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The Quality Option and Timing Option in Futures Contracts

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ABSTRACT

Often futures contracts contain quality options whereby the short position has the choice of delivering one of an acceptable set of assets. We explore the implications of the quality option on the futures price. We develop a method for pricing the quality option for the general case of n deliverable assets and provide numerical illustrations of its significance. Even when the asset prices are very highly correlated, this option can have nontrivial value, especially when there is a large number of deliverable assets. We analyze the impact of the timing option and its interaction with the quality option. A procedure is developed for valuing the timing option in the presence of the quality option, and some numerical estimates are obtained.

OFTEN IN FUTURES CONTRACTS, the short position has some flexibility with regard to when, where, how much, and what will be delivered. These flexibilities have been described as the timing option, the location option, the quantity option, and the quality option. There are many examples of such delivery options in traded futures contracts. For example, the Treasury Bond Futures Contract traded on the Chicago Board of Trade permits the short to deliver any one of a predefined set of long-term government bonds. The wheat futures contract on the same exchange permits the short position to deliver any one of eleven different types of wheat at either Chicago or Toledo at any time during the delivery month. These delivery options have attracted considerable research interest[1] in the last few years. The basic intuition is that these options which are available to the seller are reflected in a somewhat lower futures price.

The aim of the present paper is to develop a procedure for computing the impact of the quality option when there are several deliverable assets. In addition, we analyze the timing option and explore its interaction with the quality option. This interaction between these two types of options may be of interest. We show

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[1] Contributions to this literature include Arak and Goodman [2], Benninga and Smirlock [3], Cheng [5, 6], Garbade and Silber [9], Gay and Manaster [10, 11], Hegde [12, 13], Hemler [14], Kamara and Siegel [17], Kane and Marcus [18, 19], Kilcollin [20], Livingston [21], and Margrabe [23].

that, if there is only one deliverable asset, then the short position should optimally deliver this asset at the first available opportunity. When there is more than one deliverable asset, delivery at the first permitted opportunity need no longer be optimal.

Section I describes the approach used to value the quality option when there are several deliverable assets. We show how known results concerning order statistics of the multivariate normal can be used to obtain accurate numerical estimates of the value of the quality option. Some specimen numerical results are given to illustrate how the number of deliverable assets effects the futures price of the contract. We also illustrate the sensitivity of the value of the quality option to the correlation among the prices of the deliverable assets.

Section II of the paper uses arbitrage arguments to analyze the timing option when there is just one deliverable asset. In this case, the optimal strategy for the short is to deliver at the first permitted opportunity. The interaction between the timing option and the quality option is probed. We assume that the short position can select any time within a given period to deliver either of two deliverable assets. Arbitrage arguments are used to develop inequalities which illustrate the interaction. In addition, numerical estimates are derived which suggest that the value of the timing option is quite small for plausible parameter values.

I. Quality Option with n Deliverable Assets

We first describe our basic notation and assumptions. In the one asset case, it is well known that a long forward contract is equivalent to a long call and a short put. The generalization of this result to the n asset case plays an important role in the development of our method for the valuation of a futures contract when the short position has a quality option.

Assumptions: We assume perfect frictionless markets. We assume a known constant interest rate of R percent per annum so that futures prices are equal to the corresponding forward prices.[2]

It is assumed that there are n deliverable assets the prices of which follow a multivariate lognormal distribution. When the futures contract matures, we assume that the short will deliver the cheapest of the n assets at that time.

Notation: We use the following notation:

- t: current time.
- $t + T$: delivery date under futures (forward) contract.
- $A_i(t)$: current price of asset, i, $1 \le i \le n$.
- $[\sigma_{ij} T]$: variance-covariance matrix of asset returns over any time interval, T. The variables

$$\log_e\left[\frac{A_i(t+T)}{A_i(t)}\right]$$

[2] This assumption is often made in discussions of the quality option. Hence, we will use the terms forward price and futures price interchangeably in the sequel. For a discussion of the relationship between forward and futures prices, see Cox, Ingersoll, and Ross [7], Jarrow and Oldfield [15], and Richard and Sundaresan [25].

