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Source: *The Journal of Finance*, Vol. 43, No. 4 (Sep., 1988), pp. 881-892
Published by: Blackwell Publishing for the American Finance Association
Stable URL: http://www.jstor.org/stable/2328141
Accessed: 26/11/2009 07:40

THE JOURNAL OF FINANCE • VOL. XLIII, NO. 4 • SEPTEMBER 1988

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K. C. JOHN WEI*

ABSTRACT

This study shows that the competitive-equilibrium version of the APT may be extended to develop an exact model if idiosyncratic risks obey the Ross separating distribution. The results indicate that one only need add the market portfolio as an extra factor to the factor model in order to obtain an exact asset-pricing relation. Thus, this study presents an extension and integration of the CAPM and APT. The "empirical" APT is also generalized to allow for some factors to be omitted from the econometric model employed to test the theory. The developed model is extremely robust and may be reduced to the CAPM or expanded to approximate Ross's APT depending upon the number of omitted factors. Further, the importance of the market portfolio is shown to be a monotonic increasing function of the number of omitted factors. Finally, the study demonstrates that, in a finite economy, the pricing-error bound of the Ross APT in a correlated-residuals factor structure is an increasing function of the absolute value of market-residual beta, rather than the weight of the asset in the market portfolio as is the case of uncorrelated factor residuals. However, under the normality assumption, the pricing error becomes an extra component related to the market-portfolio factor, and the exact asset-pricing relation is once again obtained.

THE ARBITRAGE PRICING THEORY (APT), originally introduced by Ross [18, 19] and later extended by Huberman [14], Chamberlain and Rothschild [3], Ingersoll [16], Connor [6], Chen and Ingersoll [4], and numerous other researchers, has recently attracted considerable attention as a testable alternative to the Capital Asset Pricing Model (CAPM) of Sharpe [25] and Lintner [17]. The APT states that, under certain assumptions, the single-period expected return on any risky asset is approximately linearly related to its associated factor loadings (i.e., systematic risks). The assumptions that are generally employed in the derivation of the APT are that (a) all investors exhibit homogeneous expectations that the stochastic properties of capital asset returns are consistent with a linear structure of k factors, (b) either there are no arbitrage opportunities in the capital markets or the capital markets are in competitive equilibrium, and (c) the number of securities in the economy is either infinite or so large that the law of large numbers may be applied. An additional assumption necessary to test the theory correctly is that (d) the number of factors, k, is known in advance or can be correctly estimated by the investigator.

* Department of Finance, School of Business Administration, University of Miami (Florida). The author gratefully acknowledges the kind and helpful comments and suggestions of Carolyn Carroll, Chi-fu Huang, Cheng F. Lee, Keith Womer, Matthew Wong, and especially Steve Pruitt and two anonymous referees of the Journal. The usual disclaimer applies.

Of these four assumptions, the two that have generally been felt to be the most restrictive are the assumptions of an infinite number of securities and the previously known number of factors affecting security returns. The assumptions that asset returns are generated by a factor structure and that no arbitrage opportunities exist (or that capital markets exist in a state of competitive equilibrium) are typically regarded as acceptable approximations to reality. Connor [6] employs a competitive-equilibrium assumption to show that the elimination of the infinite-securities assumption does not change the pricing relation if the market portfolio is well diversified in a given factor structure. Chen and Ingersoll [4] reach the same conclusion provided that a well-diverisified portfolio exists in a given factor structure and this portfolio is the optimal portfolio for at least one utility-maximizing investor. More specifically, the pricing relation of the APT, given either of these diversified-portfolio assumptions, is exact in a finite economy.

Unfortunately, as noted by Connor [6, p. 28], the assumption that the market portfolio is well diversified in a given factor structure is very restrictive in a finite economy since asset supplies must exist in exactly the correct proportion for their nonsystematic risks to net to zero in the market portfolio. The assumption is also nongeneric since even a minor perturbation of asset supplies would almost certainly destroy the net-zero position. Similarly, the assumption of the existence of a well-diversified portfolio that is at least one utility-maximizing investor's optimal portfolio in a given factor structure also appears too restrictive. These problems, combined with the fact that the assumption of a known number of factors in empirical tests of the APT is also not likely to be satisfied, have led some researchers to question the merits of the APT from both theoretical and empirical standpoints. (See, for example, Shanken [23, 24].)[1]

