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The Implications of Nonmarketable Income for Consumption-Based Models of Asset Pricing

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ABSTRACT

A new representation of nonmarketable (NM) income is introduced in this essay. Using this representation and continuous trading, there exists a set of individuals who do not participate in the asset market and who consume at the rate of nonmarketable income derived from human capital. Because these individuals remain nonparticipants for a range of stochastic processes governing the NM income, consumption betas are not generally unique in value and the consumption-based CAPM (CCAPM) does not obtain. However, the intertemporal CAPM (ICAPM) of Merton remains valid.

NONMARKETABLE INCOME FROM HUMAN capital is a fact of life. Employers and financial intermediaries will not lend an individual the net present value of full-employment, lifetime earnings unless collateral is available. For many young and/or poor individuals, such collateral is not to be had. Personal credit constraints exist because the size of a personal loan will affect the individual's willingness to work or to "skip town"; an increase in the size of the loan reduces the probability that full payment will be returned at a later date. This phenomenon has not as yet been captured by models of asset pricing.

Breeden [2], Grossman and Shiller [4], Mayers [8, 9], Merton [11], Stapleton and Subrahmanyam [15], and others include "nonmarketable" income in their models. However, because they do not constrain individuals' holdings of marketable assets, the potential assets for selling the future income via trading strategies. A brief example demonstrates this fact.

Suppose an individual's rate of income $y(t)$ is instantaneously certain but that the instantaneous change in the range of income, $dy(t)$, is (possibly) stochastic; this is the assumption Breeden uses. Also assume that there exists a trading strategy using marketable assets that has a zero value at the terminal date and pays a dividend rate of $y(t)$ at all times t.[1] If the individual short-sells this portfolio, a positive payment will be received now and the future random income stream will be sufficient to pay the promised dividends. Thus, the individual can effectively sell the "nonmarketable" income by trading in marketable assets.

* School of Business, Indiana University. I am indebted to Doug Breeden, David Besanko, Wayne Ferson, and Mike Gibbons for their comments on an earlier version of this essay. Any remaining errors are mine.

[1] If markets are complete, as is commonly assumed in models of asset pricing, e.g., Breeden [2], Kraus and Litzenberger [6], and Rubinstein [12], the existence of the trading strategy is guaranteed. Furthermore, if the $dy(t)$ is certain at all dates, as assumed by Merton [10], the existence of a riskless asset is sufficient for the sale of the income.

It should be noted that the nonmarketability of income from human capital arises because of incentive problems created by the existence of asymmetric information and/or the existence of legal constraints on contracts (such as the right in the United States to declare bankruptcy). However, it is not the goal of this work to explain the phenomenon of nonmarketable income, and no claim is made that its essence is captured here in a definitive manner. Instead, this work examines the implications of its existence for measures of risk of asset prices. This is done using a tractable definition of a nonmarketable asset: *Marketable wealth of an individual is defined as the aggregate value (in terms of present consumption) of all assets owned by an individual that can be traded for present consumption. Marketable wealth is constrained to be non-negative at all times. A nonmarketable asset is anything of value to the individual that is not included in marketable wealth.* The term "nonmarketable income" (or "NM income") will be used to denote the income derived from a nonmarketable asset. An individual with NM income is said to be "liquidity constrained."

Given this definition, the next section shows that an individual with a source of NM income and with the opportunity to trade only a riskless marketable asset may decide to deplete marketable wealth and consume at a rate equal to income afterwards. Roughly, a high rate of growth in income, a high rate of time discount, or a low rate of interest is sufficient for this behavior. This simple scenario provides the best setting in which to distinguish, in a rigorous fashion, the new definition of nonmarketable assets from the prior notion of the concept.

Analysis of asset risk measures appears in Section II. Into an economy of P unconstrained, log-utility individuals are placed N individuals who find the constraint against the sale of NM income binding throughout their lives. Because consumption/portfolio choice is endogenous, conditions for the existence of the N nonparticipants, so called because they do not participate in the securities market, must be demonstrated. This is achieved by using the limiting arguments of Grossman and Shiller [4].

The existence of the nonparticipants implies that the CCAPM of Breeden does not obtain. The nonparticipants' rates of consumption are equal to their rates of income for a range of parameter values. This implies that consumption betas are not unique in value and that ratios of excess returns are not necessarily equal to ratios of the betas.

