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Valuing Commercial Mortgages: An Empirical Investigation of the Contingent-Claims Approach to Pricing Risky Debt

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ABSTRACT

This paper empirically investigates a contingent-claims model of commercial mortgage pricing. We find that the magnitude of the observed default premia for a sample of nonprepayable fixed rate bullet mortgages can be explained by the contingent-claims model. In addition, the model explains a significant proportion of the period-to-period changes in the default premia. However, given an assumed negative correlation between building value changes and interest rate changes, the model's risk structure tends to increase less steeply with increasing maturity than the observed risk structure.

SINCE THE SEMINAL WORK of Black and Scholes (1973), corporate liabilities have been modeled as options on the total value of the firm. This contingent-claims approach to bond pricing was refined by Merton (1974) to allow for cash dispersals prior to the bonds' maturities and by Brennan and Schwartz (1980) and Ingersoll (1987) to allow for interest rate uncertainty.

Although variants of these bond pricing models are currently being used extensively on Wall Street, they have not been tested extensively. Moreover, empirical studies have not successfully accounted for the observed spread between the rates on risky and default-free debt. Jones, Mason, and Rosenfeld (1984) empirically investigate a model that assumes a nonstochastic term structure of interest rates, while Ramaswamy and Sundaresan (1986) examine a model with stochastic interest rates to price floating rate notes. In both cases, the models could not explain the observed default premia.

The relative lack of empirical analysis is due partly to the complicated nature of corporate bonds. Major corporations generally have a large number of different bond issues outstanding that have different priorities in the event that the firm goes bankrupt. Moreover, the bonds can have a variety of covenants specifying sinking fund provisions, conditions under which technical default will occur, and other restrictions that can have an impact on their value. Although these complications do not preclude pricing various bond issues on a case-by-case basis, they make it difficult to test systematically the accuracy of the pricing model.

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With respect to the theoretical pricing models, commercial mortgages are almost identical to corporate bonds. However, relative to corporate bonds, the convenants on these instruments are generally very simple and homogeneous. In particular, most commercial buildings are financed with only one mortgage. Moreover, a number of these mortgages are not prepayable. For these reasons, commercial mortgages provide an excellent opportunity to investigate empirically the contingent-claims approach to pricing risky debt.

Although residential mortgages have been examined extensively, little research has been directed to the valuation of commercial mortgages.[1] There are about 450 billion dollars of these mortgages currently outstanding. This represents about twenty percent of all outstanding mortgages in the United States and about fifteen percent of all private debt in this country. With respect to their valuation, commercial mortgages are probably closer to corporate bonds than to the residential mortgages examined in previous research. In particular, default risk has an important impact on their value, and, in contrast to residential borrowers, it is reasonable to assume, as a first approximation, that commercial borrowers are value maximizers that default only when doing so is optimal from a purely financial perspective.

This paper focuses on the valuation effects of default risk. The issue of prepayment is avoided since the empirical analysis is limited to the pricing of "bullet" mortgages that cannot be prepaid. By empirically analyzing bullet mortgages, as opposed to residential mortgages, we focus upon fundamental pricing considerations rather than the adequacy of a particular prepayment model. These commercial mortgages have maturities of three, five, seven, or ten years, are non-recourse loans, and are either nonamortizing or are amortized over thirty years. A majority of all commercial real estate transactions are financed with this type of mortgage.

The model we examine is an application of the Brennan and Schwartz (1980) and Ingersoll (1987) models for pricing corporate bonds. In particular, we assume that the commercial mortgage value is determined by two state variables, the instantaneous risk-free interest rate and the value of the mortgaged building. The assumed dynamics of these state variables, together with an absence of arbitrage opportunities, imply that the value of the mortgage satisfies a partial differential equation that can be solved numerically subject to appropriate boundary and initial conditions. The specification of these dynamics, along with the boundary and initial conditions that characterize commercial mortgages, is described in the next section.

The contingent-claims approach to pricing commercial mortgages is investigated by examining empirically the differences between mortgage rates generated by our model and corresponding quoted rates on these mortgages. By analyzing mortgages with different maturities, and on different dates where default-free

[1] Starting with the work of Dunn and McConnell (1981a, 1981b), financial economists have used the contingent-claims approach to value mortgages or securities representing pools of mortgages. Although most have examined government-backed securities where default risk does not exist, recent working papers by Rosenberg (1986) and Kau, Keenan, Muller, and Epperson (1986) provide simulations from a model of residential mortgages that takes into account default as well as the prepayment option considered in the earlier work.

interest rates differ, we are able to examine how both the model and actual quotes are affected by these factors.

Section I presents the contingent-claims model that is used to price the commercial mortgages. The data which are used in the model's analysis are described in Section II. Testing the contingent-claims model requires estimates of both instantaneous risk-free interest rate process parameters and parameters of the underlying building value process. Section III describes how our estimates of the instantaneous risk-free interest rate process parameters are attained. Section IV, which contains the empirical analysis of the contingent-claims model, employs exogenously specified building value process parameters as well as those building value process parameters that are implied by the quoted commercial mortgage rates. Our conclusions are summarized in Section V.

I. Pricing Commercial Mortgages

A. Contingent-Claims Model

This section presents a contingent-claims model for pricing nonamortizing commercial mortgages. The model assumes that uncertainty about the returns from a mortgage can be summarized by two state variables: the instantaneous risk-free interest rate, r, whose dynamics follow a mean-reverting square root diffusion process (as in Cox, Ingersoll, and Ross (1985)), and the value of the mortgaged building, B, whose dynamics follow a lognormal diffusion process. These stochastic processes are described in equations (1) and (2):

$$dr = K(\mu - r)dt + \sigma_r \sqrt{r} dZ_r, \tag{1}$$

where

$K \equiv$ speed-of-adjustment coefficient,
$\mu \equiv$ long-term mean of r,
$\sigma_r \equiv$ instantaneous standard deviation,
$Z_r \equiv$ standardized Wiener process,

and

$$dB = (\alpha - b)Bdt + \sigma_B \, BdZ_B, \tag{2}$$

where

$\alpha \equiv$ instantaneous expected return on building,
$b \equiv$ continuous rate of building payout,
$\sigma_B \equiv$ instantaneous standard deviation,
$Z_B \equiv$ standardized Wiener process.

The unanticipated changes in the value of the building are assumed to be correlated with unanticipated changes in the instantaneous risk-free interest rate, $dZ_r dZ_B = \rho dt$, where ρ denotes the instantaneous correlation coefficient.

Itô's Lemma can be applied to obtain the dynamics for the value of the mortgage and for the value of a default-free bond. The mortgage values are generated as a function of the instantaneous risk-free interest rate process and the building value process, while the default-free bond's value is generated solely by the instantaneous risk-free interest rate process.

Given that the values of the three assets, the default-free bond, the building, and the mortgage, are determined as functions of just two state variables, they can be combined into a portfolio that is instantaneously risk-free. This risk-free portfolio can be formed by hedging the interest rate risk of the mortgage with a short position in the default-free bond and by hedging its risk due to changes in the value of the building with a short position in the building. An absence of arbitrage opportunities requires that the instantaneous return on this risk-free portfolio by the same as the instantaneous risk-free rate of interest. This no-arbitrage requirement allows us to derive the following partial differential equation whose solution, subject to appropriate boundary and initial conditions, yields the value of the mortgage, V:

$$\begin{aligned} 1/2\ \sigma_B^2 B^2 \frac{\partial^2 V}{\partial B^2} + \sigma_B \sigma_r \sqrt{r} B \rho \frac{\partial^2 V}{\partial B \partial r} + 1/2\ \sigma_r^2 r \frac{\partial^2 V}{\partial r^2} \\ + (r - b)B \frac{\partial V}{\partial B} + (K(\mu - r) - \lambda r) \frac{\partial V}{\partial r} + \frac{\partial V}{\partial t} - rV + m = 0, \end{aligned} \tag{3}$$

where λ denotes the market price of interest rate risk and m is the continuous rate of mortgage payment.

The above equation corresponds to the partial differential equation given by Brennan and Schwartz (1980) and Ingersoll (1987) to value corporate debt in the presence of stochastic interest rates. The differences between the commercial mortgages that are valued in this study and the corporate bonds examined by these authors are captured in the boundary and initial conditions that are specified to solve the partial differential equation.

