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Exact Arbitrage Pricing and the Minimum-Variance Frontier

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ABSTRACT

The author examines the relationship between the Arbitrage Pricing Theory of Ross and mean-variance analysis. In particular, conditions are derived on the vector of the factor risk premia that are equivalent to the existence of a strictly positively weighted portfolio on the minimum-variance frontier. Also, a sufficient condition is given under which the existence of a positive minimum-variance portfolio of all the assets in the economy will imply the existence of a positive minimum-variance portfolio on a subset. This means that rejection of the hypothesis of the existence of a positive minimum-variance portfolio on a subset satisfying this condition implies rejection for the whole set.

A GROWING BODY OF literature explores the relationship between the Arbitrage Pricing Theory (APT) of Ross [13, 14] and the mean-variance model. Among the early work in this direction were the papers of Chamberlain and Rothschild [2] and Chamberlain [1]. These papers examine a sequence of economies and characterize the relationship in the limit between the mean and portfolio cost functionals and the minimum-variance frontier. Chamberlain [1] also gives a condition under which exact arbitrage pricing will hold. Dybvig [4] and Grinblatt and Titman [7] examine equilibrium models in an APT context for a finite economy. They derive sharp bounds on the deviation from exact arbitrage pricing.

Recent papers by Huberman, Kandel, and Stambaugh [9] and Grinblatt and Titman [8] explore the relationship between the APT and the mean-variance model by studying portfolios that can be used as proxies for the factors and examining the relationship these portfolios bear to the minimum-variance frontier. These two papers also deal with equilibrium models.

In spite of the number and variety of equilibrium-APT models, the question of the pricing of factor risk in equilibrium remains largely unexplored. This paper makes some progress in addressing that issue by looking at a given equilibrium restriction and studying the constraints it imposes on the vector of factor premia itself.

The most familiar equilibrium restriction we might study is the central implication of the Capital Asset Pricing Model (CAPM), that the market portfolio is efficient. In particular, no portfolio with the same expected return as the market portfolio has a smaller variance. The restriction we actually study is weaker than

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this. Since the market portfolio has strictly positive elements, the existence of a strictly positive portfolio on the minimum-variance frontier is a necessary condition for the CAPM. We study the behavior of the factor-premia vector under the restriction that there exists a minimum-variance portfolio with all-positive components.

The existence of such a portfolio is an interesting issue in its own right for several reasons. First, an empirical test that rejected the hypothesis that a positive minimum-variance portfolio exists would reject the hypothesis that the CAPM holds. In addition, there are other, practical reasons for being interested in positive portfolios. Professional portfolio managers tend to invest their clients' wealth almost exclusively in long positions. Indeed, many managers, such as those managing pensions, are legally precluded from taking short positions. Also, as Green [6] points out, mutual funds and index funds tend to have the same feature. This suggests that a large number of investors believe that portfolio efficiency does not require short positions, and a finding to the contrary would be interesting. In addition, short-sale restrictions and high transaction costs for short sales may represent constraints that are not present in the usual asset-pricing models, and an examination of the existence of positive efficient portfolios can shed light on the issue of whether these constraints are binding.

This paper studies the joint implications of two conditions: that exact arbitrage pricing holds and that some positive portfolio lies on the minimum-variance frontier. The first major result provides conditions on the factor-premia vector λ that are equivalent under exact arbitrage pricing to the existence of such a portfolio. Roughly speaking, the result is that a positive minimum-variance portfolio will exist if the factor premia lie in a range that compensates investors for bearing the factor risk of some long position in each asset. The second major result in the paper concerns the empirical matter of testing the mean-variance model under exact arbitrage pricing when we observe only a subset of the assets in the economy. It gives conditions on the relationship between the missing assets and the included assets that imply that rejection of the model on the subset implies rejection on the whole set. The condition is that the included assets be representative of the missing ones in the sense that the covariance vector of the missing asset can be replicated with a positive portfolio of the included ones.

Section I of the paper specifies the form of the APT under study and reviews a result due to Green [6]. Section II gives the basic characterization result and Section III an interpretation. Section IV then discusses testing on subsets and gives the second major result. Section V provides concluding remarks.