The present paper represents an extension and integration of Connor's [6], Cragg and Malkiel's [8], Dybvig's [9], and Grinblatt and Titman's [11] contributions on the APT and Stapleton and Subrahmanyam's [26] contribution on the CAPM. The purposes of this paper are two-fold. First, it is shown that the competitive-equilibrium version of the APT may be extended to an exact model without reliance upon the assumption that the market portfolio is well diversified in a given factor structure, if idiosyncratic risks obey the Ross [20] separating distribution. The results demonstrate that one need only add the market portfolio as an extra factor in order to obtain an exact asset-pricing relation.[2] Thus, this unified asset-pricing theory combines both the "ex post" and "ex ante" aspects of the CAPM and APT into a testable equilibrium-pricing model. Second, the techniques developed in the paper are employed to generalize the Ross APT to allow for some factors to be omitted from an econometric model utilized in tests of the theory. Thus, many earlier theoretical and empirical limitations of the Ross APT are effectively sidestepped.

Since the APT does not define the number of factors prevailing in the economy, most previous tests of the APT have assumed that the number of factors either

[1] See also Dybvig and Ross [10] for different viewpoints about the testability of the APT.

[2] This result is consistent with that of Grinblatt and Titman [13] and Huberman, Kandel, and Stambaugh [15] on the relationship between the existence of the exact APT and the mean-variance efficiency, in that the market portfolio is assumed to be mean-variance efficient.

is known in advance or may be correctly estimated. Unfortunately, there is no clear consensus as to the true number of factors generating asset returns. Thus, this study also investigates the impact of omitting factors in empirical tests of the APT's pricing relation. It is found that, with all of the factors omitted, the generalized theory reduces to the CAPM and that, if the number of factors is correctly estimated, the generalized theory reduces to the Ross APT in an infinite economy. Finally, it is shown that the relative importance of the market portfolio in the pricing relation depends upon the number of factors omitted from the econometric model.

Essentially, the Ross APT pricing relation is merely an approximation. Recently, Dybvig [9] and Grinblatt and Titman [11] have investigated the explicit-upper-error bound of the theory in a finite economy characterized by an uncorrelated-residuals factor structure. They found that the pricing error is small relative to the measurement error in the expected returns. This paper generalizes their results to allow for a correlated-residuals factor structure. The results show that the new pricing-error bound is an increasing function of the absolute value of market-residual beta. However, under the normality assumption, the pricing error becomes an extra component related to the market-portfolio factor and an exact asset-pricing relation is achieved, a finding consistent with the model derived from the Ross separating-distribution assumption.

The paper is organized as follows. In Section I, Ross's APT and Connor's competitive-equilibrium version of the APT are briefly reviewed. The Connor competitive-equilibrium version of the APT is then extended to develop an exact asset-pricing theory by assuming that idiosyncratic risks obey the Ross separating distribution. The developed theory is then generalized to allow for some relevant factors to be omitted from an econometric model employed to test the theory. In Section II, Dybvig's and Grinblatt and Titman's explicit-error bound of the APT is extended to a world of correlated factor residuals. Finally, an exact asset-pricing relation with the pricing error as an extra component is derived under the normality assumption. The paper concludes with a summary of the major findings and their implications.

I. The Ross APT, Competitive-Equilibrium Versions of the APT, and the APT with Some Factors Omitted from an Econometric Model

In this section, Ross's APT and Connor's equilibrium version of the APT are briefly reviewed. The equilibrium version of the APT is then extended to the case where the market portfolio is not well diversified in a given factor structure. Finally, it is shown that the idiosyncratic risks of the assets ($\tilde{e}_i$s) and the market portfolio ($\tilde{e}_m$) may be viewed as hidden factors when some of the true factors are omitted from the econometric model.