have a multivariate normal distribution with variance covariance matrix $[\sigma_{ij} T]$. Instead of σ_{ij}, we sometimes use $\sigma_i \sigma_j \rho_{ij}$.
- K: strike price of European options [both call and put].
- $F(t, [A_1, \cdots, A_n], (t + T))$: forward (futures) price at time t of contract where short can deliver any one of assets $1 \cdots n$.
- $EC[t, [A_1, \cdots, A_n], K, (t + T)]$: price of a European call option at time t on the minimum of the n assets, with a strike price of K and an expiration date of $(t + T)$. At maturity, the value of this call is $\max[[\min(A_1, A_2, \cdots, A_n) - K], 0]$.
- $EP[t, [A_1, A_2, \cdots, A_n], K, (t + T)]$: current price at time t of a European put option on the minimum of the n assets, with strike price K and expiration date $(t + T)$. The value of this option at expiration is $\max[[K - \min(A_1, A_2, \cdots, A_n)], 0]$.

To motivate the n-dimensional case, it may be helpful to start with the one- and two-dimensional cases. Note that, when there is just one deliverable asset (with delivery date $(t + T)$), we can replicate a long forward contract to purchase one unit of the underlying asset with a portfolio consisting of a long European call option and a short European put option on one unit of the same underlying asset. In this case, both option contracts expire at $(t + T)$, and the exercise price of each option is equal to the equilibrium forward price. This can be written as

$$\begin{aligned} &VF(t, [A_1], F, (t + T)) \\ &\qquad = EC[t, [A_1], F, (t + T)] - EP[t, [A_1], F, (t + T)], \end{aligned} \tag{1}$$

where $VF(t, [A_1], F, (t + T))$ is the current value of the forward contract at time t and $F = F(t, [A_1], (t + T))$ is the forward price.

At inception, the value of the forward contract is zero, and so in this case the call option value is equal to the value of the put option. In fact, one method of deriving the forward price in this case would be to find the exercise price that made the two options have the same value.[3] We recall from the put-call parity theorem that the option portfolio on the right-hand side of equation (1) is equal in value to

$$A_1(t) - F \exp(-RT).$$

Since this is zero for the equilibrium forward price, we have

$$F(t, [A_1], (t + T)) = A_1(t)\exp(RT). \tag{2}$$

Equation (2) depicts the well-known cost-of-carry relationship when there is just one deliverable asset and no frictions.

In the case of two deliverable assets, the value of the forward contract is equal to a long call on the minimum of the two assets plus a short put position on the minimum of the two assets. Thus, the value of the forward contract in this case is equal to

$$EC[t, [A_1, A_2], K, (t + T)] - EP[t, [A_1, A_2], K, (t + T)]. \tag{3}$$

[3] It turns out that, while this appears a little awkward in the one-asset case, it is a useful approach when there are two (or more) underlying assets, and it is also helpful in the analysis of the timing option.

At inception, the value of the forward contract is zero, and so the value of the option combination is also zero where the strike price, K, of the options is equal to the forward price

$$F(t, [A_1, A_2], (t + T)).$$

Stulz [27] developed an extension of the put-call parity theorem for option combinations such as expression (3). He noted that (3) could also be expressed as[4]

$$EC[t, [A_1, A_2], 0, (t + T)] - Ke^{-RT}, \tag{4}$$

where the European call option in expression (4) has a zero exercise price and entitles its owner to the minimum of asset one and asset two at time $(t + T)$. We note that, in general, the current value of this option is less than the minimum of $A_1(t)$ and $A_2(t)$ and that it corresponds to the "correct" spot price to use when there are two deliverable assets. Margrabe [22] noted that options of this nature on the minimum of two assets with a zero strike price could be further simplified.