I. Preliminaries

We model asset returns as arising from a factor model, in which some number, say k, of zero-mean (or de-meaned) factors f influence asset returns according to (1):

$$\tilde{R}_i = E_i + \sum \beta_{ij} \tilde{f}_j + \tilde{e}_i, \tag{1}$$

where $\tilde{R}_i$ is the random return on the ith asset, E_i is its expected return, the $\tilde{f}_j$ are the k factors, $\tilde{e}_i$ is an independent noise term for asset i, and β_{ij} is the factor loading pertaining to asset i and factor j. This linear structure implies that the matrix $\mathbf{V}$ of the covariances of the assets in the market will be

$$\mathbf{V} = \mathbf{BB}' + \mathbf{D}, \tag{2}$$

where the i, jth element of $\mathbf{B}$ is β_{ij} and $\mathbf{D}$ is a diagonal matrix with an ith diagonal element that is $\text{var}(\tilde{e}_i)$. Ross [13, 14] shows that, in the absence of arbitrage, the expected return vector E will be approximately

$$E = \rho \underline{1} + \mathbf{B}\lambda, \tag{3}$$

where λ is the vector of factor premia and $\underline{1}$ is a vector of ones. We will assume throughout that $\mathbf{B}$ is of full rank, that $\mathbf{V}$ is positive definite and therefore nonsingular, and that the pricing relation (3) is exact. When (3) holds, we shall say that the APT holds. A portfolio will be a vector y satisfying $y'\underline{1} = 1$ (where $\underline{1}$ is a vector of ones), and an arbitrage portfolio or hedge portfolio will be a vector y satisfying $y'\underline{1} = 0$.

In this framework, we examine the problem of the existence of minimum-variance portfolios with only strictly positive components. We start with a result of Green [6] that gives conditions on $\mathbf{V}$ and E in a general framework under which there does not exist a minimum-variance portfolio with all-positive portfolio weights. We repeat Green's main theorem below.

THEOREM 1: *There exists some strictly positive minimum-variance portfolio if and only if there exists no y satisfying*

$$\mathbf{V}y > 0,$$

$$y'E = 0,$$

and

$$y'\underline{1} = 0. \tag{4}$$

In vector expressions, we write "$\geq$" to mean that in every component the relation is $\geq$, "$>$" to mean that the inequality is strict in some component, and "$\gg$" to mean that the inequality is strict in every component.

Theorem 1 implies that, if there exists a zero-mean hedge portfolio that has a non-negative covariance with every asset (and positive covariance with some asset), then there can be no positively weighted portfolio on the minimum-variance frontier. The idea is that, if y satisfies (4), then, for any positive portfolio a, $a - \gamma y$ has the same mean return as a and a smaller variance for sufficiently small, positive γ. The proof of the result is essentially an exercise in duality and may be obtained by applying Farkas' Lemma.

Although a one-factor model always has a positive minimum-variance portfolio, a multiple-factor APT model need not. Green shows that the following model with two factors and three assets has no positive portfolio on the minimum-variance frontier. Suppose that (2) and (3) hold with

$$\mathbf{B} = \begin{bmatrix} 6 & 4 \\ 7 & 2 \\ 11 & 0 \end{bmatrix}, \quad \lambda = [1 \quad 1]', \quad \rho = 0.1, \quad \text{and} \quad \mathbf{D} = \mathbf{I}.$$

In this market, all hedge portfolios with mean zero are proportional to the vector $y = [-1, \frac{1}{2}, \frac{1}{2}]'$, but this vector has a non-negative covariance with every asset. By Theorem 1, then, no positive minimum-variance portfolio exists. The reason for this lies in the choice of the factor risk premia, λ. In this model, the two factors command the same premium, even though the first factor has heavier loadings. However, a mean-variance investor will hedge or diversify risk that does not offer adequate compensation. If such an investor is to hold a positive amount of every asset, the return to bearing factor risk must reflect a premium for bearing that risk.

To see why in a mean-variance framework we might expect factors with heavier loadings to command greater premia, consider the model of Dybvig [4]. As an illustrative example, normalize the factors so that $\mathrm{E}\,\tilde{f}_j = 0$, $\mathrm{E}\,\tilde{f}_j^2 = 1$, and $\mathrm{E}\,\tilde{f}_j\tilde{f}_k = 0$ when $j \neq k$. From Dybvig's development we have that