A. The Ross APT

The APT begins with the assumption that asset returns are generated by a linear k-factor model

$$\tilde{R}_i = E_i + b_{i1}\tilde{f}_1 + \cdots + b_{ik}\tilde{f}_k + \tilde{e}_i, \qquad i = 1, \cdots, n, \tag{1}$$

or, in vector notation, $\tilde{R} = E + B\tilde{f} + \tilde{e}$, where E_i is the mean return on asset i, $\tilde{f}$ are factor scores, and $\tilde{e}_i$ is an error term. It is assumed that $\tilde{f}$ are uncorrelated both with each other and with $\tilde{e}$ and that the means of $\tilde{f}$ and $\tilde{e}$ are zero. Ross [18, 19] has shown that, if the $\tilde{e}_i$ are uncorrelated with each other and have bounded variances (i.e., a "strict" factor structure applies) and if there are no arbitrage opportunities, then, as n approaches infinity, there exist numbers $\delta_0, \delta_1, \cdots, \delta_k$ such that

$$E_i \approx \delta_0 + \delta_1 b_{i1} + \cdots + \delta_k b_{ik}, \tag{2}$$

where $\approx$ represents "approximately equals" and δ_0 is the riskless rate of return or the return on a zero-beta asset.

Chamberlain and Rothschild (CR) [3] and Ingersoll [16] have extended Ross's result by showing that equation (2) holds even for an "approximate" factor structure. In an approximate factor structure, it is assumed that the $\tilde{e}_i$ in equation (1) are correlated with each other and that the eigenvalues of the covariance matrix of $\tilde{e}$ are uniformly bounded from above by some finite number. The notion of an approximate factor structure seems to be a significantly weaker restriction on the return-generating process than the Ross strict factor structure. However, Grinblatt and Titman [12] illustrate that any infinite economy satisfying the CR approximate factor structure may be transformed into another infinite economy satisfying the Ross strict factor structure in a manner that does not alter the characteristics of investors' portfolios. In other words, a strict factor structure is equivalent to an approximate factor structure in an infinite economy, and vice versa.

B. Competitive-Equilibrium Versions of the APT

A competitive equilibrium consists of a set of portfolios such that all portfolios are budget-constrained optimal for every investor and security supply equals security demand. In a competitive equilibrium, there exists an exactly linear pricing relation in each asset's factor betas or sensitivities (i.e., equation (2) holds exactly) if the following assumptions hold:[3]

- (A1) There exists an infinite number of investors, all of whom are risk averse and possess von Neumann-Morgenstern utility functions.
- (A2) There exists a finite number of risky assets with bounded variances the returns of which obey the linear k-factor model of (1), with k known and $(B'B)$ nonsingular, and the return covariance matrix of which is nonsingular.
- (A3) There exists a riskless asset.
- (A4) All the factors may be represented by traded assets.
- (A5) $\mathrm{E}[\tilde{e} \mid f] = 0$, $\mathrm{E}(\tilde{f}) = 0$, $\mathrm{E}[\tilde{f}'\tilde{f}] = I$, and $\mathrm{E}[\tilde{e}\tilde{e}' \mid f] = \Omega$, where Ω need not be diagonal and may be singular.[4]

[3] The following assumptions are similar to those used by Connor [6] and Connor and Korajczyk [7]. In addition, assumption (A4) has also been employed by Chamberlain [2] and Admati and Pfleiderer [1].

[4] If the matrix Ω is nonsingular, it is not possible completely to eliminate idiosyncratic risk from portfolios with a finite number of assets. See Connor [6, p. 18] for more detail.

(A6) The market portfolio has no idiosyncratic risk in the given factor model of (1).

Throughout the paper, we define a portfolio as well diversified if it has no idiosyncratic risk in a given factor model. The purpose of this subsection is to study equilibrium factor pricing when the market portfolio has idiosyncratic risk in the given factor model of (1). This requires replacing assumption (A6) with either (A6a) or (A6b).[5]

(A6a) For each asset i, there exists a number b_{im} such that, for all realizations of the market portfolio's idiosyncratic risk $\tilde{e}_m$,

$$\tilde{e}_i = b_{im}\tilde{e}_m + \tilde{u}_i, \tag{3}$$

with $\tilde{u}_i$ distributed independently of $\tilde{e}_m$.

(A6b) The vector of factors $\tilde{f}$ is distributed independently of (each other and of) the vector of idiosyncratic residuals $\tilde{e}$ and $\mathrm{E}(\tilde{u}_i \mid e_m) = 0$ in equation (3).

Either assumption (A6a) or (A6b) guarantees that the pure idiosyncratic risks, $\tilde{u}$s, obey the "separating distribution" in the Ross [20] sense (and, of course, they are also "increasing risk" in the Rothschild-Stiglitz [21] sense). Either assumption is analogous to that employed by Stapleton and Subrahmanyam [26] in deriving the CAPM from the market model. Neither assumption includes any parameter that is associated with investors' preferences; nor are they explicitly related to the normal distribution between $\tilde{e}_i$ and $\tilde{e}_m$.