The generalization of Stulz's and Margrabe's results to the n-dimensional case has been discovered by several authors. In the case of a forward contract where the short can deliver any one of n assets, the value of the forward contract can be expressed as a long call plus a short put:

$$EC[t, [A_1, A_2, \cdots, A_n], K, (t + T)] - EP[t, [A_1, A_2, \cdots, A_n], K, (t + T)]. \tag{5}$$

At inception, the value of the forward contract is zero, and, in this case, the strike price, K, is equal to the forward price,

$$F(t, [A_1, A_2, \cdots, A_n], (t + T)).$$

In analogy with the two-dimensional case, expression (5) may be written as

$$EC[t, [A_1, A_2, \cdots, A_n], 0, (t + T)] - Ke^{-RT}. \tag{6}$$

Since the value of the forward contract, at inception, is zero, this implies that

$$F(t, [A_1, A_2, \cdots, A_n], (t + T)) = e^{RT}EC[t, [A_1, A_2, \cdots, A_n], 0, (t + T)]. \tag{7}$$

Hence, the forward price can be obtained if we can value the call option on the minimum of the n assets, with strike price zero and expiration date $(t + T)$.

For small numbers of deliverable assets, $n = 2$ or 3, it is fairly straightforward to evaluate the option on the right-hand side of equation (7). However, for larger numbers of assets, the computations involve integrations over $(n - 1)$-dimensional[5] multivariate normal integrals. By making some simplifying assumptions, we can develop a procedure for evaluating the option on the minimum of the n assets. This enables us to obtain the forward price of a contract where the short can deliver any one of n assets.

To explore the behavior of the quality option as the number of assets gets

[4] To obtain the equilibrium forward price in the two-asset case, we can find the value of K which makes expression (4) zero. This value of K represents the required forward price.

[5] The n-dimensional integral in this case can be converted into an $(n - 1)$-dimensional integral using the technique suggested by Margrabe [22].

large, it is convenient to assume that all the assets have the same current price and the same standard deviation and are equicorrelated. This symmetry makes the analysis simpler.[6]

To compute the value of the quality option, we use known results for the multivariate normal distribution. The option on the minimum of the n assets can be analyzed in terms of order statistics. Apart from a discount factor, the option (with zero strike price) on the minimum of the n assets is equal to the expected value of the lowest extreme order statistic. In the case of the standardized multivariate normal distribution, the moments of the order statistics when the variables are independent have been published by Ruben [26] and Teichroew [28]. Owen and Steck [24] developed simple expressions for the moments of the order statistics in the case of an equicorrelated multivariate normal distribution in terms of the moments of the order statistics of the uncorrelated multivariate normal distribution.[7] Owen and Steck [24] give the numerical values of the first four moments of the extreme (highest) order statistic from the multivariate normal distribution with zero correlation for values of n ranging from 2 to 50. The moments of the lowest order statistic are obtained by changing the sign of the odd order moments in their Table I.

Let $X^{(i)}$ denote the ith order statistic from the multivariate normal distribution in the uncorrelated case where

$$X^{(1)} \geq X^{(2)} \geq \cdots \geq X^{(n)}.$$

Let $Z^{(i)}$ be the corresponding order statistics in the case of the equicorrelated multivariate normal distribution with common correlation coefficient, ρ. Then Owen and Steck show that the moments of the $X^{(i)}$ and the $Z^{(i)}$ are related as follows:

$$E(Z^{(i)}) = \sqrt{(1-\rho)}E(X^{(i)}), \tag{8a}$$

$$E[Z^{(i)} - E(Z^{(i)})]^2 = \rho + (1-\rho)[E(X^{(i)} - E(X^{(i)}))^2], \tag{8b}$$

$$E[Z^{(i)} - E(Z^{(i)})]^3 = (1-\rho)^{3/2}[E(X^{(i)} - E(X^{(i)}))^3], \tag{8c}$$

$$\begin{aligned} E[Z^{(i)} - E(Z^{(i)})]^4 = {} & 3\rho^2 + 6\rho(1-\rho)E[X^{(i)} - E(X^{(i)}))^2] \\ & + (1-\rho)^2 E[(X^{(i)} - E(X^{(i)}))^4]. \end{aligned} \tag{8d}$$

From these relations, we can find the moments in the equicorrelated case.

In the case of the quality option with n deliverable assets, we are seeking the expected value of the lowest order statistic in the case of an equicorrelated multivariate normal distribution. We can use the moments of the equicorrelated multivariate lognormal case to obtain a series expansion for this expectation. The essential idea is that, if we know the central moments of a random variable

[6] We suspect that this symmetry should simplify the integrals developed by Johnson [16], but we do not pursue this point here.