I. Nonmarketable Income and Consumption Paths

Consider an individual, with a certain time of death, T, who has one investment possibility, a riskless asset. Let r be the constant instantaneous rate of interest, and let $y(t)$ be the individual's rate of NM income at time t; $y(t)$ is assumed to be continuous and differentiable. Assume that the individual's goal is to choose the rate of consumption at each future point in time, $c(t)$, in order to maximize lifetime utility;

$$\int_0^T \exp(-\rho t)\,\frac{(c(t)^{1-B}-1)}{1-B}\,dt, \tag{P1}$$

subject to the constraints;

$$dW(t) = rW(t) + y(t) - c(t),$$
$$W(0) = W_0 \geq 0 \quad \text{and}$$
$$W(t) \geq 0 \quad \text{for all} \quad t.$$

$W(t)$ is the individual's time-t level of marketable wealth, and the strictly positive parameters ρ and B are the individual's rate of time discount and measure of relative risk aversion, respectively. The first constraint is the budget constraint, while the second defines the initial endowment of marketable wealth. The third constraint—a liquidity constraint—prohibits the sale of future NM income; together, the first and third constraints imply that the rate of consumption is no greater than the rate of NM income when marketable wealth is depleted.

The problem stated is one of optimal control with a constraint on the state variable, $W(t)$. Let $[t_i, t_{i+1})$ be an interval on which the constraint is binding. Using the results of Seierstad and Sydsaeter [13], one can verify that, if t is in the interval $[t_i, t_{i+1})$, the optimal policies satisfy

$$c(t) = y(t) \tag{1}$$

$$W(t) = 0, \tag{2}$$

and, if $W(t)$ is strictly greater than zero for all t in an interval $[t_i, t_{i+1})$ such that $W(t_{i+1}) = 0$, then

$$c(t) = \frac{\sigma \exp(\eta t - rt_i)(Y(t_{i+1}) + W(t_i))}{\exp(\sigma t_{i+1}) - \exp(\sigma t_i)} \tag{3}$$

and

$$W(t) = \exp(r(t - t_i)) \times \left[Y(t) + W(t_i) - \frac{\exp(\sigma(t - t_i)) - 1}{\exp(\sigma(t_{i+1} - t_i)) - 1} (Y(t_{i+1}) + W(t_i)) \right] \tag{4}$$

are optimal. We have used the definitions:

$$Y(t) \equiv \int_{t_i}^{t} \exp(-r(s - t_i)) y(s) ds, \qquad t_i \leq t \leq t_{i+1},$$

$$\sigma \equiv \frac{\rho - (1 - B)r}{-B},$$

$$\text{and } \eta \equiv \frac{\rho - \mathrm{r}}{-\mathrm{B}}.$$

Except for one characteristic, the solution to problem (P1) is a specialization of Merton's [10] results to the case of isoelastic utility and no risky investment opportunities. Merton found that an individual with a certain income stream and HARA utility capitalizes the income stream at the riskless rate of interest and chooses consumption and investment as if the capitalized value were an addition to marketable wealth. If NM income exists and if t_i and t_{i+1} are the end points of an interval of time during which the individual's marketable wealth is strictly positive, then from equation (3) it can be seen that a result analogous to Merton's obtains. Rather than capitalizing the lifetime income stream, the individual capitalizes the income stream up to the first time at which marketable wealth is depleted, t_{i+1}. The individual then chooses consumption as if this value were an addition to marketable wealth.

The path of consumption over any interval of time of positive wealth is either exponentially increasing or decreasing depending upon whether the rate of time discount, ρ, is less than or greater than the rate of interest, r. Paths c_1 and c_2 of Figure 1 provide examples of this phenomenon. Individuals with time discount