The bullet mortgages that we empirically examine cannot be prepaid. However, a borrower may default on a bullet mortgage. We assume that a borrower will default on a mortgage if, at any time prior to maturity, the value of the mortgaged building, $B(t)$, falls below the value of the mortgage, $V(B,r,t)$. Prior to maturity, the value of the mortgage thus satisfies

$$V(B, r, t) \leq B(t). \tag{4}$$

If $B^*(t)$ is the critical building value below which default becomes optimal,

$$V(B^*, r, t) = B^*(t), \tag{5}$$

given that continuous mortgage payments are assumed, then the following high contact condition must also be satisfied:

$$\frac{\partial V(B^*, r, t)}{\partial B} = 1. \tag{6}$$

At the mortgage's maturity date, T, the mortgage will be valued as

$$\min(B(T), M(T)), \tag{7}$$

where $M(T)$, the remaining principal on the mortgage at maturity, equals zero if the mortgage is fully amortizing. Solving the partial differential equation, expression (3), subject to these boundary and initial conditions yields the corresponding equilibrium value of the bullet mortgage.[2]

B. Naive Pricing Model

For comparison purposes, we also consider a naive pricing model which incorporates default risk. The naive model assumes a constant conditional probability of default, p, whereupon mortgage payments cease and the lender receives a fraction f, $0 < f < 1$, of the mortgage's face value. Under risk neutrality, commercial mortgage prices are then given by

$$\textstyle\sum_{k=1}^{n} (1 - p)^{k-1} p P_d(r, t; m, k) + (1 - p)^n P(r, t; m, n), \tag{8}$$

where the total number of promised mortgage payments is given by n, $P_d(r, t; m, k)$ is the time-t price of a default-free bond paying promised mortgage payments of m per period through the kth payment date when default occurs and the lender receives a fraction f of the mortgage's face value, and $P(r, t; m, n)$ is the time-t price of a default-free bond with cash flows equal to the mortgage's promised payments. Default-free bond prices are assumed to be generated by the Cox, Ingersoll, and Ross (1985) model. Hence, the naive model prices are weighted averages of Cox, Ingersoll, and Ross (1985) default-free bond prices.

II. Mortgage Data

The data we examine are rates for commercial mortgages quoted by insurance companies. While these rates do not necessarily represent transaction rates, for our purposes they are probably more reliable.

Actual transaction rates for particular mortgages are determined by building-specific factors that vary, systematically, across mortgages. For example, since lenders often require a larger down payment for mortgages on riskier buildings, loan-to-value ratios may be related to building value volatilities. However, since data on volatility and other building-specific parameters are not readily available, it is difficult to control for these effects, making an analysis of transaction rates

[2] To complete the specification, the following boundary conditions must also be provided. We assume that the desired solution has bounded derivatives. At $B = \infty$, the mortgage becomes default-free for all values of r, including $r = 0$ and $r = \infty$, and the mortgage value V is given by the corresponding Cox, Ingersoll, and Ross (1985) solution for the value of default-free debt. At $r = \infty$ then $V = 0$ for all values of B, including $B = 0$ and $B = \infty$. At $r = 0$, assuming that $\partial V/\partial r$ and $\partial^2 V/\partial r^2$ are bounded, V satisfies the following partial differential equation:

$$1/2\ \sigma_B^2 B^2 \frac{\partial^2 V}{\partial B^2} - bB \frac{\partial V}{\partial B} + K\mu \frac{\partial V}{\partial r} + \frac{\partial V}{\partial t} + m = 0.$$

When solving this partial differential equation numerically, we replace $\partial V/\partial r$ by the explicit backward difference:

$$\partial V(B, 0, t)/\partial r \approx [V(B, \Delta r, t) - V(B, 0, t)]/\Delta r.$$

The (approximated) partial differential equation (in B and t) is subject to the free boundary condition delineating optimal default when B is small and the boundary condition that V must equal the corresponding Cox, Ingersoll, and Ross (1985) solution when $B = \infty$.

impractical. By contrast, the quoted rates are better suited to test the contingent-claims model since they are based on a standard building and a mortgage with standard terms and, hence, are less likely to vary over time and across mortgages.

We analyze the quoted rates for three-, five-, seven-, and ten-year bullet mortgages that have penalties which preclude prepayment. The standard bullet mortgage either has no amortization or has a thirty-year amortization schedule and requires a twenty percent down payment. More than half of all commercial real estate transactions are financed with standard bullet mortgages. Since most transactions are financed with mortgages having five, seven, or ten year maturities, the quoted rates for three-year maturity mortgages may be slightly less reliable than the others.

The quotes we analyze are from Aetna, Massachusetts Mutual, Equitable, State Farm, Metropolitan, and Cigna and are for selected dates between December 1985 and January 1987.[3] A typical building that is financed at the quoted rates would be a five- to twenty-million-dollar office building. Shopping centers and hotels are considered riskier and generally require higher rates.

George Smith Financial and Morgan Stanley provided the data. Individuals at these firms indicated that quotes from the above insurance companies are for mortgages with similar risk characteristics and that most transactions occurred at the quoted rates. However, in cases where the loans are considered to be riskier than the norm, higher rates are negotiated.

III. Estimating the Interest Rate Parameters

The parameters of the instantaneous risk-free interest rate process are estimated from observed government bond prices taken from the *Wall Street Journal*, adjusted for accrued interest. We employ a nonlinear least-squares procedure, previously used by Brown and Dybvig (1986), which provides parameter estimates that minimize the sum of squared deviations between model and market prices of a cross-section of government bonds for each date that we analyze.[4] Although allowing the parameters of the risk-free interest rate process to vary from period to period is inconsistent with the underlying equilibrium model of the term structure, it ensures minimum mispricing of the default-free debt on each date and, hence, allows us to concentrate on investigating the spread between the rates on commercial mortgages and default-free securities.

As noted by Brown and Dybvig (1986), while it is possible to identify the parameters r and σ_r, it is not possible to identify separately the parameters μ, K, and the market price of interest rate risk, λ.[5] However, the unidentified param-

[3] We attempted to select dates that were approximately one month apart. Unfortunately, due to the unavailability of data, some of the periods between selected dates are longer than one month while some are shorter.

[4] Since *Wall Street Journal* prices are not necessarily transactions prices, the resultant estimates may be sensitive to errors in reported government bond prices.

[5] As noted by Brown and Dybvig (1986), the term structure parameters are not identified by their static estimation procedure for exactly the same reason that the expected return on a stock is not identified by the Black-Scholes option pricing formula. This lack of identification does not arise within Gibbons and Ramaswamy's (1986) dynamic estimation strategy in which successive term structures permit identification of K.

Table I

Estimates of Identifiable Instantaneous Risk-Free Interest Rate Process Parameters

We provide estimates of the instantaneous risk-free interest rate, r, the instantaneous standard deviation of changes in $r, \sigma_r\sqrt{r}$, and the implied long-term default-free interest rate, r_L, of the Cox, Ingersoll, and Ross (1985) default-free bond pricing model based on a cross-section of treasury bonds for each of the dates that commercial mortgage quotes are available.

Date	$\hat{r}$	$\hat{r}_L$	$\hat{\sigma}_r\sqrt{\hat{r}}$
12/31/85	0.075	0.100	0.021
01/28/86	0.074	0.107	0.021
03/03/86	0.072	0.080	0.019
03/25/86	0.070	0.074	0.019
04/21/86	0.063	0.074	0.018
06/09/86	0.067	0.103	0.020
07/14/86	0.065	0.077	0.018
08/11/86	0.063	0.082	0.018
08/25/86	0.059	0.086	0.018
09/24/86	0.060	0.100	0.019
10/22/86	0.060	0.099	0.019
11/20/86	0.059	0.090	0.018
12/18/86	0.059	0.086	0.018
01/18/87	0.059	0.088	0.018

eters enter the commercial mortgage valuation equation in precisely the same manner they enter the valuation equation for default-free debt (expression (6) in Brown and Dybvig (1986)), implying that the separate identification of these parameters is not necessary to value commercial mortgages. Without loss of generality, we then set $\lambda = 0$ and specify the parameters μ and K consistent with observed government bond prices.

We use the nonlinear least-squares procedure to estimate these parameters for each of the dates that mortgage quotes are available. Since the longest maturity mortgage that we examine is ten years, the treasury bonds used to estimate the parameters have maturities less than or equal to ten years. This provides parameter estimates that minimize pricing errors for the maturities of interest. The sample, at each date, consists of approximately eighteen treasury bonds evenly spread among maturities between one and ten years.[6]

Table I reports our estimates of the identifiable parameters of the underlying instantaneous risk-free rate process: the instantaneous risk-free interest rate, r, the instantaneous standard deviation of changes in r, $\sigma_r\sqrt{r}$, as well as the implied long-term rate, r_L, of the Cox, Ingersoll, and Ross (1985) model. The estimates of r are consistent with a decline in short-term interest rates over our sample period. By contrast, the estimates of r_L are consistent with a more erratic behavior

[6] Across the quote dates, all the sampled treasury bonds save one were selling above par. For each quote date, sampled bonds with longer maturities were priced to yield a greater return, implying that the yield curve was upward sloping over the maturities considered.

in long-term interest rates. The estimates of the standard deviation of changes in r are fairly stable over the sample period.

Table II summarizes the pricing performance of the Cox, Ingersoll, and Ross (1985) model. Panel A provides summary statistics on the dollar mispricing,

Table II

Pricing Properties of the Cox, Ingersoll, and Ross Default-Free Bond Pricing Model

This table summarizes the mispricing of a sample of treasury bonds by the Cox, Ingersoll, and Ross (1985) default-free bond pricing model. Mispricing is defined as actual price minus model price assuming \$100 face value. Panel A provides summary statistics on mispricing by maturity of treasury bonds; Panel B correlates the mispricing across maturities; Panel C correlates the mispricing with respect to estimates of the identifiable parameters of the Cox, Ingersoll, and Ross (1985) default-free bond pricing model: the instantaneous risk-free rate, r, the implied long-term default-free rate, r_L, and the instantaneous standard deviation of changes in r, $\sigma_r\sqrt{r}$.