[7] Afonja [1] provides expressions for the moments of the order statistics in the general case of a multivariate normal distribution with unequal means, variances, and correlations. However, his expressions involve multiple integrals, and it would probably be as simple in the general case to evaluate the options directly using Johnson's [16] results.

X, then the expected value of a function g of X can be obtained as follows:[8]

$$E\{g(X)\} = g\{E(X)\} + \frac{g^{ii}}{2}\{E(X)\}\operatorname{var} X + \frac{g^{iii}}{6}\{E(X)\}\mu_3(X)$$

$$+ \frac{g^{iv}}{24}\{E(X)\}\mu_4(X) + \cdots. \tag{9}$$

Numerical Examples: The standard assumptions used for our numerical examples are

$$T = 0.75 \text{ years},$$
$$R = 10\% \text{ per annum},$$
$$\sigma_i = 25\% \text{ per annum}, \quad 1 \le i \le n,$$
$$\rho_{ij} = 0.95, \quad 1 \le i \le j \le n,$$
$$A_i(t) = 40, \quad 1 \le i \le n.$$

With just one deliverable asset, the futures price under the above assumptions is 43.1154. This futures price is used as a benchmark against which to measure the impact of the quality option. When there are just two deliverable assets, the futures price given by equation (4)[9] is 41.9380, so that the ratio relative to the one asset benchmark futures price is 97.27 percent. We use this ratio as the metric to illustrate the significance of the quality option.

In order to compute the impact of the quality option when there are several deliverable assets, we used the isomorphism between the lowest order statistic and the call option on the minimum of the n assets with zero strike price. Equation (9) is used to obtain the order statistics of the multivariate normal distribution in terms of the known order statistics of the multivariate lognormal distribution. The results are given in Table I. As a check on our procedure, the computed results for $n = 2$ and $n = 3$ agreed to three decimal places with the exact results obtained by independent methods. For $n = 4$, we used Monte Carlo simulations to confirm the accuracy of the approximation.

Note from Table I that the impact of the quality option increases as the number of deliverable assets increases. Thus, for five deliverable assets, the impact is twice as large as if there were just two assets. As the correlation coefficient increases, the impact of the quality option is reduced. However, note that, even when the correlation coefficient is as high as 0.995, the impact of the quality option is nontrivial, especially if there is a large set of deliverable assets.

II. The Timing Option

In several types of futures contracts, the short position has some flexibility regarding the timing of the actual delivery. For example, delivery may take place

[8] This expression can be obtained from a Taylor series expansion. The symbols $\mu_3(x)$ and $\mu_4(x)$ denote the third and fourth central moments of X.

[9] We use the formulas given by Stulz [27] to evaluate the European call option in equation (4).

Table I

Relationship of Quality Option to the Number of Deliverable Assets[a]

Number of Deliverable Assets (n)	Correlation Coefficient = 0.95: Value ($) of Option on the Minimum of the n Assets Strike Price = 0	Correlation Coefficient = 0.95: Ratio of Futures Price to Reference Futures Price	Correlation Coefficient = 0.995: Value ($) of Option on the Minimum of the n Assets Strike Price = 0	Correlation Coefficient = 0.995: Ratio of Futures Price to Reference Futures Price
2	38.908	97.27	39.654	99.14
3	38.374	95.94	39.483	98.71
4	38.033	95.08	39.372	98.43
5	37.786	94.46	39.292	98.23
10	37.100	92.75	39.066	97.66
20	36.511	91.28	38.869	97.17
30	36.201	90.50	38.765	96.91
40	35.994	89.99	38.695	96.74
50	35.840	89.60	38.643	96.61

[a] Assets are assumed to have an equicorrelated multivariate lognormal distribution. All assets are assumed to have same current price (=$40) and same standard deviation (=25% per annum). Two basic situations are considered: first, the correlation coefficient = 0.95; second, the correlation coefficient = 0.995.

on any business day within the delivery month. In this section, we examine how the timing option affects the futures price and we analyze the interaction between the timing option and the quality option. We again assume a frictionless market and that the underlying assets pay no dividends and have no associated service flow. Initially, we analyze the timing option when there is only one deliverable asset and obtain a simple expression for the futures price in this case. When there is more than one deliverable asset, there is an interaction between the timing option and the quality option. We discuss this relationship and develop a numerical procedure to evaluate the timing option in this case. Some numerical examples are presented to illustrate the size of the timing option and to indicate the circumstances where it is likely to have the greatest value.