The main result in the paper is now presented in the following theorem.

THEOREM 1: *Suppose that assumptions* (A1) *to* (A5) *and either assumption* (A6a) *or* (A6b) *hold. Then,*

(i) *all investors hold well-diversified portfolios in the sense of equation* (6) *below,*
(ii) *there exists a $k + 2$-fund-separation theorem, and*
(iii) *the extended-equilibrium version of the APT must hold exactly in competitive equilibrium:*

$$E_i = \delta_0 + \delta_1 b_{i1} + \cdots + \delta_k b_{ik} + \delta_m b_{im}, \tag{4}$$

with

$$\delta_m > 0. \tag{5}$$

Proof: The result of equation (5) will be shown in equation (14). We show the other results as follows. Substituting equation (3) into equation (1) yields[6]

$$\tilde{R}_i = E_i + b_{i1}\tilde{f}_1 + \cdots + b_{ik}\tilde{f}_k + b_{im}\tilde{e}_m + \tilde{u}_i. \tag{6}$$

[5] The independence assumption (A6a) or (A6b) is required since the assumpitons that $\mathrm{E}(\tilde{u}_i \mid e_m) = 0$ and that $\mathrm{E}(\tilde{e}_i \mid f) = 0$, for all i, are not sufficient to obtain $\mathrm{E}(\tilde{u}_i \mid f, e_m) = 0$. Furthermore, the independence assumption (A6a) implies that $\mathrm{E}(\tilde{u}_i \mid e_m) = 0$, for all i, which is consistent with Ross's separating-distribution assumption.

[6] Notice that the b_{ik} in equation (6) is identical to those in equation (1) and that b_{im} in equation (6) is identical to that in equation (3). b_{im} in equation (6) is defined as $\mathrm{cov}(\tilde{R}_i, \tilde{e}_m)/\mathrm{var}(\tilde{e}_m)$, which reduces to $\mathrm{cov}(\tilde{e}_i, \tilde{e}_m)/\mathrm{var}(\tilde{e}_m)$ given the fact that $\tilde{e}_m$ is uncorrelated with $\tilde{f}$.

Recognizing the fact that $E(\tilde{u}_i \mid f, e_m) = \tilde{u}_m = 0$ in equation (6) from assumption (A5) and either assumption (A6a) or (A6b), the first assertion that all investors hold well-diversified portfolios in the sense of the factor model of (6) is immediately obtained from Connor's [6] Remark 1. The $k + 2$-fund-separation theorem is obtained from the separating distribution of equation (6) (see Ross [20] for greater detail) or Connor's Corollary 2.1. The result of equation (4) that the expected returns are beta linear is obtained from Conner's Theorem 3.[7] Q.E.D.

An obvious corollary to Theorem 1, with the linear repackaging of the factors and $\tilde{e}_m$ into possibly correlated factors and the market return, is presented below without need for proof.[8]

COROLLARY 1: *Given the assumptions in Theorem* 1, *the extended-equilibrium version of the APT, equation* (4), *may also be represented as*

$$E_i = \delta_0 + \delta_1' b_{i1}' + \cdots + \delta_k' b_{ik}' + \delta_m' b_{im}', \tag{7}$$

where $\{b_{i1}', \cdots, b_{ik}', b_{im}'\}$ *are the multiple-regression betas of the asset's return* $\tilde{R}_i$ *against the k factors and the market return:*[9]

$$\tilde{R}_i = E_i + b_{i1}'\tilde{f}_1 + \cdots + b_{ik}'\tilde{f}_k + b_{im}'\tilde{r}_m + \tilde{u}_i, \tag{8}$$

where $\tilde{r}_m = \tilde{R}_m - E_m$, *the mean adjusted market return.*

The mutual-fund-separation theorem in Theorem 1 states that investors are *indifferent* between a mutual-fund portfolio and any other optimal portfolio. The separating portfolios are the riskless asset, the overall market, and k portfolios that are perfectly correlated with the pricing factors. Assertion (i) and the $k + 2$-fund separation of Theorem 1 that lies behind either equation (4) or (7), which indicates that the pure idiosyncratic risk $\tilde{u}_i$ is not priced, has simple intuition: no risk-averse investor will want to hold $\tilde{u}$ risk if it is not priced. This could only occur in equilibrium if the net supply of $\tilde{u}$ risk were zero, which, of course, it is, since such risk is not contained in the market portfolio. Thus, no investor is forced to bear $\tilde{u}$ risk in order to equate supply and demand. In this sense, $\tilde{e}_m$ is merely another factor in positive net supply.