$$\lambda \propto \mathrm{E}[\tilde{f}_j\, u'(w)],$$

where

$$w = a'x$$

for some efficient portfolio a, and

$$x_i = E_i + \textstyle\sum_j \beta_{ij}\tilde{f}_j + \tilde{e}_i$$

represents the payoff on an investment of one dollar in asset i. The simplest CAPM-motivated example relevant here is the case of quadratic utility. With quadratic utility, $u'(w)$ is proportional to w. Thus, λ_j is proportional to $\mathrm{E}[\tilde{f}_j w]$, which equals

$$\begin{aligned} \mathrm{E}[\tilde{f}_j w] &= \mathrm{E}[\tilde{f}_j \textstyle\sum a_i \sum \beta_{ik}\tilde{f}_k] + \mathrm{E}[\tilde{f}_j] \sum a_i(E_i) + \mathrm{E}[\tilde{f}_j \sum \tilde{e}_i] \\ &= \mathrm{E}[\textstyle\sum a_i\beta_{ij}\tilde{f}_j^2] \\ &= \textstyle\sum a_i\beta_{ij}. \end{aligned}$$

If a is a positive efficient portfolio, then this means that λ_j must be a positive combination of the β_{ij}. This paper studies a model somewhat different from Dybvig's in that, in this paper, we assume that the arbitrage-pricing equation holds exactly. The results in the next section thus differ correspondingly from the conclusion that any positive combination of the assets' factor-loading vectors will be a factor-premia vector that admits the existence of a positive efficient portfolio. The difference lies in the importance of the own-variance matrix **D**. This should not be surprising since exact arbitrage pricing depends in some models (see, for instance, Chamberlain [1]) on the ability to diversify idiosyncratic risk away, that is, on **D**'s being somehow inconsequential.

II. Characterization of Risk-Compensating Factor Premia

Naturally, Green's example gives rise to the question of whether there will always be a set of factor premia that permits the existence of a positive portfolio on the minimum-variance frontier. The first result in this section shows that such a vector of premia will always exist.

Because the existence of a positive minimum-variance portfolio depends on the condition that each factor premium rewards investors sufficiently for taking on factor risk so that some mean-variance investor will hold a positive portfolio rather than try to hedge the factor risk away with a short position, we say that, when the APT holds, a vector λ of factor premia is factor-risk compensating or risk compensating if and only if there exists a positive minimum-variance portfolio when the factor premia are λ. This allows us to state succinctly the first result:

PROPOSITION 1: *If the APT holds, then the factor-premia vector $\lambda = \mathbf{B}'\mathbf{D}^{-1}\underline{1}$ is risk compensating. Furthermore, if $\lambda = \mathbf{B}'\mathbf{D}^{-1}\underline{1}$, then an equal idiosyncratic-variance index $\mathbf{D}^{-1}\underline{1}/(\underline{1}'\mathbf{D}^{-1}\underline{1})$ will be mean-variance efficient.*

Proof: We require that, if $\lambda = \mathbf{B}'\mathbf{D}^{-1}\underline{1}$, then there must be no y satisfying $\mathbf{V}y > 0$, $y'E = 0$, and $y'\ \underline{1} = 0$. Suppose that y is a zero-mean hedge portfolio. The requirements that $y'\underline{1} = 0$ and $y'E = 0$, together with (3), imply that

$$y'\mathbf{B}\lambda = y'\mathbf{BB}'\mathbf{D}^{-1}\underline{1} = 0. \tag{5}$$

The requirement that

$$y'\mathbf{V} > 0$$

implies that

$$y'\mathbf{VD}^{-1}\underline{1} > 0.$$

However,

$$\begin{aligned} y'\mathbf{VD}^{-1}\underline{1} &= y'\mathbf{BB}'\mathbf{D}^{-1}\underline{1} + y'\mathbf{DD}^{-1}\underline{1} \\ &= y'\mathbf{DD}^{-1}\underline{1} \\ &= y'\underline{1} \\ &= 0, \end{aligned}$$

which is a contradiction. To show that the second statement is true, let $a = \underline{1}'\mathbf{D}^{-1}\underline{1}$. If the vector $a\mathbf{V}^{-1}E + a(1-\rho)\mathbf{V}^{-1}\underline{1}$ is a portfolio, then it is mean-variance efficient since $a > 0$. Substituting in $E = \mathbf{B}\lambda + \rho\underline{1}$, we have

$$\begin{aligned} a\mathbf{V}^{-1}E + a(1-\rho)\mathbf{V}^{-1}\underline{1} &= \mathbf{V}^{-1}[(a\rho + a(1-\rho))\underline{1} + a\mathbf{BB}'\mathbf{D}^{-1}\underline{1}] \\ &= a\mathbf{V}^{-1}[(\mathbf{I} + \mathbf{BB}'\mathbf{D}^{-1})\underline{1}] \\ &= \mathbf{D}^{-1}\underline{1}/(\underline{1}'\mathbf{D}^{-1}\underline{1}). \end{aligned}$$

However, this is the required portfolio, which completes the proof. Q.E.D.