A. Timing Option with One Deliverable Asset

To discuss this case, it is convenient to introduce some additional notation. We assume that the futures contract expires at time $(t + T_3)$. The short position can deliver the underlying asset at any time during the interval $[t + T_1, t + T_3]$. Let us denote the time that delivery occurs by $[t + T_2]$. We have

$$T_1 \le T_2 \le T_3.$$

The current price of the deliverable asset is $A_1(t)$, and the riskless interest rate is R percent per annum (continuously compounded). We assume that the choice of the delivery time is made by the short position in such a way as to maximize his or her benefits under the contract. We also assume that forward prices and futures prices are equal as before. Let $F(t, [A_1], (t + T_1)) = F_1$ denote the futures price, at time t, of a futures contract where delivery takes place at time $(t + T_1)$. Thus, there is no timing option in this case. Let $F(t, [A_1], (t + T_2)) = F_2$ denote the futures price, at time t, of a futures contract where delivery can occur at time $(t + T_2)$; $T_1 \leq T_2 \leq T_3$. Thus, F_2 corresponds to the case where there is a timing option.

We can show that, to prevent arbitrage, F_2 must equal F_1. First, suppose that $F_2 > F_1$. Consider the following investment strategy at time t. Take a long position in the contract with no timing option and a short position in the contract with the timing option. Note that the short position decides when the asset is to be delivered under the contract with the timing option. At time $(t + T_1)$, the long position is worth

$$A_1(t + T_1) - F_1.$$

The short position can be liquidated at time $(t + T_1)$, at which time it is worth

$$-(A_1(t + T_1) - F_2).$$

The net profit on the strategy is

$$F_2 - F_1,$$

which by assumption is strictly positive. This strategy yields a strictly positive payoff with zero initial investment and is accordingly in conflict with the no-arbitrage principle. Hence, our assumption that $F_2 > F_1$ is untenable, and so

$$F_2 \leq F_1. \tag{10}$$

To prove[10] the converse, assume that $F_1 > F_2$. Consider the following strategy. At current time, t, an investor goes long one futures contract with the timing option and simultaneously goes short a futures contract which has no timing option and calls for delivery at time $(t + T_1)$. At the same time, the investor purchases one unit of the underlying asset (planning to sell it at time $(t + T_1)$). He or she also goes short one unit of the asset (planning to liquidate the position at time $(t + T_2)$) when delivery occurs under the futures contract with the timing option.

The time pattern of cash flows under this portfolio strategy is illustrated in Table II.

The table illustrates the cash flows to the investor under the assumed strategy. At time $(t + T_1)$, the investor takes delivery of the asset under the futures contract without the timing option and simultaneously sells the asset. The proceeds from this are $(F_1 - A_1(t + T_1))$ and $A_1(t + T_1)$, i.e., F_1. Assume that the investor places this amount in a risk-free account until time $(t + T_2)$. At time $(t + T_2)$, it will have accumulated to

$$F_1 \exp(R[T_2 - T_1]).$$

[10] The author thanks Mark Rubinstein for suggestions that improved an earlier proof of this result.