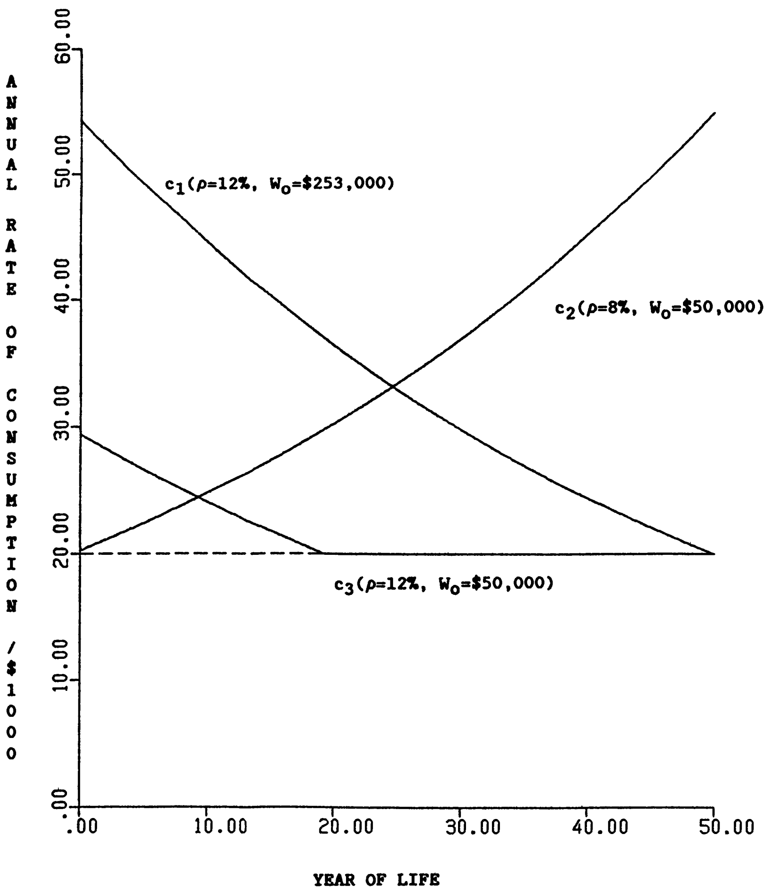

Figure 1. The instantaneous rates of consumption during the lifetime of three distinct individuals. These consumption functions, functions of time, are obtained from the solutions to problem (P1). In each case, the length of life, T, is equal to fifty years. The instantaneous rates of interest, r, and income, y, are constant and equal to 10 percent and $20,000 per year, respectively. Log utility is assumed ($B = 1$). Initial wealth, W_o, and the rate of time discount, ρ, differ across individuals. For the first and third individuals, c_1 and c_3, respectively, the rate of time discount is twelve percent, while, for the second, c_2, ρ = eight percent. The second and third individuals have initial wealth of $50,000, while W_o is $253,300 for the first.

less than the rate of interest, c_2, find the riskless asset an attractive means of saving and therefore choose higher consumption in the latter part of their lives. Conversely, impatient individuals (relative to the riskless rate), c_1, prefer to eat now and pay later; borrowing is attractive to them. Neither c_1 nor c_2 are constrained in their choices of consumption at any time during their lifetimes; it is of no consequence that their income is nonmarketable.

Figure 1 also displays the rates of consumption for an individual, c_3, who is affected by the prohibition of the sale of future NM income. One can see that the rate of consumption is equal to the rate of income, \$20,000 per year, from year twenty until the time of death. Unconstrained, the rate of consumption would be higher initially and decrease exponentially as does that of c_1. This individual would like to consume more earlier in life. Because this is not possible, all marketable wealth is consumed during the early part of life and neither borrowing nor lending is observed during the later years.

For all three of these examples the investment opportunity set is assumed to be constant and to contain only the riskless asset. Moreover, it is assumed that the time of death is certain, the rate of income is deterministic and constant, and utility is time-additive logarithmic. (The parametric values for the particular examples are listed in the legend to Figure 1.) Nevertheless, it remains true in a general model that, for some individuals, the solution to their consumption-portfolio problem prescribes zero levels of marketable wealth in some states of the world at some times during their lives. In the sequel, sufficient conditions are found for individuals to deplete marketable wealth when risky assets and stochastic rates of income exist.

II. Nonmarketable Income and Consumption-Based Risk Measures

To begin, consider an economy of P individuals, each maximizing the expected value of time-additive, state-independent utility of consumption:

$$\mathrm{E}_0\left\{\int_0^T \exp\{-\rho\tau\}\log(c_\tau)d\tau\right\}, \tag{5}$$

where the expectation operator $\mathrm{E}_0\{\cdot\}$ is conditioned upon a common time-zero information set. Each of the individuals has the opportunity to trade continuously a single consumption good and the shares of a finite number of real and financial assets. Each individual must choose a strategy specifying a rate of consumption c_τ and a vector of portfolio holdings $\underline{X}_\tau$ (in units of shares) for each time τ and each possible state of the world at that date.[2] The P individuals do *not* face liquidity constraints.