A: Summary Statistics for Dollar Mispricing of Matching Maturity Treasury Bonds

Maturity	Mean	Standard Error of Mean	Median
3 yr	−0.107	0.120	0.024
5 yr	−0.511	0.135	−0.529
7 yr	−0.138	0.112	−0.086
10 yr	0.224	0.093	0.266

B: Correlation Matrix for Dollar Mispricing of Matching Maturity Treasury Bonds

	3-yr Maturity	5-yr Maturity	7-yr Maturity	10-yr Maturity
3-yr Maturity	1.000			
5-yr Maturity	0.815	1.000		
7-yr Maturity	0.257	0.251	1.000	
10-yr Maturity	−0.420	−0.358	−0.371	1.000

C: Correlation between Dollar Mispricing of Matching Maturity Treasury Bonds and Estimates of Identifiable Interest Rate Process Parameters

	$\hat{r}$	$\hat{r}_L$	$\hat{\sigma}_r\sqrt{\hat{r}}$
3-yr Maturity	−0.257	−0.409	−0.368
5-yr Maturity	−0.227	−0.289	−0.319
7-yr Maturity	0.548	−0.018	0.392
10-yr Maturity	0.084	0.514	0.409

market price minus model price, for the sample of treasury bonds with cash flows and maturities matching (as closely as possible) the commercial mortgages we analyze.[7] The table documents a slight tendency to overprice short-term bonds, especially with five years to maturity, and a tendency to slightly underprice bonds with ten years to maturity. However, on average, the mispricing of these bonds is minimal: the root mean square pricing error is 0.51 percent.

From Panel B of Table II, we note that the magnitudes of the mispricing of bonds with three and five years to maturity are highly positively correlated over time, while the mispricing of bonds with ten years to maturity is negatively correlated with the mispricing of shorter term-to-maturity bonds. In Panel C of this table, we correlate the mispricing of the Cox, Ingersoll, and Ross (1985) model with our estimates of the model's identifiable parameters. Notice that, when $\hat{r}_L$ is low (high), the model tends to underprice (overprice) bonds with three years to maturity and overprice (underprice) bonds with ten years to maturity. These results suggest that a one-factor model may not adequately describe the term structure. However, since the magnitude of the Cox, Ingersoll, and Ross (1985) model's mispricing is small, this systematic misspecification will have a minimal impact on the empirical properties of our commercial mortgage pricing model.

IV. Comparison of Model and Actual Mortgage Rates

In this section we investigate the pricing properties of the mortgage valuation models and compare rates generated by them with actual quoted rates. To implement the contingent-claims model, estimates of the parameters of the building value process are required. These include the volatility of the building value, σ_B, the correlation between changes in that value and changes in the instantaneous risk-free interest rate, ρ, and the rate of building payout, b. Unfortunately, time-series data pertaining to building values are difficult to obtain and are generally of poor quality. For these reasons, we take two approaches for specifying these building parameters.

The first method examines an exogenous range of parameter values in which the true building value parameters most likely fall. This range of plausible parameter values generates a corresponding range of mortgage rates that can be compared with the actual quoted rates to evaluate the accuracy of our pricing model. Examining a range of parameter values also enables us to examine the sensitivity of mortgage rates to changes in the model's parameters.

The second method examines the parameters of the building value process that are implied by the quoted rates. These are the building value process parameters that minimize the sum of squared deviations between model and market prices of the commercial mortgages. A comparison of the implied parameter values with the range of parameters that are considered reasonable a priori provides additional insights into the model's consistency with observed quotes.

[7] For each quote date, we restrict our attention to treasury bonds with maturities close to but never greater than the corresponding commercial mortgage. If more than one treasury bond with comparable maturity is available, we select that bond with yield as close as possible to the commercial mortgage rate.

A. Exogenous Building Process Parameters

We use estimated common stock volatilities and correlations between stock returns and interest rate changes as guidelines for selecting the range of exogenous building value volatilities and correlation parameters. In particular, parameter estimates from the returns of two Real Estate Investment Trusts, Hubbard and Washington REIT, are examined.[8] The parameter values chosen are −0.1, −0.2, and −0.3 as correlation coefficients and 0.15, 0.175, 0.20, and 0.225 as annualized standard deviations of the building returns. The values of the volatility parameter can be compared to the estimated Hubbard and Washington REIT return volatilities of 0.18 and 0.22, respectively, the estimated volatility of an index of residential real estate of 0.115 (Rosenberg (1986)), the estimated volatility of the S&P 500 of approximately 0.20, and the average volatility of (leveraged) common stocks on the order of 0.30. The range of correlation parameters should be compared with the estimated correlations of Hubbard and Washington REIT returns with changes in the one-month treasury bill return which are −0.16 and −0.02, respectively.

For the third parameter, the rate of building payout, the values 0.07, 0.085, and 0.10 are selected. Conversations with mortgage brokers indicated that the ratio of before-tax and before-depreciation cash flows to market value generally fell within this range over the time period of our investigation.

The above set of building parameter values are used as inputs in our pricing model to generate mortgage rates. To obtain these mortgage rates, we numerically solve the partial differential equation given by equation (3) subject to the appropriate boundary and initial conditions and determine iteratively the mortgage payment rate for which the mortgage sells at par. The numerical procedure used was Gourlay and McKee's (1977) line hopscotch algorithm. Their extensive numerical calculations indicate that this algorithm is more efficient to implement than the alternating-direction implicit method employed by others to numerically price contingent claims.[9] We describe this numerical procedure in the Appendix.

[8] Equity Real Estate Investment Trusts (REITs) are publicly traded portfolios of real estate assets. The advantage of the returns from these securities over the returns from available commercial real estate price indices is that they are based on market rather than appraised prices. A disadvantage is that the REITs are leveraged investments and, hence, provide biased estimates of the volatilities of building values and their correlations with changes in the treasury bill yield. Moreover, the properties owned by REITs are generally riskier than properties typically collateralizing mortgages held by insurance companies. Offsetting these upward biases in the estimated building value volatilities is the fact that the volatility of these investments reflects the riskiness of a diversified portfolio rather than a single building; however, the REITs are not as diversified as any of the available commercial real estate price indices.

The REITs selected, Washington REIT and Hubbard Real Estate Investments, were chosen because they have returns available during the last ten years (1976–1985) and because they are not highly leveraged. Of the sixteen REITs examined in Titman and Warga (1986), these are the only two that satisfy these criteria. Hubbard has an unusually low debt level for an REIT, approximately ten percent of total assets, and Washington REIT has a debt level that averaged approximately twenty-five to thirty-five percent of their total assets over this period. Washington REIT has the added advantage that they are not highly diversified: all of the property they own is located in the Washington, D.C. area. A disadvantage of Hubbard is that they hold a diversified portfolio of real estate assets that include some mortgages.

[9] Gane and Gourlay's (1977) Theorem 4.1 suggests that the line hopscotch algorithm has a

We first examine mortgage rates generated for August 25, 1986. From Table I, this date corresponds to when the estimated instantaneous risk-free interest rate is lowest over our sample period. Table III presents the mortgage rates and the default-free rates generated by our model, given the parameters of the interest rate process estimated with treasury bond prices on this date, and the exogenous range of building process parameters. The table also provides the range of actual mortgage rate quotes and treasury bond rates on this date for comparison purposes. In addition, the table presents both actual and model spreads between the mortgage rates and the rates on treasury bonds with the same maturity, expressed in terms of basis points.

As expected, the mortgage rates calculated from the model increase substantially as the volatility of the building value increases. This has been previously demonstrated for the case of residential mortgages by Rosenberg (1986) and Kau et al. (1986). Our analysis also illustrates the importance of the building payout parameter. As expected, increases in this parameter increase equilibrium mortgage rates. The magnitude of this increase is approximately eighty basis points for an increase in the rate of payout from seven to ten percent. The third building value parameter, the correlation between building value changes and interest rate changes, has less of an impact on mortgage rates because the estimated standard deviation of changes in the instantaneous risk-free interest rate is rather small. The difference between the estimated mortgage rates for the highest and lowest correlations averaged only about fifteen basis points and is higher for the longer term maturities than it was for the shorter term maturities.

A comparison with actual mortgage rate quotes suggests the following. The mortgage rates generated with the lowest volatilities and the lowest payout rates are lower than observed mortgage rate quotes. Conversely, our model with all but the lowest volatility parameter in conjunction with the highest payout rate generated mortgage rates that are higher than observed mortgage rate quotes. However, the mortgage rates generated with a building volatility of $\sigma_B = 0.15$ and a payout rate of $b = 0.10$, or a building volatility of $\sigma_B = 0.175$ and a payout rate of $b = 0.085$, are generally within thirty basis points of the observed quoted rates, suggesting that the observed spreads between commercial mortgages and treasury bonds can be explained by our model with reasonable parameter values.[10]

convergence rate of 0 ($\Delta t + \Delta B^2 + \Delta r^2$), where Δt represents the time increment while ΔB and Δr represent building value and interest rate increments, respectively. For a particular mortgage contract given a particular specification of interest rate process and building value process parameters, we tested our implementation of this algorithm by doubling the density of nodes in the building value and interest rate directions and quadrupling it in the time direction. By numerically solving on three nested grids we determined that, as expected, successive differences between numerically generated solutions decreased approximately by a factor of four.

