

$$2^{n(n+1)/4+n\log_2 k} < |E[Z_{n,f}]| < E[Z_{n,s}] < 2^{n(n+1)/4+n\log_2 K}. \tag{17}$$

Thus, the doubling strategy has unbounded expected gains and losses, conditional on success or failure by the final step $n = N$ as $N \to \infty$. Moreover, these conditional gains and losses increase more rapidly than their corresponding probabilities diminish, so that expected gains and expected losses are both infinite. In Appendix B, we show that the expected gains outweigh the expected losses, so that the result is an expected value of plus infinity.

The doubling strategy will therefore be accepted by all risk-neutral agents. However, suppose alternatively that all agents are risk averse. Whether an agent pursues the doubling strategy depends on his or her expected utility with and without the strategy. Let $u(W)$ be the agent's utility function. Assume an initial portfolio consisting of W_o invested in the risk-free asset. Then, defining the expected utility of the doubling strategy EU_{DS} to be the increment in expected

[3] See Appendix A for a derivation of the bounds.

utility from engaging in the strategy, we have

$$EU_{DS} = \lim_{N\to\infty} \textstyle\sum_{n=1}^{N} (1/2)^n E[u(Wo + 1 + Z_{n,s})]$$

$$+ (1/2)^N E[u(Wo + 1 + Z_{N,f})] - u(Wo). \quad (18)$$

If Jensen's inequality for strictly concave functions is applied to each conditional expected utility in the sum, we have the upper bound

$$EU_{DS} < \lim_{N\to\infty} \textstyle\sum_{n=1}^{N} (1/2)^n u(Wo + 1 + E[Z_{n,s}])$$

$$+ (1/2)^N u(Wo + 1 + E[Z_{N,f}]) - u(Wo). \quad (19)$$

A second application of Jensen's inequality, this time applied to the first N terms representing successful outcomes, gives

$$EU_{DS} < \lim_{N\to\infty} [1 - (1/2)^N] u(Wo + 1 + E[Z_s])$$

$$+ (1/2)^N u(Wo + 1 + E[Z_{N,f}]) - u(Wo). \quad (20)$$

The inequality above has a simple interpretation: the doubling strategy has two sources of randomness. The first is whether the agent succeeds or fails; the second is the randomness in the associated gains and losses Z_s and $Z_{N,f}$. If we consider a hypothetical variation of the doubling strategy in which the random gains and losses are replaced with their expected values $E[Z_s]$ and $E[Z_{N,f}]$, the second source of randomness has been eliminated. The expected utility of a risk-averse agent engaging in the doubling strategy is therefore less than the expected utility of the same agent playing the hypothetical variation.

In order to calculate either the expected utility of the doubling strategy or its upper bound, the expected utility of the hypothetical variation, a minimum requirement is that the utility function be defined for all $W \in [-\infty, +\infty]$, with $u' > 0$ and $u'' < 0$ everywhere. The only utility function in the HARA class with these properties is the negative exponential, for which the doubling strategy results in infinite disutility. This utility function obviously "stacks the deck" against the doubling strategy since the utility of gains is bounded whereas the disutility of losses approaches infinity with extreme rapidity. To construct a fairer test, consider a utility function which is a power function of W at both extremes; i.e.,

$$u(W) \to AW^a \qquad \text{as } W \to \infty, \quad \text{with } A > 0 \text{ and } 0 < a < 1, \quad (21a)$$

$$u(W) \to -B(-W)^b \qquad \text{as } W \to -\infty, \quad \text{with } B > 0 \text{ and } b > 1. \quad (21b)$$

The restrictions on the exponents a and b are necessary to ensure the proper signs for u' and u''. By letting $a = 1 - \varepsilon_a$ and $b = 1 + \varepsilon_b$, where ε_a, $\varepsilon_b > 0$ are arbitrarily small, we can maximize the rate at which utility increases with gains and minimize the rate at which it decreases with losses and, at the same time, make the agents with this class of utility functions arbitrarily close to being risk neutral, all of which favors the doubling strategy. In Appendix C, we show that, for this broad class of utility functions, EU_{DS} approaches minus infinity as $N \to \infty$; i.e., the doubling strategy results in infinite disutility.