The continuous evolutions of the values of investments in the assets are given by

$$\begin{aligned} dV_t^i &= V_t^i\mu_i dt + V_t^i\sigma_i dz_i, \qquad i = 1, \cdots, A + F, \\ dV_t^0 &= V_t^0 r dt, \end{aligned} \tag{6}$$

[2] Individuals may have heterogeneous incomes and initial wealth levels. However, no confusion is caused by leaving the heterogeneous individual values unspecified.

where the z_i are correlated Brownian motions and the μ_i and σ_i are constants. The first A risky assets, $i = 1, \cdots, A$, are real production possibilities; the remainder of the risky assets and the instantaneously riskless asset are financial assets. For all assets, the V_t^i represent investment values with all cash flows, e.g., dividends, reinvested.

In addition to owning marketable securities with aggregate value W_o, the individuals receive the instantaneously certain flow of income y. This income rate changes stochastically and continuously:

$$dy = y\mu_y dt + y\sigma_y dz_y, \tag{7}$$

where z_y is a Brownian motion and μ_y and σ_y are constants.

One well-known implication of these assumptions (see Merton [10]) is that portfolio separation obtains: All individuals hold (only) the market portfolio of risky securities 1 through A and the riskless asset in nonzero amounts. This implies, in turn, that Pareto-optimal capital markets need consist of only the market portfolio of risky assets and the riskless assets. For this reason, when individual investment choices are analyzed in the sequel, it is assumed that the riskless asset is the only financial asset available; i.e., F is set to zero.

Another well-known result that will be used in the sequel is that the percentage change in aggregate consumption C^P is equal to the rate of return on the market portfolio:

$$dC^P/C^P = \mu_m dt + \sigma_m dz_m, \tag{8}$$

where z_m is a Brownian motion equal to a weighted sum of the z_i, and μ_m and σ_m are constants.

A. Analysis of a Single Liquidity-Constrained Individual

The necessary conditions for the optimum consumption/portfolio choices of a single liquidity-constrained individual coexisting with the P unconstrained investors are investigated using the technique of Grossman and Shiller [4]. The individual's consumption/portfolio problem is posed for each of a sequence of "h-economies"; h is the trading interval of this individual. The original P log-utility individuals and the investment opportunities are identical across economies.

Unlike the original P investors, the liquidity-constrained individual is also required, in any h-economy, to trade only at discrete dates and to consume and receive income at constant rates during the intervals of time between the trading dates. Stochastic changes in the rate of consumption and income may occur at the trading dates. It is assumed that the individual's optimal rate of consumption follows an Itô process in the limit ($h = 0$) and that this function of time is equal to the limit of the individual's consumption functions in the h-economies with probability one. These assumptions allow us to argue, as do Grossman and Shiller, that the necessary conditions of the optimal policy of the individual with NM income in an economy with continuous trading are characterized by the limit of the first-order conditions for the optima of the h-economies.

In each h-economy, the individual under study maximizes the expectation of time-additive utility function:

$$E_0\{\textstyle\sum_{j=0}^{(T-h)/h} \delta^{hj} \log(c_{jh}^h)h\}, \tag{9}$$

where c_{jh}^h is the rate of consumption from time jh to time $(j + 1)h$, δ is the discount factor, $\delta < 1$, and $\log(c_t^h)$ is the flow of utility.[3] This maximization is subject to the budget constraint:

$$\underline{X}_{(j-1)h}^h \cdot \underline{V}_{jh} + y_{jh}^h h = c_{jh}^h h + \underline{X}_{jh}^h \cdot \underline{V}_{jh}, \tag{10}$$

where y_t^h is the instantaneous rate of NM income at time t in economy h. The individual's rate of income between two trading dates jh and $jh + 1$ is set equal to the value of y_{jh}. That is, given h:

$$y_\tau^h = y_{jh}, \qquad jh \leq \tau < jh + h, \tag{11}$$

where y_t is the rate of income in the limit economy and is governed by the stochastic differential equation (7).