[10] It should be noted that the simulated mortgage rates in this table assume that the mortgages are nonamortizing. The mortgage brokers claimed that, although the quote sheets do not specify any of the mortgages as thirty-year amortization, some of the mortgages are in fact amortized in this manner. In other unreported simulations we examined the impact of amortization on equilibrium mortgage rates. We found that ten-year mortgages that were amortized over thirty years had rates that averaged twenty-five basis points less than the reported nonamortizing rates. For five- and seven-year mortgages the difference in rates averaged eleven and fourteen basis points, respectively. For three-year mortgages the difference was only two basis points.

Table III

Commercial Mortgage Rates and Spreads over Treasury Bond Rates Generated by the Contingent-Claims Model for August 25, 1986

This table provides commercial mortgage rates and spreads over treasury bond rates for August 25, 1986 generated by the contingent-claims model given exogenous ranges for the building volatility, σ_B, the rate of building payout, b, and the correlation between unanticipated changes in building value and unanticipated changes in the instantaneous risk-free rate, ρ. For comparison purposes, actual commercial mortgage rates, actual treasury bond rates, and actual spreads are also provided.

A: Model Commercial Mortgage Rates

		$\tau = 3$			$\tau = 5$			$\tau = 7$			$\tau = 10$		
		$b =$ 0.070	$b =$ 0.085	$b =$ 0.100	$b =$ 0.070	$b =$ 0.085	$b =$ 0.100	$b =$ 0.070	$b =$ 0.085	$b =$ 0.100	$b =$ 0.070	$b =$ 0.085	$b =$ 0.100
	$\rho = -0.1$	0.0772	0.0802	0.0842	0.0805	0.0838	0.0884	0.0821	0.0860	0.0906	0.0841	0.0879	0.0926
$\sigma_B = 0.150$	$\rho = -0.2$	0.0766	0.0798	0.0838	0.0796	0.0833	0.0878	0.0814	0.0852	0.0899	0.0832	0.0870	0.0917
	$\rho = -0.3$	0.0762	0.0794	0.0833	0.0791	0.0827	0.0872	0.0807	0.0845	0.0891	0.0824	0.0861	0.0908
	$\rho = -0.1$	0.0821	0.0857	0.0899	0.0847	0.0887	0.0933	0.0861	0.0902	0.0949	0.0876	0.0917	0.0964
$\sigma_B = 0.175$	$\rho = -0.2$	0.0817	0.0853	0.0895	0.0841	0.0881	0.0927	0.0854	0.0894	0.0942	0.0867	0.0908	0.0955
	$\rho = -0.3$	0.0812	0.0848	0.0890	0.0834	0.0874	0.0920	0.0846	0.0886	0.0934	0.0898	0.0946	0.0946
	$\rho = -0.1$	0.0878	0.0917	0.0961	0.0896	0.0938	0.0986	0.0905	0.0948	0.0996	0.0916	0.0958	0.1007
$\sigma_B = 0.200$	$\rho = -0.2$	0.0874	0.0912	0.0956	0.0889	0.0931	0.0979	0.0897	0.0940	0.0988	0.0906	0.0948	0.0996
	$\rho = -0.3$	0.0869	0.0907	0.0951	0.0882	0.0924	0.0972	0.0889	0.0931	0.0979	0.0896	0.0938	0.0986
	$\rho = -0.1$	0.0939	0.0980	0.1026	0.0948	0.0992	0.1040	0.0953	0.0997	0.1046	0.0959	0.1002	0.1052
$\sigma_B = 0.225$	$\rho = -0.2$	0.0935	0.0975	0.1021	0.0941	0.0985	0.1033	0.0944	0.0988	0.1037	0.0948	0.0992	0.1041
	$\rho = -0.3$	0.0930	0.0970	0.1016	0.0934	0.0978	0.1026	0.0935	0.0979	0.1028	0.0937	0.0981	0.1030
Model Treasury Bond Rate			6.40%			6.66%			6.92%			7.09%	
Actual Treasury Bond Rate			6.41%			6.84%			6.95%			7.10%	
Actual Mortgage Quote			7.88%–8.75%			8.63%–8.88%			9.00%–9.13%			9.38%	

B. Commercial Mortgage Spreads over Treasury Bonds

		$\tau = 3$			$\tau = 5$			$\tau = 7$			$\tau = 10$		
		$b =$ 0.070	$b =$ 0.085	$b =$ 0.100	$b =$ 0.070	$b =$ 0.085	$b =$ 0.100	$b =$ 0.070	$b =$ 0.085	$b =$ 0.100	$b =$ 0.070	$b =$ 0.085	$b =$ 0.100
$\sigma_B = 0.150$	$\rho = -0.1$	132	162	202	139	172	218	129	168	214	132	170	217
	$\rho = -0.2$	126	158	198	130	167	212	122	160	207	123	161	208
	$\rho = -0.3$	122	154	193	125	161	206	115	153	199	115	152	199
$\sigma_B = 0.175$	$\rho = -0.1$	181	217	259	181	221	267	169	210	257	167	208	255
	$\rho = -0.2$	177	213	255	175	215	261	162	202	250	158	199	246
	$\rho = -0.3$	172	208	250	168	208	254	154	194	242	149	189	237
$\sigma_B = 0.200$	$\rho = -0.1$	238	277	321	230	272	320	213	256	304	207	249	298
	$\rho = -0.2$	234	272	316	223	265	313	205	248	296	197	239	287
	$\rho = -0.3$	229	267	311	216	258	306	197	239	287	187	229	277
$\sigma_B = 0.225$	$\rho = -0.1$	299	340	386	282	326	374	261	305	354	250	293	343
	$\rho = -0.2$	295	335	381	275	319	367	252	296	345	239	283	332
	$\rho = -0.3$	290	330	376	268	312	360	243	287	336	228	272	321
Actual Spread			147–234			179–204			205–218			228	

These same simulations are repeated using the parameters of the interest rate process for December 31, 1985, the date in our sample for which the estimated instantaneous risk-free interest rate is highest. Treasury bond rates on this date averaged about 180 basis points higher than on August 25, 1986, which, in turn, implied that the estimated risk-free interest rate parameters are also different. A comparison of the mortgage rates generated for this date and those generated for August 25 allows us to examine the effect of changes in the parameters of the risk-free interest rate process.

Simulations for December 31, 1985, presented in Table IV, indicate that building process parameters in a reasonable range also provide mortgage rate estimates that fit the observed quotes on this date. The mortgage rates generated with a building volatility of $\sigma_B = 0.15$ and a payout rate of $b = 0.10$, or a building volatility of $\sigma_B = 0.175$ and a payout rate of $b = 0.085$, are again generally within thirty basis points of the observed quoted rates. Comparative statics with respect to changes in the building process parameters are also similar across the two dates.

The most noteworthy difference between the rates estimated for December 31 and those estimated for August 25 is that the estimated spreads between commercial mortgage rates and corresponding default-free rates are about seventy basis points lower on December 31. This narrowing of spreads, which is predicted by our model, is consistent with the narrowing of the actual spreads between the quoted rates and treasury bond rates on these two dates.

To understand this narrowing of spreads, note that the building payout rates for the two dates are held fixed in these simulations while the prevailing nominal default-free interest rates increase. This characterization is consistent with the change in nominal interest rates being generated primarily by a change in the expected rate of inflation. If this were the case, then, in real terms, the amortization of newly originated mortgages would have decreased over the December–August period, increasing their default risk. Alternatively, a change in nominal default-free interest rates driven primarily by shifts in real rates does not result in a change in the real amortization of the mortgages but, rather, should result in an increase in the payout rate. In unreported simulations we find that, if the payout rates are increased on the earlier date to reflect higher real interest rates, the narrowing of spreads is no longer evident.

The fact that the quoted spreads actually did narrow is then consistent with payout rates remaining constant over this period and the expected inflation rate declining between the two dates. Discussions with mortgage bankers indicated that payout rates did indeed remain fairly constant over this time period. Moreover, forecasted rates of future inflation, compiled by *Blue Chip Economic Indicators*, declined an average of 1.2 percent between December 1985 and August 1986.

B. Implied Building Value Process Parameters

The quoted commercial mortgage rates can be used to generate alternative estimates of the parameters of the building value process. This section examines parameter estimates that are implied from the quoted rates and investigates the properties of our pricing model given these estimates.

The implied parameters of the building value process are generated with a Gauss-Newton algorithm that minimizes the sum of squared deviations between model and market prices of our sample of commercial mortgages.[11] The procedure assumes that the parameters of the building value process are constant through time and across maturities and uses the previously estimated instantaneous risk-free interest rate process parameters for each quote date. The market price of a mortgage is taken to be par, while the model price is computed under the assumption that the prevailing mortgage rate is the median of available quotes.

The implied building value process parameters are $\hat{\sigma}_B = 0.1550$ for the building value volatility, $\hat{b} = 0.0903$ for the rate of building payout, and $\hat{\rho} = 0.1435$ for the correlation between building value changes and changes in the instantaneous risk-free interest rate.[12] With the exception of the implied correlation, the implied parameters of the building value process fall within the corresponding plausible parameter ranges analyzed in the previous section.[13] Table V summarizes the pricing performance of the contingent-claims model given the implied parameters of the building value process. Panel A provides summary statistics describing the deviations between model and market commercial mortgage prices. On average, the contingent-claims approach does not systematically misprice the sampled commercial mortgages. The accuracy of the contingent-claims approach is reflected in the fact that the model's root mean square pricing error evaluated at the implied parameters is on the order of 1.6 percent. This compares with a corresponding pricing error of 2.3 percent given the exogenously specified building parameter values of $\sigma_B = 0.15$, $b = 0.10$, and $\rho = -0.2$.