Table II

Portfolio Strategy to Illustrate That F_2 Must Be Greater Than or Equal to F_1, Where F_2 Is Futures Price with Timing Option and F_1 Is Futures Price without Timing Option[a]

		Cash Flow to Investor	
Action	Time t	$t+T_1$	$t+T_2$
Long Futures (with Timing Option), Delivery at $(t+T_2)$	0	—	$A_1\ (t+T_2)-F_2$
Short Futures (without Timing Option), Delivery at $(t+T_1)$	0	$F_1-A_1\ (t+T_1)$	
			$F_1e^{R(T_2-T_1)}$
Buy Asset (Sell at $t+T_1$)	$-A_1\ (t)$	$A_1\ (t+T_1)$	
Short Asset (Liquidate at $t+T_2$)	$+A_1\ (t)$	—	$-A_1\ (t+T_2)$
Total	0	0	$F_1e^{R(T_2-T_1)}-F_2$

[a] $A_i\ (t+T_i)$ is the price of the deliverable asset 1 at time $(t+T_i)$, where $i = 1, 2$.

At time $(t + T_2)$, the short position in the futures contract with the timing option delivers the asset. Simultaneously, the investor liquidates the short asset position. The net value of these two transactions to the investor is

$$[A_1(t + T_2) - F_2)] - A_1(t + T_2) = -F_2.$$

Hence, the total value of the investor's portfolio at time $(t + T_2)$ is

$$F_1e^{R(T_2-T_1)} - F_2.$$

By our initial assumption, this is strictly positive (since R is assumed positive). However, this result violates the no-arbitrage principle, so our initial assumption is untenable.

Hence,

$$F_1 \leq F_2. \tag{11}$$

Combining (10) and (11), it must be that $F_1 = F_2$. Thus, when there is just one deliverable asset, the timing option has no value[11] and the short position should optimally deliver the asset at the first permitted opportunity.

Recall that F_1 can be obtained from the cost-of-carry model so that

$$F_1 = A_1(t)\exp[RT_1]. \tag{12}$$

When there is just one deliverable asset, this is also equal to F_2. The intuition is that the seller is committed to deliver the asset during the delivery window and that there are no gains to waiting since the asset pays no dividend.

B. *Timing Option with Two Deliverable Assets*

When the terms of the futures contract permit more than one deliverable asset, there is an interaction between the timing option and the quality option. In this

[11] It has no value when the comparison is made with the contract that calls for delivery at time $(t + T_1)$. Of course, the forward price for the contract with the timing option, F_2, will generally be less than F_3 when there is just one asset.

case, it is no longer true that the futures price when there is a timing option is equal to the futures price assuming earliest delivery. It is instructive to see where the arbitrage arguments used for one deliverable asset fail. The first stage of the argument leading to equation (10) still goes through. To see where the argument leading to inequality (11) breaks down, assume that there are two deliverable assets denoted by i and j. Assume that $F_1(t, [A_i, A_j], (t + T_1)) = F_1$ denotes the current futures price of a contract which involves the delivery of the asset with the lowest price at time $(t + T_1)$. Let $F(t, [A_i, A_j], (t + T_2)) = F_2$ denote the futures price of the contract where delivery of either asset i or asset j can take place at any time during the interval $[t + T_1, t + T_3]$ at the option of the short position.

When there is uncertainty as to which asset will be delivered, then the argument leading to equation (11) no longer goes through. It could happen that the first asset is the cheapest to deliver under the futures contract without the timing option whereas the second asset is the cheaper at time $(t + T_2)$. Under these circumstances it is not possible to guarantee a riskless profit to the previous strategy. Hence, when there is a timing option and more than one deliverable asset, there is an interaction between the timing option and the quality option.

To explore the magnitude and significance of the timing option when there are two underlying deliverable assets, we have developed a numerical algorithm. This involves an extension of the Cox, Ross, and Rubinstein [8] approach to option pricing developed recently by Boyle [4]. A binomial lattice framework is constructed to value contingent claims the payoffs of which are a function of two state variables. In our case, the state variables are the prices of the two deliverable assets. The method proceeds by obtaining the value of the contingent claim on the boundary and working recursively backwards through the lattice to arrive at its current value. The method lends itself to the incorporation of any type of early exercise policy. In the case of a futures contract with a timing option, we assume that the asset is delivered when it is to the advantage of the short position to do so.