The single important feature distinguishing the current analysis from that of Grossman and Shiller is the liquidity constraint:

$$\underline{X}_t^h \cdot \underline{V}_t \geq 0, \qquad \text{for all } t \text{ and } h. \tag{12}$$

The level of marketable wealth of the individual under study is forced to be nonnegative, for, without this constraint, the scheme for selling NM income outlined in the introduction is possible.

Consider the marginal change in the individual's lifetime expected utility in an h-economy given a consumption strategy and given an infinitesimal reduction in the holdings of asset i between times t and $t + hj$ (for $t = hk$ and for some integers j and k):

$$V_t^i/c_t^h - \delta^{hj} E_t\{V_{t+hj}^i/c_{t+hj}^h\},$$

where $E_t[\cdot]$ is the expectation operator conditional on the information at time t. The first term is the magnitude of the infinitesimal increase in the time-t utility of consumption due to the time-t sale of shares, and the second term is the decrease in the time-t expected value of the $t + hj$ utility of consumption due to the time-$t + hj$ repurchase of shares. When constraint (12) is not binding, an interior optimum obtains for every information set at every time t, and this expression for marginal value is equal to zero. Using this equality, Grossman and Shiller provide an alternative derivation of the CCAPM of Breeden: For any two continuously traded assets i and j,

$$E\{dV^i/V^i\} - r = \frac{\beta_{iCq}}{\beta_{jCq}} (E\{dV^j/V^j\} - r), \tag{13}$$

where $\beta_{iCq} \equiv \text{cov}\{d\ln C^q, dV^i/V^i\}$ and C^q is the rate of consumption of any individual of subset of individuals.

[3] Note that the limit of (9) as h declines to zero, given a consumption policy, is the function (5).

For the individual with NM income, the assumption of interior optimality must be dropped. The individual, when liquidity constraints do not exist, could choose to sell future income perhaps by short-selling marketable assets. The existence of the constraint (12) provides Euler conditions that are inequalities:

$$V_t^i/c_t^h - \delta^{hj}\mathrm{E}_t\{V_{t+hj}^i/c_{t+hj}^h\} \geq 0. \tag{14}$$

Condition (14) as a strict inequality does not imply that asset i cannot be short-sold but only that it cannot be used, perhaps in combination with other assets, to sell nonmarketable income.

It is not difficult to find parameter values such that the inequality (14) is strict throughout the individual's lifetime $[0, T]$. To do so, first note that Itô's Lemma provides[4]

$$d(V_t^i/y_t) = (V_t^i/y_t)(\theta_i dt + \sigma_i dz_i - \sigma_y dz_y), \qquad i = 1, \cdots, A,$$

and

$$d(V_t^0/y_t) = (V_t^0/y_t)(\theta_o dt - \sigma_y dz_y), \tag{15}$$

where

$$\theta_i \equiv \mu_i - \mu_y + \sigma_y^2 - \sigma_{yi}, \qquad i = 1, \cdots, A,$$

$$\theta_o \equiv r - \mu_y + \sigma_y^2,$$

and

$$\sigma_{yi} \equiv \sigma_y \sigma_i \rho_{iy}$$

$$\equiv \sigma_y \sigma_i \mathrm{E}\{dz_y dz_i\}/dt.$$

Knowledge of bivariate normal random variables (see Rubinstein's [12] Appendix) then provides

$$\mathrm{E}_t\{V_{t+hj}^i/y_{t+hj}\} = (V_t^i/y_t)\exp\{\theta_i hj\}, \qquad i = 0, 1, \cdots, A. \tag{16}$$

Now consider the Euler condition (14) in the case that the individual consumes at the rate of income.[5] Using the expectation (16), the Euler condition is

$$1 - \delta \exp\{\theta_i\} \geq 0, \qquad i = 0, 1, \cdots, A. \tag{17}$$

Examination of (17), which is an expression independent of h, indicates that the inequality will hold strictly for all assets i under a number of conditions. One of these is that the value of δ is sufficiently small; i.e., the individual under study is sufficiently impatient. Furthermore, it is demonstrated in the Appendix that, for any h-economy, the condition, "Relation (17) is a strict inequality for assets $i = 0, \cdots, A$," is sufficient to imply the condition, "The individual under study finds the liquidity constraint (12) binding and optimally consumes at the rate of

[4] Apply Itô's Lemma to the ratios V_t^i/y_t, where V and y are governed by (6) and (7), respectively.