This result serves to demonstrate the existence of a risk-compensating factor-premia vector. Intuitively, the proof proceeds by showing that, if $\lambda = \mathbf{B}'\mathbf{D}^{-1}\underline{1}$, then the index portfolio proportional to $\mathbf{D}^{-1}\underline{1}$ is both minimum variance and uncorrelated with every zero-mean arbitrage portfolio. We will see later that the expression for λ in Proposition 1 figures in our complete characterization of the set of risk-compensating factor-premia vectors.

Under the mean-variance model, we know that the minimum-variance frontier corresponds to the range space of $[\mathbf{V}^{-1}E \,|\, \mathbf{V}^{-1}\underline{1}]$, a fact used in proving Proposition 1. Thus, the existence of a positive minimum-variance portfolio is equivalent to the condition that this range space intersects the positive orthant. Thus, if $\mathbf{V}^{-1}\underline{1} \gg 0$, then the global minimum-variance portfolio is positive, and so any factor premia will be risk compensating. Failing that, λ will be risk compensating if and only if there exist numbers $\alpha \neq 0$ and γ and a vector $z^* \gg 0$ such that

$$\begin{aligned} \alpha\mathbf{B}\lambda + \gamma\underline{1} &= \mathbf{V}z^* \\ &= (\mathbf{D} + \mathbf{BB}')z^*. \end{aligned} \tag{8}$$

We will assume that $\alpha > 0$ since this is the case in which the positive portfolio is on the upward-sloping portion of the frontier. We can rewrite, saying that λ is risk compensating if and only if for some number γ there is a $z \gg 0$ such that

$$\mathbf{B}\lambda + \gamma\underline{1} = (\mathbf{I} + \mathbf{BB}'\mathbf{D}^{-1}\underline{1})z,$$

or

$$\mathbf{B}\lambda = \mathbf{BB}'\mathbf{D}^{-1}z + (z - \gamma\underline{1}). \tag{9}$$

This $z = \mathbf{D}z^*/\alpha$. This suggests that risk-compensating λ's look like $\mathbf{B}'\mathbf{D}^{-1}z + q$, where $\mathbf{B}q = z - \gamma\underline{1}$. In fact, this is exactly the case. If equation (9) holds, then $z - \gamma\underline{1}$ is in the range space of $\mathbf{B}$. This implies that there is some q such that $\mathbf{B}q = z - \gamma\underline{1}$, or $\mathbf{B}q + \gamma\underline{1} = z$. However, if z is strictly positive, then $\gamma > -\min(\mathbf{B}q)_i$. Hence, equation (9) implies that, for some q and some $\gamma > -\min(\mathbf{B}q)_i$,

$$\mathbf{B}\lambda = \mathbf{BB}'\mathbf{D}^{-1}(\mathbf{B}q + \gamma\underline{1}) + \mathbf{B}q. \tag{10}$$

Since $\mathbf{B}$ has full column rank, it also has a left inverse, so we can rewrite (10) as

$$\lambda = (\mathbf{B}'\mathbf{D}^{-1}\mathbf{B} + \mathbf{I})q + \gamma\mathbf{B}'\mathbf{D}^{-1}\underline{1}. \tag{11}$$

We can also reverse the calculation so that, if we choose any vector q and any $\gamma > -\min(\mathbf{B}q)_i$ and if we calculate λ using (11) and let $z = \mathbf{B}q + \gamma\underline{1}$, then λ, z, and γ will satisfy (9).

If $\alpha < 0$, then equation (9) is the same, except that we require z^* to be strictly negative, and $-z^*$ will be the positive portfolio on the minimum-variance frontier. The same argument goes through as above, but now we need $\gamma < -\max(\mathbf{B}q)_i$. In fact, λ is risk compensating by this argument, using q and γ in equation (11) if and only if $-\lambda$ is risk compensating by the first argument using $-q$ and $-\gamma$. We state all of this as Theorem 2.