We can think of no utility function which is risk averse everywhere that would be a counterexample to this result. For example, assuming that utility is given by ln W as $W \rightarrow +\infty$ would strengthen the result, since W^a ultimately overtakes ln W, and $-\ln(-W)$ is not suitable for $W \rightarrow -\infty$ since it represents risk-loving behavior for $W < 0$. Bounding a utility function from below would invalidate the result since the disutility of failure would then be finite. While Arrow [1] has argued that utility functions ought to be bounded from both above and below, Samuelson [11] has taken the opposite position, and none of the commonly used utility functions in financial economics is bounded from below.[4]

II. Summary and Conclusions

As earlier authors have noted, the doubling strategy generates a positive net return on borrowed funds within a finite period of time with probability one. In addition, we have shown that its expected value is infinite. Nonetheless, it does not appear to be an arbitrage opportunity. The reason is that, though the probability of failure vanishes in the limit, the losses from failure are infinite in the limit, producing infinite disutility unless the potential arbitrager's utility function is bounded from below. A direct expected utility calculation is therefore indeterminate since it multiplies a zero probability by an infinite disutility. Consequently, the expected utility of an agent adopting the doubling strategy must be calculated by taking the limit, in order to compare the rate at which the probability of failure decreases with the rate at which the disutility of failure grows. For a broad class of utility functions, the strategy results in infinite disutility, and we can think of no counterexample utility functions which are risk averse everywhere.

It is only fair to point out that the preceding analysis is not completely general in terms of the specification of either the trading strategy or the utility function. Nonetheless, the results suggest that future research in this area be directed toward either generalizing the above results or constructing a counterexample. Occam's razor indicates that only in the latter case should we turn to the task of designing constraints to render the strategy infeasible. Perhaps more important is the methodological point at issue: events with vanishing probabilities cannot be safely ignored unless their utility consequences are known to be finite.

Appendix A

To derive the bounds (16),

$$k[2^{n/2}] < \beta_{n,f} < |\beta_{n,s}| < K[2^{n/2}], \qquad K > k > 0, \tag{A1}$$

we begin by defining

$$f(y) = E_x[e^{xy}: x > 0] - 1 \tag{A2}$$

[4] See Samuleson [11] for a complete discussion. Arrow's argument that utility functions must be bounded from both above and below for a complete transitive ordering of alternatives follows from Menger's super-St. Petersburg Paradox. Samuelson both challenges the paradox itself and notes the loss of risk aversion everywhere (strict concavity for utility functions) and general diversification theorems if Arrow's proposition is accepted.

and

$$g(y) = 1 - E_x[e^{xy}: x < 0], \tag{A3}$$

where $E_x[\cdot]$ denotes the expectation of the function in brackets with respect to a standardized random normal variable x. Then $f(y)$ is convex and $g(y)$ is concave since

$$f'(y) = E_x[xe^{xy}: x > 0] > 0, \tag{A4}$$

$$f''(y) = E_x[x^2e^{xy}: x > 0] > 0, \tag{A5}$$

$$g'(y) = -E_x[xe^{xy}: x < 0] > 0, \tag{A6}$$

$$g''(y) = -E_x[x^2e^{xy}: x < 0] < 0. \tag{A7}$$

In addition,

$$f(0) = g(0) = 0 \tag{A8}$$

and

$$f'(0) = g'(0) = \sqrt{2/\pi}\,. \tag{A9}$$

The geometry of $f(y)$ and $g(y)$ therefore produces the inequalities

$$g(yo)(y/yo) < g(y) < g'(0)y = f'(0)y < f(y) < f(yo)(y/yo) \tag{A10}$$

for $0 < y < y_o$. Letting $y = \sigma\sqrt{\Delta_n}$ and $y_o = \sigma$,

$$g(\sigma)\sqrt{\Delta_n} < g(\sigma\sqrt{\Delta_n}) < \sqrt{2/\pi}\,\sigma\sqrt{\Delta_n} < f(\sigma\sqrt{\Delta_n}) < f(\sigma)\sqrt{\Delta_n} \tag{A11}$$