[5] It is *not* assumed that the optimal rate of consumption is equal to income. It is to be shown that (14) is a strict inequality for all assets i, under conditions to be identified, when income is substituted for consumption. Also, it is to be shown that this implies that the optimal consumption rate is the income rate.

income throughout the lifetime [0, T]." Because the consumption function of the individual in the limit economy is equal to the limit of the consumption functions in the h-economies, the impatient individual will consume at the rate of income in the limit economy as well as in each of the h-economies. Therefore, if inequality (17) is strict, the stochastic behavior of the rate of the individual's consumption when trading is continuous is determined by the stochastic differential equation (7).

B. An Economy in Which Aggregate Consumption Betas Are Not Unique

Consider the introduction of a large number, N, of liquidity-constrained individuals, one by one, into the economy. Like the original P citizens of the economy, each of the N individuals is assumed to have the opportunity to trade continuously. However, each of the N individuals is identical to the individual analyzed above; preferences are given by (5) and incomes by (7). An important point is that each of the N individuals faces the constraint (12). Let C^N be the total consumption of the N individuals.

Assume that the inequalities (17) for the N individuals are strict prior to the introduction of the first liquidity-constrained individual. The addition of a single individual to the economy of Section II, Subsection A, preserves the equilibrium. The reason is not the fact that the individual is small relative to the economy. Instead, as has been demonstrated, it is that the individual does not trade in the securities market, i.e., is a nonparticipant. The equilibrium prices remain unchanged, and the conditions (17) remain strict inequalities. Furthermore, the original P citizens remain participants, and their optimal policies are unchanged.

The addition of the N identical individuals constrained by (12) to the economy of Section II, Subsection A, does not alter equilibrium prices. Upon the introduction of any single individual, price behavior and the optimal policies of the participants do not change when inequalities (17) are strict. Therefore, the conditions (17), provided that they are strict inequalities when the first of the individuals is added, are strict inequalities when each of the N individuals is considered. It follows that each of the N individuals, upon addition to the economy, is a nonparticipant in the securities market and consumes at the rate of income throughout life.

The total consumption of the nonparticipants must solve the stochastic differential equation:

$$dC^N = C^N \mu_y dt + C^N \sigma_y dz_y. \tag{18}$$

Because the optimal policies of the participants do not change with the introduction of the nonparticipants, the stochastic behavior of C^P does not change from that specified in (8). Therefore, the consumption beta of any asset i, with respect to aggregate consumption C, can be written:

$$\begin{aligned}\beta_{iC} &= \text{cov}\{d\ln C,\, dV^i/V^i\}\\ &= (C^P/C)\text{cov}\{d\ln C^P,\, dV^i/V^i\} + (C^N/C)\text{cov}\{d\ln C^N,\, dV^i/V^i\}\\ &= (C^P/C)\sigma_m\sigma_i \text{E}\{dz_m,\, dz_i\}/dt + (C^N/C)\sigma_y\sigma_i\rho_{iy}.\end{aligned} \tag{19}$$

It is now possible to prove, in a direct fashion, the following:

THEOREM 1: *When individuals in an economy have nonmarketable income and the liquidity constraint* (12) *is imposed, the CCAPM* (13) *does not necessarily obtain for C^q equal to aggregate consumption.*

Proof: Examination of expressions (17) and (19) allows the demonstration of this result. First, note that (17) remains a strict inequality for a range of values of ρ_{iy}. Indeed, provided that

$$\rho_{iy} \geq [\mu_i + \sigma_y^2 + \log \delta - \mu_y]/\sigma_i \sigma_y$$

and has absolute value bounded by one, the N individuals remain nonparticipants and (19) remains a valid expression for consumption betas. Holding the other exogenous parameters constant while varying ρ_{iy}, the ratio β_{iC}/β_{jC} can be altered. Nonetheless, participants' optimal policies and assets' expected rates of return are not changed; rates of return on real assets are exogenous. Therefore, the CCAPM does not necessarily obtain. Q.E.D.