THEOREM 2 (*Characterization Theorem*): *A vector λ is risk compensating if and only if $-\lambda$ is risk compensating as well. They will be risk compensating if and only if, for one or the other (say λ), there exist a vector q and a number γ, such that*

$\gamma > -\min(\mathbf{B}q)_i$, *that satisfy*

$$\lambda = (\mathbf{B}'\mathbf{D}^{-1}\mathbf{B} + \mathbf{I})q + \gamma\mathbf{B}'\mathbf{D}^{-1}\underline{1}. \tag{11}$$

This is a complete characterization of the set of risk-compensating prices. Note that Proposition 1 is the special case with $q = 0$ and $\gamma = 1$. We can add to this the observation that the set of risk-compensating λ's is a cone omitting the origin since, if λ, q, and γ satisfy equation (11) and $\gamma > -\min(\mathbf{B}q)_i$, then, for any positive number θ, $\theta\lambda$, θq, and $\theta\gamma$ will solve (11) also, and $\theta\gamma > -\min(\mathbf{B}(\theta q))_i$.

We also note that, if the original α in equation (8) is positive, then the positive efficient portfolio z^* is a combination of the global minimum-variance portfolio and the tangency portfolio proportional to $\mathbf{V}^{-1}E$ and includes a positive weighting of that portfolio. Therefore, these λ correspond to the existence of a positive portfolio on the upward-sloping portion of the minimum-variance frontier—a positive, mean-variance efficient portfolio. If $\alpha < 0$, then the positive minimum-variance portfolio $-z^*$ lies on the downward-sloping portion of the frontier. In other words, one half of the cone of risk-compensating λ's corresponds to minimum-variance portfolios on the mean-variance efficient portion of the minimum-variance frontier, and the other half corresponds to the downward-sloping branch.

It is also the case that each half of the cone is convex. Suppose that λ_1, q_1, γ_1 and λ_2, q_2, γ_2 are two triplets satisfying (11) and that we have $\gamma_1 > -\min(\mathbf{B}q_1)_i$ and $\gamma_2 > -\min(\mathbf{B}q_2)_i$, so that λ_1 and λ_2 are both risk compensating. Then for $0 \le \theta \le 1$, $\lambda = \theta\lambda_1 + (1 - \theta)\lambda_2$, $q = \theta q_1 + (1 - \theta)q_2$, and $\gamma = \theta\gamma_1 + (1 - \theta)\gamma_2$ satisfy (11), and $\gamma > -\min(\mathbf{B}q)_i$, so λ is risk compensating as well.

III. Interpretation of the Characterization Theorem

This section provides two ways of interpreting (11). Proposition 1 states that one risk-compensating factor-premia vector is $\lambda = \gamma\mathbf{B}'\mathbf{D}^{-1}\underline{1}$, which is proportional to an idiosyncratic risk-weighted average of the assets' vectors of factor loadings. In particular, if one factor tends, "on average" in this sense, to have a much heavier loading on the assets than another, then risk-compensating factor pricing requires that the more heavily weighted factor command a larger premium. Going back to Green's example, the first factor has a larger loading than the second on every asset. As Green shows, a factor-premia vector giving each factor the same premium is not risk compensating. However, $\lambda = \gamma\mathbf{B}'\mathbf{D}^{-1}\underline{1} = [4\ 1]'$ is risk compensating.

Theorem 2 specifies exactly the extent to which factor premia can deviate from this average and still be risk compensating. Recall that the risk-compensating λ's lie in a convex cone, so that, if λ is risk compensating, then so is any vector proportional to λ. Consider the vector $\lambda = \gamma\mathbf{B}'\mathbf{D}^{-1}\underline{1}$, which is risk compensating as long as $\gamma \neq 0$. Noting that, for any q, we can write $q = \mathbf{B}'p$ for some (not necessarily unique) p, rewrite (11) as

$$\lambda = (\mathbf{B}'\mathbf{D}^{-1}\mathbf{B} + \mathbf{I})\mathbf{B}'p + \gamma\mathbf{B}'\mathbf{D}^{-1}\underline{1}, \tag{12}$$

with the restriction on p being that $\mathbf{B}\mathbf{B}'p + \gamma\underline{1} \gg \underline{0}$. Then if we think of p as an investment position after the fashion of Huberman, Kandel, and Stambaugh [9] (not necessarily a portfolio since $p'\underline{1}$ need not equal one), the restriction requires us to consider only investment positions with a factor-borne covariance with each asset (that is, $(\mathbf{BB}'p)_i$ for asset i) that is greater than $-\gamma$. We can further rewrite (12) as