There appears to be no relation between the mispricing of the mortgages and the mispricing of the treasury bonds documented previously in Table II. This is probably due to the small pricing errors in the treasury bond pricing model. In particular, while the Cox, Ingersoll, and Ross [5] model tends to slightly overprice our sampled five-year treasury bonds and to slightly underprice our sampled ten-year treasury bonds, there is no evidence to suggest overpricing of five-year mortgages or underpricing of ten-year mortgages. This can be seen more clearly in the two-dimensional scatterplots of commercial mortgage residuals versus matching treasury bond residuals presented in Figure 1. For each maturity, the commercial mortgage residuals appear to be randomly distributed about the average matching treasury bond residual.

Panel B of Table V presents correlations between the pricing errors of different maturity mortgages across the different dates in our sample. The correlations

[11] For further details on the Gauss-Newton algorithm, see Philips (1976), especially pages 102–105.

[12] Necessary conditions for a local maximum were satisfied at the implied building value process parameters. The first-order partial derivatives of the sum-of-squared-deviations function with respect to each of the building value process parameters were less than 10^{-5} when evaluated at the implied parameters.

[13] The fact that the implied correlation parameter is positive needs to be interpreted cautiously given problems with model biases and, in particular, with the possible inconsistency in static estimation of dynamic problems.

Table IV
Commercial Mortgage Rates and Spreads over Treasury Bond Rates Generated by the Contingent-Claims Model for December 31, 1985

This table provides commercial mortgage rates and spreads over treasury bond rates for December 31, 1985 generated by the contingent-claims model given exogenous ranges for the building volatility, σ_B, the rate of building payout, b, and the correlation between unanticipated changes in building value and unanticipated changes in the instantaneous risk-free rate, ρ. For comparison purposes, actual commercial mortgage rates, actual treasury bond rates, and actual spreads are also provided.

A: Model Commercial Mortgage Rates

		$\tau = 3$			$\tau = 5$			$\tau = 7$			$\tau = 10$		
		$b =$ 0.070	$b =$ 0.085	$b =$ 0.100	$b =$ 0.070	$b =$ 0.085	$b =$ 0.100	$b =$ 0.070	$b =$ 0.085	$b =$ 0.100	$b =$ 0.070	$b =$ 0.085	$b =$ 0.100
	$\rho = -0.1$	0.0885	0.0913	0.0946	0.0920	0.0951	0.0989	0.0945	0.0977	0.1015	0.0974	0.1006	0.1045
$\sigma_B = 0.150$	$\rho = -0.2$	0.0880	0.0907	0.0940	0.0912	0.0943	0.0980	0.0936	0.0967	0.1005	0.0963	0.0994	0.1032
	$\rho = -0.3$	0.0875	0.0901	0.0933	0.0904	0.0934	0.0971	0.0926	0.0956	0.0994	0.0951	0.0981	0.1019
	$\rho = -0.1$	0.0935	0.0966	0.1003	0.0962	0.0996	0.1036	0.0982	0.1017	0.1058	0.1006	0.1040	0.1081
$\sigma_B = 0.175$	$\rho = -0.2$	0.0929	0.0960	0.0996	0.0953	0.0987	0.1026	0.0971	0.1005	0.1046	0.0993	0.1026	0.1067
	$\rho = -0.3$	0.0922	0.0953	0.0989	0.0943	0.0977	0.1016	0.0960	0.0933	0.1033	0.0980	0.1012	0.1052
	$\rho = -0.1$	0.0989	0.1024	0.1063	0.1008	0.1045	0.1087	0.1023	0.1060	0.1103	0.1042	0.1078	0.1121
$\sigma_B = 0.200$	$\rho = -0.2$	0.0982	0.1017	0.1056	0.0998	0.1035	0.1077	0.1011	0.1047	0.1090	0.1028	0.1063	0.1105
	$\rho = -0.3$	0.0975	0.1009	0.1048	0.0988	0.1024	0.1066	0.0998	0.1034	0.1076	0.1013	0.1048	0.1089
	$\rho = -0.1$	0.1048	0.1086	0.1127	0.1058	0.1097	0.1140	0.1068	0.1107	0.1151	0.1080	0.1120	0.1163
$\sigma_B = 0.225$	$\rho = -0.2$	0.1041	0.1078	0.1119	0.1047	0.1086	0.1129	0.1055	0.1093	0.1137	0.1065	0.1104	0.1147
	$\rho = -0.3$	0.1033	0.1070	0.1111	0.1036	0.1075	0.1118	0.1041	0.1078	0.1123	0.1050	0.1087	0.1130
Model Treasury Bond Rate			8.16%			8.46%			8.70%			8.94%	
Actual Treasury Bond Rate			8.12%			8.57%			8.67%			8.89%	
Actual Mortgage Quote			9.25%–9.50%			9.625%–10.00%			10.00%–10.50%			10.375%–10.75%	

B: Commercial Mortgage Spreads over Treasury Bonds													
		$\tau = 3$			$\tau = 5$			$\tau = 7$			$\tau = 10$		
		$b =$ 0.070	$b =$ 0.085	$b =$ 0.100	$b =$ 0.070	$b =$ 0.085	$b =$ 0.100	$b =$ 0.070	$b =$ 0.085	$b =$ 0.100	$b =$ 0.070	$b =$ 0.085	$b =$ 0.100
$\sigma_B = 0.150$	$\rho = -0.1$	69	97	130	74	105	143	75	107	145	80	112	151
	$\rho = -0.2$	64	91	124	66	97	134	66	97	135	69	100	138
	$\rho = -0.3$	59	85	117	58	88	125	56	86	124	57	87	125
$\sigma_B = 0.175$	$\rho = -0.1$	119	150	187	116	150	190	112	147	188	112	146	187
	$\rho = -0.2$	112	144	180	107	141	180	101	135	176	99	132	173
	$\rho = -0.3$	106	137	173	97	131	170	90	123	163	86	118	158
$\sigma_B = 0.200$	$\rho = -0.1$	173	208	247	162	199	241	153	190	233	148	184	227
	$\rho = -0.2$	166	201	240	152	189	231	141	177	220	134	169	211
	$\rho = -0.3$	159	193	232	142	178	220	128	164	206	119	154	195
$\sigma_B = 0.225$	$\rho = -0.1$	232	270	311	212	251	294	198	237	281	186	226	269
	$\rho = -0.2$	225	262	303	201	240	283	185	223	267	171	210	253
	$\rho = -0.3$	217	254	295	190	229	272	171	208	253	156	193	236
Actual Spread			113–138			105–143			133–183			149–186	

Table V

Pricing Properties of Contingent-Claims Commercial Mortgage Pricing Model Given Implied Building Value Process Parameters

This table summarizes the mispricing of a sample of commercial mortgages by the contingent-claims commercial mortgage pricing model given implied building value process parameters. Mispricing is defined as actual price minus model price assuming \$100 face value. Panel A provides summary statistics on mispricing by maturity of commercial mortgages; Panel B correlates the mispricing across maturities; Panel C correlates the mispricing with respect to estimates of the identifiable parameters of the Cox, Ingersoll, and Ross (1985) default-free bond pricing model: the instantaneous risk-free rate, r, the implied long-term default-free rate, r_L, and the instantaneous standard deviation of changes in r, $\sigma_r\sqrt{r}$.

A. Summary Statistics for Dollar Mispricing of Commercial Mortgages

Maturity	Mean	Standard Error of Mean	Median
3 yr	−0.150	0.111	−0.189
5 yr	0.209	0.192	0.147
7 yr	−0.018	0.258	−0.123
10 yr	−0.019	0.266	−0.259

B. Correlation Matrix for Dollar Mispricing of Commercial Mortgages

	3-yr Maturity	5-yr Maturity	7-yr Maturity	10-yr Maturity
3-yr Maturity	1.000			
5-yr Maturity	0.781	1.000		
7-yr Maturity	0.683	0.862	1.000	
10-yr Maturity	0.272	0.606	0.755	1.000

C: Correlation between Dollar Mispricing of Commercial Mortgages and Estimates of Identifiable Interest Rate Process Parameters

	$\hat{r}$	$\hat{r}_L$	$\hat{\sigma}_r\sqrt{\hat{r}}$
3-yr Maturity	0.155	−0.689	−0.259
5-yr Maturity	0.488	−0.614	0.030
7-yr Maturity	0.274	−0.636	−0.117
10-yr Maturity	0.564	−0.021	0.465

are positive for all pairs of maturities, with their magnitudes decreasing as differences in their maturities increase. Panel C of this table presents correlations between our estimates of the identifiable parameters of the interest rate process and the dollar amount by which the commercial mortgages are mispriced. The