Theorem 1 and Corollary 1 generalize the normality models in Cragg and Malkiel [8] and Grinblatt and Titman [11] to allow for correlated residuals and (a few) non-normal distributions. They also generalize the equilibrium model in Connor [6] to allow for nonzero market residuals. Either equation (4) or (7) states that, in competitive equilibrium, if $\tilde{e}_m$ is nonzero but the $\tilde{e}_i$ obey the Ross separating distribution, then expected returns should be *exactly* linearly related to their factor loadings as well as the residual-market systematic risk b_{im} in a finite economy. Equations (4) and (7) unify the APT and CAPM in that they employ the information contained within the market portfolio in addition to the

[7] Chen and Ingersoll's [4] arguments may be applied to obtain the result given in equation (4) by replacing the competitive-equilibrium assumption with the assumption that the market portfolio is the optimal portfolio for at least one utility maximizer. Assumptions (A3) and (A4) are not required in this case.

[8] The author thanks one of the referees for suggesting this corollary and for offering some of the interpretations discussed below.

[9] It can be shown that $b_{ij}' = b_{ij} - b_{im}b_{mj}$, $j = 1, \cdots, k$, and that $b_{im}' = b_{im}$.

information contained within the relevant factors. The resulting models recognize the role of the exogenous-factor/endogenous-aggregate covariances in the asset-pricing relation.

Comparing the unified asset-pricing theory of (4) with the Ross model of (2), we see that, in addition to an extra term in equation (4), the approximate sign in equation (2) is replaced by the equals sign. Notice that, even though the factor risk premiums, $\delta_1, \cdots, \delta_k$, may be positive or negative in either equation (2) or (4), the market-residual risk premium, δ_m, must be positive. Notice also that the unified theory reduces to the Ross APT in an infinite economy. If nonsystematic risk is completely diversified away in the market portfolio (however unlikely), i.e., $\tilde{e}_m$ is zero, then all of the b_{im}s are zero and equation (4) reduces to the Connor equilibrium version of the APT.

It might be better to note that the market is not sensitive to any factor that is in zero net supply. Such a "factor" will simply not be priced. In an infinite economy, such as the Connor model, the assumption that $\tilde{e}_m$ is zero is justified because of the diversification of $\tilde{e}_i$ risk over a large number of assets. Hence, in this case, idiosyncratic risk is in zero net aggregate supply and is not priced. In the infinite-economy Ross model, which makes no such assumption, assets "large" relative to the economy as a whole might exist. The idiosyncratic risk of these assets cannot be diversified away in the market portfolio, implying that some $\tilde{e}_i$s (and hence $\tilde{e}_m$) exist in positive net supply. Only the "large" assets associated with these idiosyncratic risk terms will be mispriced by the APT, and, as a consequence, Ross only obtains a finite bound for his pricing deviation, rather than the exact pricing relation established for every asset in both the present theory and the Connor theory.

Corollary 1 has some important implications related to Theorem 1. The results from both Theorem 1 and Corollary 1 indicate that the assumptions of the k-exogenous-factor return-generating process and the Ross separating distribution are equivalent to the assumption of the k-exogenous-factor plus an extra endogenous-market-factor return-generating process.[10] In other words, one need only add the market portfolio as an extra factor in order to obtain an exact asset-pricing relation. Consequently, one would expect that residual variance, skewness, or firm size should not enter the pricing relation although residual-market risk may.

Assuming the factor independence in assumption (A6b), then e_i and e_m may be viewed as hidden factors. This is illustrated as follows. Suppose that the true number of factors is k but that j factors (less than k) are assumed in the econometric model to test the theory:

$$\tilde{R}_i = E_i + b_{i1}\tilde{f}_1 + \cdots + b_{ij}\tilde{f}_j + \tilde{e}_i^*, \tag{9}$$

or, in vector notation, $\tilde{R} = E + B^*\tilde{f}^* + \tilde{e}^*$. We now investigate the impact of omitted factors from the econometric model employed to test the theory on the asset-pricing relation in the following corollary.