$$\begin{aligned}\lambda &= (\mathbf{B}'\mathbf{D}^{-1}\mathbf{BB}' + \mathbf{B}')p + \gamma\mathbf{B}'\mathbf{D}^{-1}\underline{1} \\ &= \mathbf{B}'\mathbf{D}^{-1}(\mathbf{BB}' + \mathbf{D})p + \gamma\mathbf{B}'\mathbf{D}^{-1}\underline{1} \\ &= \mathbf{B}'\mathbf{D}^{-1}(\mathbf{V}p + \gamma\underline{1}). \end{aligned} \tag{13}$$

In other words, risk-compensating λ's must be linear combinations of $\mathbf{B}'\mathbf{D}^{-1}$, the ith weight of which can be written as γ plus the covariance of some position p with asset i, where p meets the "factor-borne" covariance restriction. Notice that this does not rule out negative weights. A negative weight may arise if $(\mathbf{BB}'p + \gamma\underline{1})_i$ is close to zero, $p_i < 0$, and d_i is relatively large. Since $\mathbf{D}^{-1}$ has all-positive diagonal elements, the set of risk-compensating λ's is nearly, but not quite, the cone of positive linear combinations of the assets' factor-loading vectors. Note further that, since a negative weight typically requires d_i to be relatively large, the presence of $\mathbf{D}^{-1}$ in expression (13) reduces the impact of a negative weight when it can arise. Conversely, it is also a simple matter to construct examples of positive weightings that do not yield risk-compensating factor premia.

A second way of looking at (11) may also be helpful. This interpretation relates to statistical estimation of the factors from realized returns, using the GLS projection that minimizes the variance of the residual ascribed to idiosyncratic (that is, $\mathbf{D}$) risk.

To see this, rewrite (11) as

$$\begin{aligned}\lambda &= \mathbf{B}'\mathbf{D}^{-1}(\mathbf{B}q + \gamma\underline{1}) + q \\ &= \mathbf{B}'[\mathbf{D}^{-1}(\mathbf{B}q + \gamma\underline{1}) + \mathbf{D}^{-1}\mathbf{B}(\mathbf{B}'\mathbf{D}^{-1}\mathbf{B})^{-1}q]. \end{aligned} \tag{14}$$

Again writing $q = \mathbf{B}'p$, the second term in brackets gives the GLS predictor of the portion of the return to p due to the factors.

Huberman, Kandel, and Stambaugh [9] show that, under exact arbitrage pricing, $\mathbf{D}^{-1}\mathbf{B}(\mathbf{B}'\mathbf{D}^{-1}\mathbf{B})^{-1}$ can be used in place of the factors to price all assets. They also note (as do Grinblatt and Titman [8]) that this matrix gives the maximum-likelihood factor-analysis portfolios. Thus, (14) provides that λ must be a linear combination of the assets' factor loadings such that the weights are a positive vector plus a specific combination, q, of the maximum-likelihood estimates of the factors.

IV. An Application to Testing on Subsets

One important practical question that arises with respect to empirical testing concerns Roll's [12] critique that we only observe a subset of the assets available in the market. At issue is the validity of empirical tests on such subsets. In this section, we show that, in the APT setting under consideration here, some tests on subsets still may produce valid results. We show that, if the factor-loading vectors of the assets included in a sample are in some sense representative of the factor loadings of those omitted, then rejection of the existence of a positive efficient portfolio on the subset implies rejection of the existence of a positive efficient portfolio on the entire set. As in Kandel [10], we may identify a collection of omitted assets with a single asset—the portfolio formed from those assets. Proposition 3 defines the sense in which the included assets must be representative; the omitted asset's factor loadings must be nearly a positive combination of the included assets', permitting a deviation from positive weights similar to that in Theorem 2. Proposition 4 ties this result to one in Kandel [10].