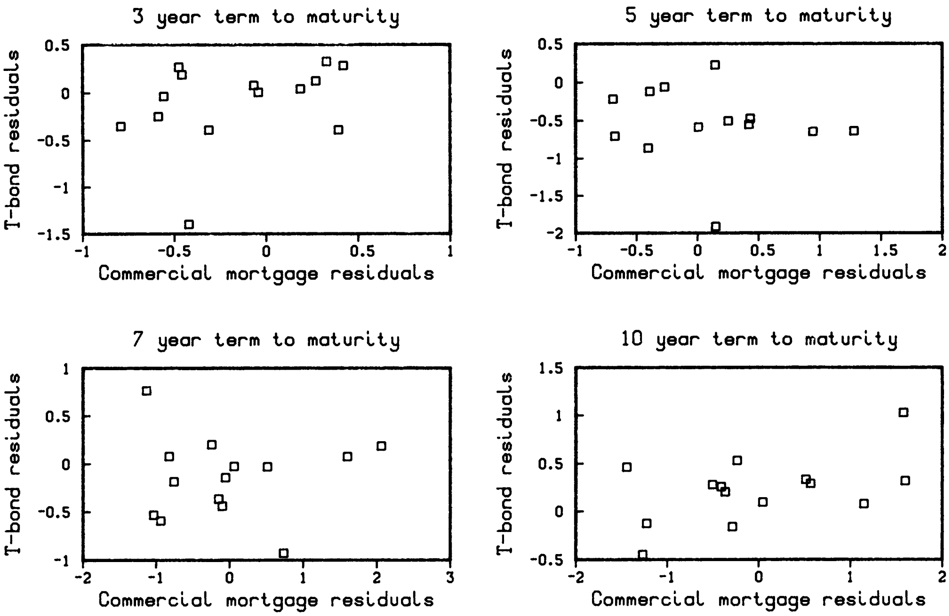

Figure 1. Treasury bond residuals in dollars, actual matching treasury bond prices minus corresponding Cox, Ingersoll, and Ross (1985) default-free bond prices, versus commercial mortgage residuals in dollars, actual commercial mortgage prices minus corresponding contingent-claims model prices, for three, five, seven, and ten year terms to maturity.

results suggest a weak positive correlation between pricing errors and estimates of the instantaneous risk-free rate, $\hat{r}$, and a more significant negative correlation between the mispricing and estimates of the implied long rate, $\hat{r}_L$. In other words, on dates where the estimated implied long rate is higher than average, the contingent-claims model tends to overprice commercial mortgages.

Table VI examines the dollar differences between actual and model spreads in commercial mortgage prices over matching treasury bond prices using the implied parameters. We note from Panel A that, except for the five-year maturity mortgages, the actual price spread does not significantly exceed the model price spread. Correlations between the differences in actual and model price spreads at different maturities, given in Panel B, are positive for all pairs of maturities. Correlations between our estimates of the identifiable parameters of the interest rate process and the differences in actual and model price spreads are provided in Panel C. These correlations are very similar to the correlations between the mispricing of commercial mortgages and estimates of r_L and r. The correlations between $\hat{r}_L$ and differences between actual and model price spreads are negative while the correlations between $\hat{r}$ and these differences are positive for each maturity.

C. Pricing Performance of the Naive Model

In addition to parameter estimates of the instantaneous risk-free interest rate process, the naive model requires an estimate of the conditional probability of

Table VI

Differences between Actual and Contingent-Claims Model Spreads in Commercial Mortgage Prices over Treasury Bond Prices

This table summarizes dollar differences between actual and contingent-claims model spreads in commercial mortgage prices over treasury bond prices given implied building value process parameters. Both commercial mortgages and treasury bonds assume \$100 face values. Panel A provides summary statistics on dollar differences by maturity; Panel B correlates the dollar differences across maturities; Panel C correlates the dollar differences with respect to estimates of the identifiable parameters of the Cox, Ingersoll, and Ross (1985) default-free bond pricing model: the instantaneous risk-free rate, r, the implied long-term default-free rate, r_L, and the instantaneous standard deviation of changes in r, $\sigma_r\sqrt{r}$.

A: Summary Statistics for Dollar Differences between Actual and Model Price Spreads

Maturity	Mean	Standard Error of Mean	Median
3 yr	−0.044	0.132	−0.022
5 yr	0.720	0.232	0.676
7 yr	0.120	0.279	0.086
10 yr	−0.243	0.233	−0.346

B: Correlations Matrix for Dollar Differences between Actual and Model Price Spreads

	3-yr Maturity	5-yr Maturity	7-yr Maturity	10-yr Maturity
3-yr Maturity	1.000			
5-yr Maturity	0.769	1.000		
7-yr Maturity	0.602	0.672	1.000	
10-yr Maturity	0.498	0.791	0.650	1.000

C: Correlation between Dollar Differences between Actual and Model Price Spreads and Estimates of Identifiable Interest Rate Process Parameters

	$\hat{r}$	$\hat{r}_L$	$\hat{\sigma}_r\sqrt{\hat{r}}$
3-yr Maturity	0.362	−0.205	0.117
5-yr Maturity	0.535	−0.339	0.210
7-yr Maturity	0.033	−0.583	−0.266
10-yr Maturity	0.610	−0.230	0.367

default, p, and the fraction of the loan's face value, f, that lenders recover in the event of default. Conversations with mortgage bankers indicated that on average lenders recover approximately eighty percent of the face value of the loan in the event of its default. Given this value of f, we implement the naive model by numerically determining that value of p which minimizes the sum of the squared deviations between model and market mortgage prices.[14] As before, the previously estimated parameters of the instantaneous risk-free interest rate process are used for each quote date.

The naive model performs worse than the contingent-claims model in two respects. First, the model requires unrealistic parameter estimates to optimize its pricing performance. The implied default rate of 8.525 percent per year is substantially higher than historical default rates. The American Council of Life Insurance reports that, over the period from June 1976 to December 1985, the average foreclosure rate on nonresidential nonfarm mortgages was 0.573 percent per year. Second, the naive model systematically misprices the sampled commercial mortgages. Panel A of Table VII provides evidence indicating that the naive model tends to overprice mortgages with three and five years to maturity and to underprice mortgages with ten years to maturity. In addition, the naive model's root mean square pricing error of 2.4 percent is greater than the pricing error of the contingent-claims model. Also, from Panel B of Table VII we note that the pricing errors of the naive model are highly positively correlated across all maturities, while the results of Panel C are consistent with the naive model's mispricing being highly negatively correlated with estimates of the instantaneous risk-free rate, $\hat{r}$.

D. Comparison of Model Spreads and Actual Spreads for the Entire Sample

To complete our analysis, the actual and model spreads between mortgage rates and default-free interest rates for each of the quote dates of our sample are compared. The comparison includes the model mortgage rates generated with the implied parameters of the building value process, as well as the corresponding rates generated with the previously specified exogenous parameters that had minimal mispricing on the August 25 and December 31 dates: $\rho = -0.2$ for the correlation between changes in building values and changes in the risk-free rate, and either $\sigma_B = 0.15$ for the standard deviation of building returns and $b = 0.10$ for the payout rate or $\sigma_B = 0.175$ for the standard deviation of building returns and $b = 0.085$ for the payout rate. For comparison purposes, we also present a corresponding analysis of naive model spreads.

Table VIII summarizes our results. For each date, Panel A presents the spreads generated by the contingent-claims model, for each maturity and set of assumed parameters of the building value process, together with the corresponding average spreads. Panel B provides this information for the spreads between the median commercial mortgage quote for each date and the corresponding treasury bond rate. For each maturity, the average absolute difference between model and actual spreads given the exogenously specified building value process parameters is less than forty basis points. As expected, the implied parameters generate model mortgage rates that are closest, on average, to the actual rates. The corresponding

[14] It is not possible to jointly imply relevant estimates of both f and p, since the naive model fits perfectly when both parameters are set equal to one. The sum of squared pricing errors can be made arbitrarily small by taking both f and p arbitrarily close to one. In other words, mortgage values generated by the model always equal actual values of newly issued mortgages if we assume they will default immediately and repay the lender in full.

Table VII

Pricing Properties of Naive Commercial Mortgage Pricing Model

This table summarizes the mispricing of a sample of commercial mortgage by the naive commercial mortgage pricing model. Mispricing is defined as actual price minus model price assuming \$100 face value. Panel A provides summary statistics on mispricing by maturity of commercial mortgages; Panel B correlates the mispricing across maturities; Panel C correlates the mispricing with respect to estimates of the identifiable parameters of the Cox, Ingersoll, and Ross (1985) default-free bond pricing model: the instantaneous risk-free rate, r, the implied long-term default-free rate, r_L, and the instantaneous standard deviation of changes in r, $\sigma_r\sqrt{r}$.