The conditional expectations $\beta_{n,s}$ and $\beta_{n,f}$ of equations (11a) and (11b) can be rewritten

$$|\beta_{n,s}| = -\beta_{n,s} = \frac{e^{\mu\Delta_n}(E_x[e^{x\sigma\sqrt{\Delta_n}}: x > 0] - 1)}{e^{\mu\Delta_n} - 1} = \frac{e^{\mu\Delta_n}f(\sigma\sqrt{\Delta_n})}{e^{\mu\Delta} - 1}, \tag{A12}$$

$$\beta_{n,f} = \frac{e^{\mu\Delta_n}(1 - E_x[e^{x\sigma\sqrt{\Delta_n}}: x < 0])}{e^{\mu\Delta_n} - 1} = \frac{e^{\mu\Delta_n}g(\sigma\sqrt{\Delta_n})}{e^{\mu\Delta_n} - 1}. \tag{A13}$$

Using the inequalities above and the additional inequality

$$\mu\Delta_n < e^{\mu\Delta_n} - 1 < \mu\Delta_n e^{\mu\Delta_n}, \tag{A14}$$

the inequality

$$k(1/\sqrt{\Delta_n}) < \beta_{n,f} < |\beta_{n,s}| < K(1/\sqrt{\Delta_n}) \tag{A15}$$

is easily derived, where

$$K = e^{\mu}f(\sigma)/\mu, \quad k = g(\sigma)/\mu, \quad \text{and } K > k > 0. \tag{A16}$$

Noting that the trading interval $\Delta_n = (½)^n$, this simplifies to the bounds to be derived.

Appendix B

To show that $EV_{DS} \to +\infty$, consider equation (15) and "throw away" the gains from all winning outcomes except a win at the last step $n = N$. Then

$$EV_{DS} > \lim_{N\to\infty} (1/2)^N E[Z_{N,s}] + (1/2)^N E[Z_{N,f}], \tag{B1}$$

$$EV_{DS} > 2^{-N} E[Z_{N-1,f}](\beta_{N,s} + \beta_{N,f}), \tag{B2}$$

$$EV_{DS} > 2^{-N} E[Z_{N-1,f}]E(b_N). \tag{B3}$$

From equations (10) and (13b), $E[b_N]$ and $E[Z_{N-1,f}]$ are both negative; the lower bound on EV_{DS} given by the right-hand side above is therefore positive. From equation (10), the limiting value of $E[b_N]$ is

$$\lim_{N\to\infty} E[b_N] = (\mu - \alpha)/\mu = (-\sigma^2/2)/\mu. \tag{B4}$$

EV_{DS} therefore goes to plus infinity if $2^{-N} | E[Z_{N-1,f}] |$ does. The inequality

$$2^{-N} | E[Z_{N-1,f}] | > 2^{(N-1)N/4+(N-1)\log_2 k - N} \tag{B5}$$

follows from equation (17) and ensures this result.

Appendix C

To show that $EU_{DS} \to +\infty$ for utility functions in the class

$$u(W) \to AW^a \qquad \text{as } W \to \infty, \text{ with } A > 0 \text{ and } 0 < a < 1, \tag{C1}$$

$$u(W) \to -B(-W)^b \qquad \text{as } W \to -\infty, \text{ with } B > 0 \text{ and } b > 1, \tag{C2}$$

begin with the upper bound of equation (20):

$$EU_{DS} < \lim_{N\to\infty} [1 - (1/2)^N] u(Wo + 1 + E[Z_s]) + (1/2)^N u(Wo + 1 + E[Z_{N,f}]) - u(Wo). \tag{C3}$$