The nature of the discussion preceding Theorem 1 may disguise the fact that it is proved by example. An economy is constructed in which ratios of aggregate consumption betas are not unique in value across equilibria while ratios of expected excess rates of return are unique. Perhaps an apology should be offered for the fact that the economy is a relatively simple one in which the investment opportunity set is constant and utility is logarithmic. It is quite apparent from Breeden's discussion that consumption-based risk measures are, in principle, most useful in an economy with stochastic investment opportunities. As Merton [11] shows, the instantaneous version of the Sharpe [14], Lintner [7], Black [1] CAPM obtains when all individuals have logarithmic utility.

However, the simplicity of the example does not diminish the force of the argument, nor does it obscure the characteristics of a more general case in which the CCAPM does not obtain. The constancy of the investment opportunity set provides a means to demonstrate the existence of and to identify a set of N individuals who are nonparticipants throughout their lives. In the general case, participation in the securities market depends upon stochastic state variables and upon time. Hence, the distinction between participants and nonparticipants can be made only after conditioning upon time and state. However, given any state/date pair in which nonparticipants exist, aggregate consumption betas are not unique and the CCAPM may not obtain.

III. Summary

Using a new representation for nonmarketable income, it has been argued that the consumption opportunity set of an individual who owns a nonmarketable asset will differ from the set available to that individual if the asset were tradable. Ceteris paribus, the rate of consumption of an individual with little present income, little marketable wealth, and significant future income may be low relative to those with higher present incomes. In particular, economic recessions are perhaps marked by an increase in the number of consumers who are constrained to consume at a rate less than if all income were marketable. The findings of Ferson and Merrick [3] are consistent with this conjecture. Their evidence "indicates that aggregate consumption growth during recessions is lower *relative to* real treasury bill returns than would be predicted by the marginal rate of substitution model . . . assuming interior optimality of consumption-investment choice and a representative agent with constant time preference" (their emphasis).

If the existence of nonmarketable income alters the covariance of changes in aggregate consumption with assets returns, the consumption-correlatedness model using aggregate consumption can fail to explain differences in risk premia. Nevertheless, the consumption-correlatedness model is correct if aggregate consumption is replaced by the total consumption of any subset of individuals who are active in the market for the traded securities. This result is obtained by the aggregation of the first-order conditions, as in Breeden, of the securities market participants. Therefore, tests of the consumption-correlatedness model require either that (a) estimates of the total consumption of subsets of the population be used or (b) as Ferson and Merrick's results indicate, if aggregate consumption is used, test periods be chosen to minimize the effects of the nonmarketable income.

One finds that the ICAPM of Merton [11] obtains in general. The asset demands of liquidity-constrained individuals are identities, $0 = 0$, and these same individuals' wealths are zero provided that trading is continuous and asset prices follow continuous paths. Hence, the asset demands of all economic actors can be aggregated to obtain the ICAPM. However, the continuous-time version of the Mayers [8] model may not obtain even in the case that the investment opportunity set is constant. Securities market participants who face potentially binding constraints in the future choose portfolios to hedge against adverse changes in future NM income; the portfolios do not hedge only against adverse changes in the current rate of income.

Appendix

Consider economy h. If the individual under study has zero wealth at the final trading date $T - h$, the consumption rate is equal to the rate of income during the final period $[T - h, T]$. No other choice is available.

Induction is now used to demonstrate that the conditions (17) (as strict inequalities) are sufficient for the optimal policy of the individual under study, provided that initial wealth is zero, to dictate that the rate of consumption is equal to the rate of income throughout the life $[0, T]$. The inductive proof proceeds as follows:

Define an individual to be "penniless" at trading date t is the value of the portfolio of marketable securities, purchased at date $t - h$, is zero. Choose a trading date t. Assume that, if the individual is penniless at trading date $t + h$, then the consumption rate is equal to the rate of income at every date subsequent to $t + h$. Show that, whenever this assumption holds and the individual is penniless at time t, then the consumption rate is equal to the rate of income for every date subsequent to time t. Because the individual is penniless at time zero, i.e., initial marketable wealth is zero, and because the consumption rate is equal to the rate of income subsequent to date $T - h$, whenever the individual is penniless at that date, inductive reasoning implies that the consumption rate and rate of income are equal throughout the lifetime. This will complete the proof.