A: Summary Statistics for Dollar Mispricing of Commercial Mortgages

Maturity	Mean	Standard Error of Mean	Median
3 yr	−0.864	0.134	−0.840
5 yr	−0.664	0.258	−0.645
7 yr	0.021	0.296	0.175
10 yr	0.721	0.378	0.975

B: Correlation Matrix for Dollar Mispricing of Commercial Mortgages

	3-yr Maturity	5-yr Maturity	7-yr Maturity	10-yr Maturity
3-yr Maturity	1.000			
5-yr Maturity	0.902	1.000		
7-yr Maturity	0.804	0.925	1.000	
10-yr Maturity	0.658	0.828	0.884	1.000

C: Correlation between Dollar Mispricing of Commercial Mortgages and Estimates of Identifiable Interest Rate Process Parameters

	$\hat{r}$	$\hat{r}_L$	$\hat{\sigma}_r\sqrt{\hat{r}}$
3-yr Maturity	−0.745	0.185	−0.448
5-yr Maturity	−0.865	0.116	−0.575
7-yr Maturity	−0.781	0.175	−0.500
10-yr Maturity	−0.850	−0.274	−0.812

average absolute difference between model and actual spreads, for this set of building value process parameters, is less than twenty basis points for each maturity.

It should be noted that commercial mortgage rates and treasury bond rates vary considerably within our sample period. As can be seen from Panel B in Table VIII, the spreads between these rates also vary within our sample period, suggesting that further analysis of the contingent-claims model can be provided by investigating how well the model predicts changes in the actual spreads. This is done by regressing changes in the actual spreads against a constant and changes in the model spreads. These regressions are reported in Panel C. The results suggest that, in almost all cases, the contingent-claims model explains a statistically significant proportion of the period-to-period change in actual spreads. In some cases, the proportion of the variation in changes in the actual spread that is explained by the model, given the implied building value process parameters, exceeds seventy percent. Moreover, the null hypothesis that, for each maturity, changes in the model spread provide an unbiased estimate of changes in the actual spread cannot be rejected. For each regression, the intercept is not significantly different from zero and the slope coefficient is not significantly different from unity.

Table VIII also provides information on naive spreads. From Panel A of this table we note that the default premia generated by the naive model change very little over our sample period. Consequently, in Panel C of Table VIII we can reject the null hypothesis that, for each maturity, changes in the naive spreads provide unbiased estimates of changes in actual spreads. However, the direction of change in the spread for a given maturity predicted by the naive model is generally correct, and, interestingly, the correlations between the changes in naive model spreads and changes in actual spreads are in most cases higher than the correlations with the changes in contingent-claims model spreads.

E. Comparison of Model and Actual Risk Structures of Commercial Mortgage Rates

In this section the comparison of the contingent-claims model spreads and the corresponding actual spreads is examined with respect to the term to maturity of the commercial mortgages. It is important to examine the pricing properties of the contingent-claims model with respect to term to maturity as this is the model's only input parameter which varies cross-sectionally and, more significantly, is directly observable.

A comparison of the average model and actual spreads reported in Table VIII, Panels A and B, shows that the model spreads either decline (for $\sigma_b = 0.175$, $b = 0.085$, and $\rho = -0.2$), flatten out (for $\sigma_B = 0.15$, $b = 0.10$, and $\rho = -0.2$), or do not increase as steeply (for $\sigma_B = 0.155$, $b = 0.0903$, and $\rho = 0.1435$) with increasing maturity as the actual risk structure. Figure 2 plots the differences between average actual and model spreads as a function of term to maturity for the assumed parameters of the building value process. Notice that, for mortgages with maturities exceeding five years, the plotted differences are increasing with increasing term to maturity, indicating that the actual spreads increase with increasing term to maturity at a greater rate than model spreads.

The risk structure generated by the contingent-claims model becomes steeper if the assumed correlation between the changes in building value and interest rate changes is increased.[15] However, since the implied parameters minimize the

[15] For example, with the implied building payout ratio and the implied building volatility but a correlation of $\rho = 0.3$, the average yield spreads are 147 basis points for the three-year loan, 178 for the five-year loan, 191 for the seven-year loan, and 206 for the ten-year loan.

Table VIII

Commercial Mortgage Spreads for Various Terms to Maturity τ (in Years)

Panel A tabulates spreads (in basis points) of commercial mortgage rates over treasury bond rates generated by the contingent-claims model for exogenous parameters $\sigma_B = 0.15$, $b = 0.10$, $\rho = -0.2$ and $\sigma_B = 0.175$, $b = 0.085$, $\rho = -0.2$, and implied parameters $\sigma_B = 0.155$, $b = 0.0903$, $\rho = 0.1435$, where σ_B represents building volatility, b represents the building payout rate, and ρ represents the correlation between unanticipated changes in building value and unanticipated changes in the instantaneous risk-free rate; as well as spreads (in basis points) generated by the naive model for a fraction $f = 0.80$ of the loan's face value recovered in default and a conditional probability of default of $p = 0.007938$. Panel B provides actual commercial mortgage spreads. Panel C summarizes the results of the linear regressions, $\Delta actual\ spread_t = \beta_0 + \beta_1 \Delta model\ spread_t + \epsilon_t$, investigating the properties of changes in contingent-claims model spreads and changes in naive model spreads as predictors of changes in actual spreads.

A: Model Commercial Mortgage Spreads (in Basis Points) over Treasury Bonds

Contingent-claims model

σ_B	τ	b	ρ	12/31	1/28	3/3	3/25	4/21	6/9	7/14	8/11	8/25	9/24	10/22	11/20	12/18	1/18	Average
0.15	3	0.1	−0.2	124	139	164	160	179	160	177	186	197	179	184	189	194	197	174
0.15	5	0.1	−0.2	134	153	178	184	204	171	197	203	212	197	201	205	214	214	190
0.15	7	0.1	−0.2	135	151	182	192	209	171	202	205	207	195	201	207	212	212	192
0.15	10	0.1	−0.2	138	156	186	197	213	173	207	208	208	199	206	210	220	220	195
0.175	3	0.085	−0.2	144	159	183	179	196	179	195	204	213	196	201	206	213	213	191
0.175	5	0.085	−0.2	141	161	184	189	205	176	201	205	215	200	204	207	215	215	194
0.175	7	0.085	−0.2	135	152	181	189	203	169	197	200	202	191	197	200	206	206	188
0.175	10	0.085	−0.2	132	152	178	188	201	167	196	197	199	190	197	199	208	208	186
0.155	3	0.0903	0.1435	120	107	134	124	157	129	134	151	157	142	140	150	156	158	140
0.155	5	0.0903	0.1435	141	131	158	152	187	152	162	178	186	168	166	178	183	187	166
0.155	7	0.0903	0.1435	151	138	172	168	201	163	177	191	193	176	177	191	195	196	178
0.155	10	0.0903	0.1435	159	148	183	178	209	173	188	200	206	186	190	202	206	210	188

Naive model

f	τ	p	12/31	1/28	3/3	3/25	4/21	6/9	7/14	8/11	8/25	9/24	10/22	11/20	12/18	1/18	Average
0.8	3	0.007398	179	178	179	171	178	178	176	179	183	183	178	181	181	180	179
0.8	5	0.007398	175	174	177	171	178	175	176	181	185	183	179	183	183	181	179
0.8	7	0.007398	173	168	175	170	176	168	177	180	181	177	178	183	181	177	176
0.8	10	0.007398	166	164	176	173	180	165	185	184	188	179	181	190	187	188	179

B: Actual Commercial Mortgage Spreads (in Basis Points) over Treasury Bonds

	12/31	1/28	3/3	3/25	4/21	6/9	7/14	8/11	8/25	9/24	10/22	11/20	12/18	1/18	Average
3-yr Loan	113	126	134	112	157	101	106	144	178	156	162	178	160	160	142
5-yr Loan	118	117	136	106	142	109	126	162	179	173	173	193	192	186	151
7-yr Loan	158	176	158	125	166	129	172	184	205	191	185	203	191	192	174
10-yr Loan	155	150	177	155	192	159	212	205	228	200	202	235	220	221	194

C: Regression Results: Δ *actual spread*$_t = \beta_0 + \beta_1 \Delta$*model spread*$_t + \varepsilon_t$

Contingent-claims model	σ_B	τ	b	ρ	β_0	$se\ (\beta_0)$	β_1	$se\ (\beta_1)$	R^2
	0.15	3	0.1	−0.2	−5.42	5.75	1.61	0.42	0.58
	0.15	5	0.1	−0.2	0.01	5.29	0.84	0.32	0.39
	0.15	7	0.1	−0.2	−1.86	6.61	0.76	0.37	0.28
	0.15	10	0.1	−0.2	−1.27	6.15	0.99	0.33	0.46
	0.175	3	0.085	−0.2	−5.32	5.93	1.68	0.46	0.55
	0.175	5	0.085	−0.2	0.31	5.48	0.85	0.36	0.34
	0.175	7	0.085	−0.2	−1.77	6.69	0.81	0.41	0.27
	0.175	10	0.085	−0.2	−1.56	6.34	1.1	0.38	0.43
	0.155	3	0.0903	0.1435	−0.05	5.28	1.25	0.32	0.59
	0.155	5	0.0903	0.1435	1.69	3.39	1	0.19	0.72
	0.155	7	0.0903	0.1435	0.37	6.35	0.68	0.33	0.28
	0.155	10	0.0903	0.1435	1.38	4.92	1.12	0.27	0.62

Naive model	τ	f	p	β_0	$se\ (\beta_0)$	β_1	$se\ (\beta_1)$	R^2
	3	0.8	0.007938	3.32	6.79	3.91	1.83	0.29
	5	0.8	0.007938	2.83	2.63	5.04	0.7	0.83
	7	0.8	0.007938	1.63	5.57	3.1	1.07	0.43
	10	0.8	0.007938	0.31	2.56	2.83	0.29	0.9