[10] Including the market portfolio as one of the factors (see, for example, Chen, Roll, and Ross [5]) is consistent with the exact asset-pricing theory derived in this paper.

COROLLARY 2: *Assume that an asset's returns follow a k-factor structure of* (1) *where assumptions* (A1) *to* (A5) *hold but a j-factor* ($j < k$) *structure of* (9) *is assumed in an econometric model utilized to test the theory. It is further assumed that assumption* (A6b) holds, with the new residuals $\tilde{e}_i^*s$ and $\tilde{e}_m^*$ *in equation* (9) *replacing old residuals* $\tilde{e}_i s$ *and* $\tilde{e}_m$ *in equation* (3). *That is,*[11]

$$\tilde{e}_i^* = b_{im}^*\tilde{e}_m^* + \tilde{u}_i^*, \quad with \quad \mathrm{E}(\tilde{u}_i^* \mid e_m^*) = 0. \tag{10}$$

Then, in competitive equilibrium, we have

(i) $E_i = \delta_0 + \delta_1 b_{i1} + \cdots + \delta_j b_{ij} + \delta_m^* b_{im}^*$, (11)
(ii) $\delta_m^* > 0$ *in* (11), *and*
(iii) δ_m^* *is an increasing function of the number of omitted factors.*

Proof: Substituting equation (10) into equation (9) yields

$$\tilde{R}_i = E_i + b_{i1}\tilde{f}_1 + \cdots + b_{ij}\tilde{f}_j + b_{im}^*\tilde{e}_m^* + \tilde{u}_i^*.$$

The conditions that $\tilde{u}_m^* = \mathrm{E}(\tilde{u}_i^* \mid f^*, e_m^*) = 0$ are obvious from assumption (A6b) of (10). Consequently, assertion (i) immediately follows from Theorem 1. Assertions (ii) and (iii) are proved in the Appendix. Q.E.D.

Clearly, equation (11) states that, with some factors omitted from an econometric model, an asset's expected return remains a linear function of the estimated factor loadings (in which the number of factors is less than the true number) and the market-residual beta (which depends upon the factor structure assumed). In addition, equation (11) states that the market-residual risk premium is positive and will decrease to a constant, δ_m, as the number of factors estimated in the econometric model utilized to test the theory increases.

Equation (11) also has some interesting implications. If all of the factors are omitted (i.e., a zero-factor model is assumed), equation (11) reduces to the CAPM (derived from the market model). In this case, the factor independence in assumption (A6b) is equivalent to a two-fund separating distribution, which immediately implies the intersection of the CAPM and a factor model. On the other hand, if the assumed or estimated number of factors is sufficient ($j = k$), then equation (11) expands to either the Ross APT approximation in an infinite economy or the unified asset-pricing theory of (4) in a finite economy. The market portfolio plays a preeminent role in the model of (11). As the number of omitted factors increases, the importance of the market portfolio increases.

The "empirical" APT of (11) is a generalization of the unified theory of (4). These equations differ in that, in addition to the testable hypothesis that the market factor may enter the pricing relation in equation (4), equation (11) offers an additional testable hypothesis that the market-residual risk premium is nonnegative and will decrease to a constant as the number of estimated factors increases.

[11] It is noteworthy that this modified assumption is necessary since this condition cannot be derived directly from assumptions (A5) and (A6b). This is an "ex post" assumption and should depend upon the assumed-factor model.

C. The Number of Factors: An Indirect Test

Since direct tests of the condition that $\tilde{e}_m = 0$ for the Connor exact APT in a finite economy are difficult (if not impossible),[12] the models proposed above offer an indirect testable alternative. That is, if a k-factor Connor APT ($k < n$) is accepted, then $\delta_m = 0$ in equation (4) may not be rejected. The rejection of $\delta_m = 0$ does imply the condition that $\tilde{e}_m = 0$ is rejected for the predetermined number of factors k. In other words, it implies that (a) k is insufficient, (b) nonsystematic risk is not completely diversified away in the market portfolio, or both. It does not, however, imply that the asset-pricing theory of either model (4) or (11) is rejected. For the unified or generalized asset-pricing theory to be rejected, we must find that extraneous variable(s) such as residual variance, firm size, and/or skewness enter into the pricing relation.