In the limit, the bounds of equation (17) ensure that $W_o + 1$ is negligible in comparison with $E[Z_s]$ and $| E[Z_{N,f}] |$, so that only the extremes of $u(W)$ as W approaches plus and minus infinity matter, and the upper bound becomes

$$EU_{DS} < \lim_{N\to\infty} u(E[Z_s]) + (1/2)^N u(E[Z_{N,f}]) - u(Wo), \tag{C4}$$

$$EU_{DS} < \lim_{N\to\infty} A(E[Z_s])^a - (1/2)^N B(| E[Z_{N,f}] |)^b - u(Wo), \tag{C5}$$

$$EU_{DS} < \lim_{N\to\infty} A(E[Z_s])^a [1 - BA^{-1} R_N] - u(Wo), \tag{C6}$$

where

$$R_N = (1/2)^N (| E[Z_{N,f}] |)^b (E[Z_s])^{-a}]. \tag{C7}$$

Since A, $B > 0$, $EU_{DS} \to -\infty$ if $R_N \to +\infty$. To show the latter, we replace the loss $| E[Z_{N,f}] |$ with its lower bound from equation (17),

$$| E[Z_{N,f}] | > 2^{N(N+1)/4+N\log_2 k}, \tag{C8}$$

and the gain $E[Z_s]$ with an upper bound. To calculate an upper bound for $E[Z_s]$, note that

$$E[Z_s] = \sum_{n=1}^{N} (1/2)^n E[Z_{n,s}], \tag{C9}$$

$$E[Z_s] < \sum_{n=1}^{N} (1/2)^n 2^{n(n+1)/4+n\log_2 K}, \tag{C10}$$

$$E[Z_s] < \sum_{n=1}^{N} 2^{n(n+1)/4+n\log_2 K-n}. \tag{C11}$$

For large N, each term is upper bounded by the last term, giving

$$E[Z_s] < N2^{N(N+1)/4+N\log_2 K-N}, \tag{C12}$$

$$E[Z_s] < 2^{N(N+1)/4+N\log_2 K-N+\log_2 N}. \tag{C13}$$

Replacing the conditional expected gain $E[Z_s]$ with its upper bound and the magnitude of the conditional expected loss $|E[Z_{N,f}]|$ with its lower bound in the expression for R_N above results in

$$R_N > 2^{b[N(N+1)/4+N\log_2 k]-N}2^{-a[N(N+1)/4+N\log_2 K-N+\log_2 N]}. \tag{C14}$$

The utility function restriction $b > 1 > a > 0$ and the dominance of the N^2 terms in the exponents as $N \to +\infty$ then ensure that

$$\lim_{N\to\infty} R_N = +\infty \tag{C15}$$

and, therefore, that $EU_{DS} = -\infty$.

REFERENCES

1. Kenneth Arrow. *Essays in the Theory of Risk-Bearing.* Chicago: Markham Publishing Company, 1971, 44–89.
2. Fischer Black and Myron Scholes. "The Pricing of Options and Corporate Liabilities." *Journal of Political Economy* 81 (May 1973), 637–54.
3. Philip Dybvig. "A Positive Wealth Constraint Precludes Arbitrage Profits (e.g., from Doubling) in the Black-Scholes Model." Unpublished manuscript, Princeton University, 1980.
4. ——— and Chi-fu Huang. "Non-Negative Wealth, Absence of Arbitrage, and Feasible Consumption Plans." MIT Working Paper, 1976–88, December 1987.
5. Michael Harrison and David Kreps. "Martingales and Arbitrage in Multiperiod Securities Markets." *Journal of Economic Theory* 20 (1979), 381–408.
6. ——— and Stanley Pliska. "Martingales and Stochsatic Integrals in the Theory of Continuous Trading." *Stochastic Processes and Their Applications* 11 (1981), 215–60.
7. David Heath and Robert Jarrow. "Arbitrage, Continuous Trading, and Margin Requirements." *Journal of Finance* 42 (December 1987), 1129–42.
8. Jonathan Ingersoll, Jr. *Theory of Financial Decision Making.* Totowa, NJ: Rowman and Littlefield, 1987.
9. David Kreps. "Three Essays on Capital Markets." Technical Report No. 298, Institute for Mathematical Studies in the Social Sciences, 1979.
10. Edward Omberg. "On the Theory of Perfect Hedging." Unpublished manuscript, Santa Clara University, August 1988.
11. Paul Samuelson. "St. Petersburg Paradoxes: Defanged, Dissected and Historically Described." *Journal of Economic Literature* 15 (March 1977), 24–55. Reprinted in *The Collected Scientific Papers of Paul A. Samuelson,* Volume V, pp. 133–164.