PROPOSITION 3: *Consider an economy with n assets, in which* (2) *and* (3) *hold, and for which* λ *is risk compensating. Suppose that asset n's factor-loading vector,* β_n, *can be written*

$$\beta_n = (\mathbf{B}'\mathbf{D}^{-1}\mathbf{B} + \mathbf{I})a, \quad \textit{where} \quad \mathbf{B}a > 0. \tag{15}$$

Here, **B**, **D**, *and* **I** *refer to the first* $n - 1$ *assets. Then* λ *is risk compensating in the economy restricted to the first* $n - 1$ *assets as well.*

Proof: Suppose that λ is risk compensating for all n assets and that β_n satisfies (15). Then for some q and γ we can write

$$\lambda = [\mathbf{B}' \quad \beta_n] \begin{bmatrix} \mathbf{D}^{-1} & 0 \\ 0 & d_n^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{B} \\ \beta_n \end{bmatrix} q + \gamma[\mathbf{B}' \quad \beta_n] \begin{bmatrix} \mathbf{D}^{-1} & 0 \\ 0 & d_n^{-1} \end{bmatrix} \underline{1},$$

where $\mathbf{B}q + \gamma\underline{1} \gg 0$ and $\beta_n'q + \gamma > 0$. Rewriting,

$$\begin{aligned} \lambda &= (\mathbf{B}'\mathbf{D}^{-1}\mathbf{B} + \mathbf{I})q + \gamma\mathbf{B}'\mathbf{D}^{-1}\underline{1} + (\gamma/d_n)(\mathbf{B}'\mathbf{D}^{-1}\mathbf{B} + \mathbf{I})a \\ &\quad + (\beta_n'q/d_n)(\mathbf{B}'\mathbf{D}^{-1}\mathbf{B} + \mathbf{I})a \\ &= (\mathbf{B}'\mathbf{D}^{-1}\mathbf{B} + \mathbf{I})(q + a(\beta_n'q + \gamma)/d_n) + \gamma\mathbf{B}'\mathbf{D}^{-1}\underline{1}. \end{aligned}$$

Now $\mathbf{B}a \gg 0$, so $\mathbf{B}a(\beta_n'q + \gamma)/d_n \gg 0$, so $\mathbf{B}(q + a(\beta_n'q + \gamma)/d_n) \gg 0$. Thus, λ is risk compensating for the subset as well. Q.E.D.

This proof is a simple calculation showing that, if (11) holds on the whole set, then (15) guarantees that (11) holds on the subset as well, with γ unchanged and q modified but still meeting the restriction $\mathbf{B}q + \gamma\underline{1} \gg 0$.

Proposition 3 is a special case (under exact arbitrage pricing) of a more general result that derives directly from Theorem 1. That result, Proposition 4, should clarify further the meaning of Proposition 3.

PROPOSITION 4: *Consider a market with n assets with mean-returns vector E and variance matrix* **V**. *Write* $\mathbf{V}_{-i}$ *and* E_{-i} *for the variance submatrix and mean vector for the subset omitting the ith asset. Suppose that there exists a positive mean-variance efficient portfolio, and suppose that* σ_i, *the vector of the covariances between the ith asset and the others, can be written*

$$\sigma_i = \mathbf{V}_{-i}a, \quad a > 0.$$

Then there exists a positive minimum-variance portfolio on the subset omitting the ith asset.

Proof: We prove the contrapositive. Suppose without loss of generality that $i = n$, and suppose that there is no positive minimum-variance portfolio on the subset. Then there is an $(n-1)$-vector y such that

$$\mathbf{V}_{-n} y > 0,$$
$$y' E_{-n} = 0,$$
$$y' \underline{1} = 0.$$

Note that

$$\mathbf{V} = \begin{bmatrix} \mathbf{V}_{-n} & \sigma_n \\ \sigma_n & \mathbf{V}_{nn} \end{bmatrix} = \begin{bmatrix} \mathbf{V}_{-n} & \mathbf{V}_{-n} a \\ a' \mathbf{V}_{-n} & \mathbf{V}_{nn} \end{bmatrix}.$$

Then

$$\mathbf{V} \begin{bmatrix} y \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{V}_{-n} y \\ a' \mathbf{V}_{-n} y \end{bmatrix} > 0 \quad \text{since} \quad a > 0.$$
$$[y' \quad 0] E = y' E_{-n} + 0 = 0,$$
$$[y' \quad 0] \underline{1} = y' \underline{1} + 0 = 0.$$

Hence, $[y' \quad 0]$ meets the conditions for Theorem 1, so there is no positive minimum-variance portfolio on the whole set. Q.E.D.