Assume that a penniless individual at trading date $t + h$ consumes at the rate of income throughout the remaining lifetime. Define $J_{t+h}(\cdot)$ to be the individual's indirect utility of wealth at time $t + h$. The penniless individual's two-period maximization problem at time t is

$$\max_{c_t^h, \underline{X}_t^h} \log(c_t^h)h + \delta \mathrm{E}_t\{J_{t+h}(\underline{X}_t^h \cdot \underline{V}_{t+h} + y_{t+h}^h h)\}$$

$$- \lambda_b[c_t^h h + \underline{X}_t^h \cdot \underline{V}_t - y_t^h h] - \int_{\underline{V}_{t+h}} \lambda_x(\underline{v})\underline{X}_t^h \cdot \underline{v} d\underline{v}. \quad \text{(AP)}$$

The multipliers λ_b and $\lambda_x(\underline{v})$ are the Lagrange multiplier on the budget constraint and the Kuhn-Tucker multipliers on the liquidity constraints, respectively. There must exist one liquidity constraint for each possible level of time-$t + h$ wealth, and so the integral is defined over the range of values of $\underline{V}_{t+h}$. There is no need for an additional constraint on the time-t wealth. The posited liquidity constraint requires that the individual enter period $t + h$ with zero liabilities and so requires that the individual leave period t with none. The liquidity constraint (a) prevents the individual from borrowing against future NM income to fund time-t consumption, and (b) prevents the individual from holding a hedge portfolio that redistributes time-$t + h$ nonmarketable income across the events defined by the levels of $\underline{V}_{t+h}$.

In the present case in which the individual has no marketable income but only a stream of NM income, the liquidity constraints are equivalent to constraints against the short sale of any asset. In the case in which the individual has a stream of marketable income, the liquidity constraints would require short sales to be limited although it would not rule out short sales entirely. An example of limited short sales appears in Hakansson [5]. In that model, the individual shorts the riskless asset, to a degree, in order to borrow against certain future marketable income.

Let λ^i be the Kuhn-Tucker multiplier on the constraint against shorting asset i. The first-order conditions with respect to consumption c_t^h and the investments $\underline{X}_t^h$ for problem (AP), with short-sale constraints replacing the liquidity constraints, can be written:

$$(c_t^h)^{-1} = \lambda_b \quad \text{(A1)}$$

$$\delta \mathrm{E}_t\{J'_{t+h}(\underline{X}_t^h \cdot \underline{V}_{t+h} + y_{t+h}^h h)V_{t+h}^i\} - \lambda_b V_t^i = \lambda^i V_t^i, \qquad i = 0, 1, \cdots, A, \quad \text{(A2}')$$

$$\lambda^i X_t^{hi} = 0; \qquad \lambda^i \le 0; \qquad X_t^{hi} \ge 0, \qquad i = 0, 1, \cdots, A. \quad \text{(A3)}$$

The envelope condition states that equality must hold between marginal utility of wealth at date $t + h$, $J'_{t+h}(\cdot)$, and the marginal utility of consumption, $(c_{t+h}^h)^{-1}$, at every *positive* level of wealth. At a level of marketable wealth of zero, the marginal utility of wealth is defined as the marginal utility of consumption. In either case, upon substitution for the marginal utility of wealth, the conditions

(A2′) become

$$\delta \mathrm{E}_t\{V^i_{t+h}/c^h_{t+h}\} - V^i_t/c^h_t = \lambda^i V^i_t, \qquad i = 0, 1, \cdots, A. \tag{A2}$$

Whenever $X^{hi}_t = 0$, for all i, the individual will be penniless at time $t + h$. In this case, the individual consumes at the rate of income at all dates subsequent to (and inclusive of) $t + h$, and one can write $c^h_{t+h} = y^h_{t+h}$. This fact, along with the assumption that inequalities (17) hold strictly, implies that

$$X^{hi}_t = 0, \quad i = 0, 1, \cdots, A,$$

$$c^h_t = y^h_t,$$

$$\lambda_b = (y^h_t)^{-1},$$

and

$$\lambda^i = \delta \mathrm{E}\{V^i_{t+h}/y^h_{t+h} V^i_t\} - (y^h_t)^{-1}, \qquad i = 0, 1, \cdots, A,$$

satisfy the first-order conditions (A1), (A2), (A3), and the budget constraint. (The relations (17) imply that the λ^i, as defined here, are negative and hence satisfy the condition (A3).)

This proves that the individual consumes at the rate of income at all dates subsequent to t and completes the proof.

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