The testing procedure for the condition that $\tilde{e}_m = 0$ is as follows. First, a predetermined number of factors, say k, is assumed. Next, equation (11) is employed to test the hypothesis that $\delta_m^* = 0$. The smallest k (denoted by k^*) for $\delta_m^* = 0$ will be the number of factors that the condition of $\tilde{e}_m = 0$ holds. If k^* is unbounded, we may conclude that $\tilde{e}_m = 0$ does not hold in a reasonable k-factor structure. Alternatively, if δ_m^* approaches a positive constant after some k^*, we might conclude that the unified theory of (4) is accepted.

II. The Explicit-Error Bound and Exact Asset Pricing

This section details the explicit-error bound of Ross's APT in a finite economy. Since the Ross APT pricing relation is an approximation, i.e., one that prices "most" assets well but permits arbitrarily large deviations from exact pricing on a finite set of assets (see Shanken [24, p. 1191]), investigations concerning the explicit-upper-error bound for individual securities in various factor structures are warranted.

Assuming (a) a competitive equilibrium, (b) a known number of factors, and (c) an uncorrelated-residuals k-factor structure, Dybvig [9] and Grinblatt and Titman [11] have derived the explicit-upper-pricing-error bound (θ_i) of the Ross APT for each security i in a finite economy as follows:

$$|\theta_i| < \bar{R} \times x_i \times \mathrm{var}(\tilde{e}_i), \tag{12}$$

where $\bar{R}$ is the upper bound for the measure of representative relative-risk aversion, x_i is the proportion of total wealth represented by asset i, and $\mathrm{var}(\tilde{e}_i)$ is the variance of $\tilde{e}_i$ defined in equation (1). The annualized-upper-error bound of (12) calculated by Dybvig and Grinblatt and Titman is 0.04 percent and 0.2 percent, respectively. As explained in Dybvig and Ross [10], this error is small relative to the measurement error in expected returns, and the bound is valid even if all investor wealth is not held in the form of stocks. Grinblatt and Titman [11, equation (6)] show that the equality in equation (12) holds if normality is assumed.

[12] There is no existing statistic that may be used to test the hypothesis that the coefficient of determination in a regression is equal to one.

The result of equation (12) may be extended to a correlated-residuals factor structure as follows. From equation (3) in Grinblatt and Titman, we have

$$|\theta_i| = |-\text{cov}[U'(\tilde{W}), \tilde{e}_i]/\text{E}[U'(\tilde{W})]| \qquad \text{(from [11, equation (3)])}$$

$$= |-\text{cov}[U'(\tilde{W}^*) + WU''(\tilde{Z})\tilde{e}_m, \tilde{e}_i]/\text{E}[U'(\tilde{W})]|$$

(Taylor expansion)

$$\leq \bar{R}\,|\,\text{cov}[\tilde{e}_m, \tilde{e}_i]|$$

$$= \bar{R} \times |b_{im}| \times \text{var}(\tilde{e}_m), \tag{13}$$

where U' is the first-order derivative (i.e., marginal utility) and U'' the second-order derivative of utility function U with respect to terminal wealth $\tilde{W}$, W is initial wealth, $\tilde{W}^* = W(1 + E_m + b_m\tilde{f}_1 + \cdots + b_{mk}\tilde{f}_k)$, $\min(\tilde{W}, \tilde{W}^*) \leq \tilde{Z} \leq \max(\tilde{W}, \tilde{W}^*)$, and b_{im} is defined as in equation (6). The inequality in equation (13) follows from $\text{E}[\tilde{e}_i\,|\,g(f)] = 0$ and the definition of $\bar{R}$.

The new pricing-error bound of equation (13) is now dependent upon representative risk aversion ($\bar{R}$), market-residual beta (b_{im}), and market-residual variance ($\text{var}(\tilde{e}_m)$). Notice that this pricing-error bound is an increasing function of the absolute value of b_{im}, rather than the weight of the asset in the market portfolio, x_i, as is the case of uncorrelated-residuals factor structure.

If we further assume that terminal wealth ($\tilde{W}$) and each factor residual ($\tilde{e}_i$) are bivariate normally distributed, then, using Stein's [27] Lemma, one can show that the inequality in equation (13) will convert to an equality without the absolute operators and with $\bar{R}$ replaced by the Rubinstein [22] measure of relative-risk aversion (R). That is,

$$\theta_i = R \times \text{var}(\tilde{e}_m) \times b_{im} = \delta_m b_{im}, \tag{14}$$

where $R = -W\text{E}[U''(\tilde{W})]/\text{E}[U'(\tilde{W})] > 0$.