This proposition is similar in flavor to Theorem 4 in Kandel [10]. That theorem examines the existence of σ_n and $\mathbf{V}_{nn}$, which give mean-variance efficiency for a portfolio consisting of a specific portfolio of the included assets (Kandel's α) and the true portfolio of the missing assets. This proposition exploits Theorem 1 to obtain a sufficient condition on σ_n for the existence of a positive minimum-variance portfolio when the true portfolio of the missing assets is added to the set. This sufficient condition is that it be possible to duplicate the covariances of the added asset by forming a non-negative portfolio of the other assets. The result says in particular that the expected return on the added asset is not important.

As in Kandel, if we have no information regarding σ_n or the variance of the missing asset, we can say nothing about the existence of a positive minimum-variance portfolio when we add the missing asset. However, it also shows that we can make some statement about the existence of the positive minimum-variance portfolio on the strength of only partial information about the missing asset.

To see that Proposition 3 is a special case of Proposition 4, note that the variance matrix for the whole set is

$$\mathbf{V} = \begin{bmatrix} \mathbf{BB}' + \mathbf{D} & \mathbf{B}\beta_n \\ \beta_n' \mathbf{B}' & \beta_n' \beta_n + d_n \end{bmatrix}.$$

If $\beta_n = (\mathbf{B}'\mathbf{D}^{-1}\mathbf{B} + \mathbf{I})a$, then

$$\begin{aligned} \sigma_n &= \mathbf{B}\beta_n \\ &= (\mathbf{B}\mathbf{B}'\mathbf{D}^{-1} + \mathbf{I})\mathbf{B}a \\ &= (\mathbf{B}\mathbf{B}' + \mathbf{D})\mathbf{D}^{-1}\mathbf{B}a, \quad \text{with} \quad \mathbf{D}^{-1}\mathbf{B}a > 0. \end{aligned}$$

Proposition 3 says that adding an asset with such factor loadings cannot widen the cone of risk-compensating λ. To see that adding such an asset can narrow it, consider Green's example. Suppose that $\lambda = [3 \;\; 2]'$. Then λ satisfies (11) with

$$q = \left[\frac{15}{111} \;\; \frac{30}{111}\right]' \quad \text{and} \quad \gamma = -\frac{163}{111}. \qquad \text{Note that } \min(\mathbf{B}q)_i = \frac{165}{111}.$$

However, suppose we add a fourth asset and suppose that $\beta_4 = [5 \;\; 1]'$ are its factor loadings. Then β_4 satisfies (15) with $a = \left[\frac{67}{2903} \;\; \frac{17}{2903}\right]'$ since $\mathbf{B}a > 0$.

Assuming that $d_4 = 1$, the augmented variance matrix is

$$\mathbf{V} = \begin{bmatrix} 53 & 50 & 66 & 34 \\ 50 & 54 & 77 & 37 \\ 66 & 77 & 122 & 55 \\ 34 & 37 & 55 & 27 \end{bmatrix}$$

and

$$\mathbf{B}\lambda = \begin{bmatrix} 26 \\ 25 \\ 33 \\ 17 \end{bmatrix}.$$

Then $y = [0 \;\; -1 \;\; 1/2 \;\; 1/2]'$ has $y'\mathbf{B}\lambda = 0$, $y'\underline{1} = 0$, and $\mathbf{V}y = [0 \;\; 3 \;\; 11.5 \;\; 4]' > 0$. Hence, $\lambda = [3 \;\; 2]'$ is no longer risk compensating on the subset.

Theorem 2 and Proposition 3 say that, if the assets that we examine are in this sense representative of the assets in the market, then rejection of the existence of a positive efficient portfolio on the subset under exact arbitrage pricing implies rejection on the whole set. The example serves to demonstrate that the reverse is not true. That is, it may be the case that a factor-premia vector would be risk compensating on the subset but not on the larger set.

V. Conclusion

In this paper, we have added to the discussion of the relationship between the Arbitrage Pricing Theory and mean-variance analysis. In particular, we have studied the conditions under which exact arbitrage pricing is compatible with the existence of a positive portfolio on the minimum-variance frontier and, therefore, (weakly) compatible with the Capital Asset Pricing Model. By approaching the issue from the standpoint of studying the behavior of the vector of factor premia, we have also contributed insight to the issue of how factor prices should look in equilibrium, an issue that has received rather little attention. We have also shown that the conditions derived here can in principle be used to reject the null hypothesis of the existence of a positive efficient portfolio when we only have in hand a subset of the total universe of assets, if we know enough about the relationship between the included assets and the excluded ones.

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