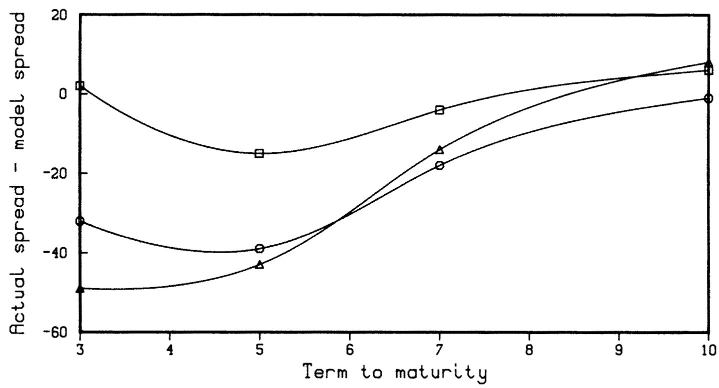

Figure 2. The difference (in basis points) between average actual spreads and average contingent-claims model spreads in commercial mortgage rates over treasury bond rates as a function of term to maturity (in years) for building volatility $\sigma_B = 0.155$, building payout rate $b = 0.0903$, and correlation coefficient $\rho = 0.1435$ (indicated by □); for building volatility $\sigma_B = 0.175$, building payout rate $b = 0.0850$, and correlation coefficient $\rho = -0.2000$ (indicated by ○); for building volatility $\sigma_B = 0.150$, building payout rate $b = 0.1000$, and correlation coefficient $\rho = -0.200$ (indicated by △).

sum of squared pricing errors, increasing the correlation parameter increases mispricing in some other respects. Moreover, a large positive correlation is inconsistent with casual observations regarding the impact of interest rate changes on real estate values. Hence, the deviation between the observed and model risk structures is probably not due to the correlation parameter being misspecified.

This discrepancy between the model and actual spreads also cannot be explained by the possible amortization of these mortgages. As mentioned previously (in footnote 10), equilibrium mortgage rates given a thirty-year amortization schedule are lower than corresponding nonamortizing rates, the difference increasing with increases in the term to maturity. This has the effect of amplifying the difference between the risk structure of the model and the risk structure of the quoted rates.

The intuition for the eventual decline in the spread with increases in the term to maturity can be understood if we consider the case where default occurs only at the maturity date. Given a positive drift in the building return process, the probability of default at this date eventually (at least weakly) decreases as the term to maturity increases. Moreover, with equal expected losses due to default, the longer maturity mortgage sells for a lower yield (above the comparable default-free security) since the expected loss due to default is offset by a higher yield spread for a longer period of time.

The increasing yield spreads for the quoted rates might be explained if we relax the maintained assumption that for each maturity the building value parameters are equal. For example, the increasing yield spread could reflect increases in future building value volatilities that might arise if the mortgaged buildings have existing leases, with fixed terms, that expire prior to the maturity of the mortgage. This may have a large impact on the riskiness of relatively long-term mortgages but should have a minor impact on shorter term mortgages.

It may also be the case that owners of particularly risky buildings are likely to choose the longer term mortgages since it will be more difficult for these individuals to get refinancing in the future. Insurance companies aware of this tendency would then charge higher premia on longer term mortgages to reflect the increase in risk revealed by the borrower's choice. (This argument is consistent with the Flannery (1986) model.)[16]

Increasing yield spreads may also arise if insurance companies erroneously use comparable corporate bond yields as a benchmark for setting their rates. The yield spread for the latter instruments may not decline with increases in the term to maturity. This may be true for a number of reasons. First, the call feature attached to most corporate bonds becomes more valuable as the term to maturity of these instruments increase. Moreover, in comparison to commercial mortgages, covenants on corporate debt place fewer restrictions on the amount of new debt the borrower may issue. Hence, a corporate borrower may be expected to issue new debt when the firm does well, offsetting the decline in the probability of default that would have otherwise occurred. This tendency, which will not have a similar impact on commercial mortgages, will have a larger impact as the term to maturity increases.

V. Conclusion

Our results suggest that, for plausible parameter values, commercial mortgage rates generated by a two-state-variable contingent-claims pricing model provide accurate estimates of commercial mortgage rates quoted by large insurance companies. Moreover, a significant portion of the observed period-to-period changes in the spread between the rates quoted for commercial mortgages and treasury bonds can be explained by this model. We also examined the effect on these mortgage rates of changes in their maturities and of changes in the instantaneous risk-free interest rate process. The model indicates that the spread between commercial mortgage rates and comparable treasury bond rates should eventually decline as the term to maturity increases. This effect is not confirmed by the actual quoted rates. The model also indicates that increases in risk-free interest rates that are not accompanied by an increase in the building payout should narrow the spread between commercial mortgage rates and comparable treasury bond rates. This effect is confirmed by the observed quoted rates.

Our analysis provides a stronger endorsement of the contingent-claims approach to pricing risky debt than previous empirical analysis of such models that examined corporate debt (e.g., Jones, Mason, and Rosenfeld (1984)). Perhaps the more accurate pricing of the commercial mortgages reflects the fact that these debt instruments are conceptually simpler than their counterparts issued by corporations. Future research should examine the ability of the contingent-claims pricing model to price other types of commercial mortgages which are more complicated than the bullet mortgages yet still simpler than most corporate bonds. The contingent-claims approach used here can be adapted to price

[16] See also Dunn and Spatt (1988) for a discussion of how self-selection by borrowers affects market prices in residential mortgage markets.

prepayable mortgages, participating mortgages, and some types of variable rate mortgages.[17]

Future research should also be directed toward obtaining more precise estimates of the parameters of the building return-generating process. Our analysis indicates that the volatility of building values and the building payout rate have major effects on commercial mortgage rates. The sensitivity of the observed effect of risk-free interest rate changes on commercial mortgage rates to assumptions about changes in the payout rate warrants further study of the relation between building payout rates and risk-free interest rate changes. Finally, given the lack of suitable time-series data on building market values, statistical procedures that extract market value volatilities from time series of appraised values should be developed.

Appendix

We briefly outline the hopscotch method used to solve our partial differential equation, expression (3), and the associated free boundary condition. For further details, see Gourlay and McKee (1977), especially pages 201–202.

We begin by transforming the partial differential equation by the following change of variables:

$$y = (1 + B)^{-1} \text{ and } z = (1 + 10r)^{-1} \tag{A1}$$

As a result, we confine our attention to the unit square. A square grid is superimposed onto the unit square, giving the set of points (ih, jh) for integers, $i, j = 0, 1, \cdots, 100$, with mesh spacing $h = 1/100$. Let $t_m = mk$ represent a plane parallel to the unit square, $m = 1, \cdots, M$, with $k = 1/52$ the mesh spacing in time. The approximate solution to the transformed partial differential equation at the grid point (ih, jh, mk) is denoted by v_{ij}^m.

We impose the initial conditions at $m = 0$ ($\tau = 0$), while the boundary conditions at $i = 0$ ($y = 0$ or $B = \infty$) and $j = 0$ ($z = 0$ or $r = \infty$) are imposed for all $m = 1, 2, \cdots, M$ ($\tau > 0$).

Gourlay and McKee's (1977) line hopscotch algorithm alternately uses explicit and implicit differences to approximate the transformed partial differential equation on lines perpendicular to the z-axis (parallel to the y-axis).

In particular, at t_{m+1}, we first use explicit differences to approximate the transformed partial differential equation for every j such that $m + j$ is even and sweep toward $j = 100$ ($z = 1$ or $r = 0$) on every alternate line perpendicular to the z-axis. For a particular j, we use the explicit method to sweep from $i = 1$ toward $i = 100$ ($y = 1$ or $B = 0$) and determine iteratively that value of i, i_c, such that $v_{i_c-1,j}^{m+1} < ((i_c - 1)h)^{-1} - 1$, $v_{i_c,j}^{m+1} \geq (i_c h)^{-1} - 1$, and set $v_{i,j}^{m+1} = (ih)^{-1} - 1$ for $i = i_c, \cdots, 100$. If $m + j$ for $j = 100$ ($z = 1$ or $r = 0$) is even, then we use explicit differences to impose the natural boundary condition given at $r = 0$ ($j = 100$).

Subsequently, at t_{m+1}, we use implicit differences to approximate the transformed partial differential equation for every j such that $m + j$ is odd and, as before, sweep toward $j = 100$ ($z = 1$ or $r = 0$) on every alternate line perpendicular

[17] See, for example, Kau, Keenan, Muller, and Epperson (1987).

to the z-axis. The totality of the implicit scheme for every alternate value of j gives a tridiagonal set of equations which is solved by a successive substitution algorithm. For a particular j, the tridiagonal sweeps start at $i = 1$ and progress toward $i = 100$ ($y = 1$ or $B = 0$), and we use the procedure of Brennan and Schwartz (1977) to determine the minimum value of i, i_c, such that $v_{i_c,j}^{m+1} = (i_c h)^{-1} - 1$. If $m + j$ for $j = 100$ ($z = 1$ or $r = 0$) is odd, then we use implicit differences to impose the natural boundary condition given at $r = 0$ ($j = 100$).

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