III. Conclusions

Broadly speaking, the APT emphasizes the role of the covariance between asset returns and exogenous factors, while the CAPM stresses the covariance between asset returns and the endogenous market portfolio. In this study, the positive aspects of each model are combined to derive a theory unifying both models. The approach is based upon Connor's competitive-equilibrium version of the APT and Ross's separating distribution. The derived results demonstrate that one need only add the market portfolio as an extra factor to the factor model in order to obtain an exact asset-pricing relation.

In addition to this derivation, we have also proved that the new approach may be applied to generalize the "empirical" APT with some factors omitted from the econometric model. This generalized theory is shown to be an integrated model of the CAPM and APT. If all factors are omitted, the new model reduces to the CAPM. When none of the factors is omitted, the new model becomes either the Ross APT in an infinite economy or the unified asset-pricing theory in a finite economy. The role played by the market portfolio depends upon the number of omitted factors. No ambiguity remains with our APT/CAPM-based approach since the omitted pervasive factors have been replaced by the market factor. The results also explore interesting empirical implications in the determination of the number of factors generating returns.

The techniques developed in this paper require only an "ex ante" factor/"ex post" aggregate-return-generating model, which is implied in the factor model and the Ross [20] separating-distribution assumption. Our techniques do not assume normally distributed returns; nor do they assume a well-diversified market portfolio as in Connor [6]. Further, the resulting model allows some factors to be omitted in an econometric model.

We have also discussed the pricing-error bound of Ross's APT in a finite correlated-residuals factor-structure economy. The new pricing-error bound in this case is an increasing function of the absolute value of market-residual beta, rather than the weight of the asset in the market portfolio as is the case of the uncorrelated-residuals factor structure. However, the pricing error becomes an extra market factor under the normality assumption. Further research will test the relationship between the importance of the market factor, and the number of omitted factors, and the number of factors generating asset returns using the results derived in this paper.

Appendix: Proof of Corollary 2

Assume that both equations (4) and (11) are true. Then, from these two equations and the definition of b^*_{im}, we have the following relationships:

$$\delta^*_m b^*_{im} = \delta_{j+1} b_{i,j+1} + \cdots + \delta_k b_{ik} + \delta_m b_{im}, \tag{A1}$$

$$b^*_{im} = \frac{[b_{i,j+1} b_{m,j+1} + \cdots + b_{ik} b_{mk} + b_{im}\mathrm{var}(\tilde{e}_m)]}{[b^2_{m,j+1} + \cdots + b^2_{mk} + \mathrm{var}(\tilde{e}_m)]}. \tag{A2}$$

Substituting (A2) into (A1) and rearranging yields

$$\delta^*_m = \frac{[\delta_{j+1} b_{i,j+1} + \cdots + \delta_k b_{ik} + \delta_m b_{im}]}{[b_{m,j+1} b_{i,j+1} + \cdots + b_{mk} b_{ik} + \mathrm{var}(\tilde{e}_m) b_{im}]} \times (b^2_{m,j+1} + \cdots + b^2_{mk} + \mathrm{var}(\tilde{e}_m)). \tag{A3}$$

Since δ^*_m is independent of individual asset information, the right-hand side of (A3) should be equal for each i. The only solutions to this condition are that

$$\delta_m = c \times \mathrm{var}(\tilde{e}_m), \tag{A4}$$

$$\delta_h = c \times b_{mh}, \qquad h = j+1, \cdots, k, \tag{A5}$$

where c is a constant. From equation (14), we know that

$$\delta_m = R \times \mathrm{var}(\tilde{e}_m), \tag{A6}$$

with $R > 0$. From (A4) and (A6), we have $c = R > 0$. Thus,

$$\delta^*_m = R[b^2_{m,j+1} + \cdots + b^2_{mk} + \mathrm{var}(\tilde{e}_m)] > 0. \tag{A7}$$

Furthermore, the term in the right-hand bracket of equation (A7) is positive and is a decreasing function of j. Thus, δ^*_m is an increasing function of omitted factors. Q.E.D.

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