Suppose that F_2 is the current futures price of the contract with a timing option when either asset i or asset j can be delivered by the short position. The value of a long position in this contract at time $(t + T_2)$ is

$$\min[[A_i(t + T_2), A_j(t + T_2)] - F_2].$$

Note that this value can be either positive or negative. The early exercise strategy conducted by the short position is designed to minimize the value of the contract held by the long position. Thus, at each node of the lattice, delivery is assumed to occur if it minimizes the contract value. The equilibrium futures price, F_2, is obtained by recognizing that the initial contract value is zero. The solution design recognizes that the short position will deliver the minimum of the two assets at the time during the permitted interval when it is in its best interest to do so.

Table III provides values of futures prices when there are two deliverable assets and a timing option. It is assumed that delivery can take place during the last three weeks of the life of the contract. We have computed futures prices for different contract maturities and different assumptions as to the correlation

Table III

Futures Prices with and without Timing Option When There Are Two Deliverable Assets[a]

	$T_3 = 6$ weeks $T_1 = 3$ weeks	$T_3 = 13$ weeks $T_1 = 10$ weeks	$T_3 = 39$ weeks $T_1 = 36$ weeks
Futures Price	$	$	$
	Correlation Coefficient = 1.00		
F_1	40.23	40.78	42.87
F_2	40.23	40.78	42.87
F_3	40.46	41.01	43.12
	Correlation Coefficient = 0.90		
F_1	39.80	39.98	41.28
F_2	39.74	39.94	41.27
F_3	39.85	40.10	41.45
	Correlation Coefficient = 0.50		
F_1	39.27	38.99	39.32
F_2	39.01	38.84	39.23
F_3	39.09	38.97	39.40

[a] Both assets are currently worth $40 and have the same standard deviation; 25% per annum. The timing option is available during the last three weeks of each contract. The futures prices are as follows:

F_1: futures price when delivery occurs at $t+T_1$.
F_2: futures price when delivery occurs during $[t+T_1, t+T_3]$ at the option of the short; i.e., there is a timing option.
F_3: futures price when delivery occurs at $t+T_3$.

between the two assets. The main conclusion from Table III is that the impact of the timing option on the equilibrium futures price is small. As the correlation between the two assets increases, the futures price of the contract with the timing option, F_2, moves closer to the futures price for the single asset case, F_1. The impact of the timing option is small—smaller than the impact of the quality option. Note that, when the correlation between the assets is less than one, the futures price for the contract with the timing option, F_2, is less than the futures price assuming delivery at the first permitted date, F_1. Note that F_2 is also less than the futures price assuming delivery at the last permitted date, F_3.

It is important to emphasize that the model developed here does not apply directly to the Treasury bond futures contracts. In the case of the Treasury bond futures contract, the underlying assets make coupon payments. In addition, stochastic interest rates drive a wedge between futures prices and forward prices, and it is slightly inconsistent to maintain that they are equal in this situation. Empirical results (Hemler [14]) indicate that participants in the Chicago T-Bond futures contract tend to postpone delivery toward the end of the delivery month.

III. Conclusion

In this paper we have explored the price implications of some of the delivery options that are often found in futures contracts. In the case of the quality option, a method was developed for analyzing the significance of this option when there are several deliverable assets. Numerical simulations were carried out to illustrate the significance of the quality option when there are several deliverable assets. Even when the underlying assets have highly correlated returns, this option continues to have nontrivial value. The timing option was also analyzed and found to be much less important. Some aspects of the interaction between the timing option and the quality option were analyzed.

The present analysis applies to the situation where the underlying deliverable assets follow a joint lognormal distribution. Hence, it is more applicable to commodity futures than to interest rate futures. An important area of further research would be the analysis of Treasury bond futures contracts. Recently Kane and Marcus [19] used a simulation approach to analyze delivery options in this case and concluded that the delivery option had a significant impact on futures prices. For the parameter values used, they concluded that the delivery option resulted in a two to six percent reduction in the futures price.[12]

Clearly, these delivery options are of economic significance in many types of futures contracts. The quality option has an impact on the futures price, and, thus, it will also effect hedging strategies. Further extensions of this work could tackle some of these issues. It would also be interesting to provide an empirical examination of the results developed in the present paper. These extensions are left for future research.

[12] Hemler [14] produces somewhat lower values for the quality option in the case of Treasury bond futures contracts.

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