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Option Bounds with Finite Revision Opportunities

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ABSTRACT

This article generalizes the single-period linear-programming bounds on option prices by allowing for a finite number of revision opportunities. It is shown that, in an incomplete market, the bounds on option prices can be derived using a modified binomial option-pricing model. Tighter bounds are developed under more restrictive assumptions on probabilities and risk aversion. For this case the upper bounds are shown to coincide with the upper bounds derived by Perrakis, while the lower bounds are shown to be tighter.

IN 1984, PERRAKIS AND RYAN [6] derived upper and lower bounds on European option prices in discrete time and under modest assumptions on preferences and return distributions.[1] Although these bounds require more information than the Black-Scholes model, they do not rely on the continuous-time riskless hedges used in the Black and Scholes [1] derivation or on the binomial assumption of Cox, Ross, and Rubinstein [2] and Rendleman and Bartter [7]. As a result, the bounds developed are particularly useful for valuing a contingent claim when the claim and/or underlying asset is thinly traded or not traded at all. Examples of such contracts include options to purchase physical assets, possibly at the expiration of a lease, and claims that result in costs being incurred in the advent of bankruptcy.

Other discrete-time trading models making similar preference assumptions have been established by Levy [4] and Ritchken [8]. Levy uses first- and second-order stochastic dominance rules to obtain option bounds, while Ritchken uses a linear-programming approach to obtain identical bounds. The bounds developed, however, assume that no opportunities exist to revise positions prior to expiration. With a finite number of opportunities to revise positions prior to expiration, the bounds should become tighter. In this paper, the single-period linear-programming option model is extended to handle multiple periods. Initially, we assume that the stock price follows a multiplicative multinomial process. Option bounds are derived under the assumption of discrete trading opportunities. If the process is binomial, then it turns out that the upper and lower bounds coincide, and the binomial option-pricing model is obtained. As such, the models presented here represent a generalization of the binomial option-pricing model. Option bounds are derived under two different sets of assumptions involving preferences and probabilities. Under assumptions similar to Perrakis, we obtain identical upper bounds and tighter lower bounds.

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[1] Perrakis [5] improved these bounds under the assumption that return distributions had a finite range. He also provided a multiple-period model for bounding American puts.

I. Notation and Assumptions

Let the current stock and option prices be S_0 and C_0, respectively. The strike price is X, the time to expiration is 1, and the instantaneous riskless rate is r^*. Let $R = e^{r^*}$. We assume that the time to expiration is partitioned into K periods of equal length. Let n_i be the number of states of nature in period i. Let s_{ij} (c_{ij}) be the value of the security (option) in period i given that state j has occurred ($i = 1, 2, \cdots, K; j = 1, 2, \cdots, n_i$). We label the states such that stock prices in period i (given the stock price in period $i - 1$) form a nondecreasing sequence with respect to the states, i.e., $s_{i,j+1} \geq s_{ij}$. With no intermediate dividends, the value of the security, S_0, can be written as

$$S_0 = \textstyle\sum_{j=1}^{n_i} s_{ij}e_{ij} \quad \text{for} \quad i = 1, 2, \cdots, K,$$

where e_{ij} is the current price of a pure state j security that pays one dollar in period i if state j occurs and zero dollars otherwise. Let B_0^* be the current value of a portfolio containing all state securities. Then

$$B_0^* = \textstyle\sum_{j=1}^{n_k} e_{kj}.$$

Since this portfolio is riskless, $B_0^* = R^{-1}$. We further assume that the risk-free rate is nonstochastic. It follows that

$$(B_0^*)^{i/k} = \textstyle\sum_{j=1}^{n_i} e_{ij}, \qquad i = 1, 2, \cdots, K.$$

Letting r denote one plus the interest rate over one period, we have

$$r = (B_0^*)^{-1/k}.$$

The call prices at expiration are given by $c_{kj} = \max(0, s_{kj} - X), j = 1, 2, \cdots, n_k$. Hence, $C_0 = \sum_{j=1}^{n_k} c_{kj}e_{kj}$.

We shall assume that the stock price, S_0, and bond price, B_0^*, are observable. The problem is to derive bounds for the call price given that portfolio revisions can occur in each period prior to expiration and given that the stock price follows a multiplicative multinomial process. Specifically, given any state, the stock price can move to n possible future states in the next period. Let $u_1, u_2, \cdots, u_n$ be the n possible price relatives for the process with $u_1 < u_2 < \cdots < u_n$. Figure 1 illustrates a typical stock price process for $n = 3$ and $K = 2$. Let $\tilde{U}$ be the random

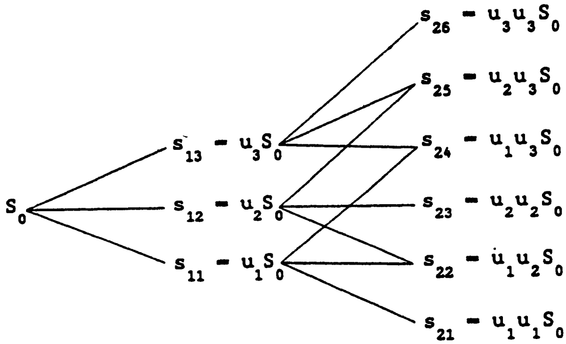


Figure 1. A Sample Stock Price Process

variable representing the actual price relative in a period. Then

$$\hat{u} = \mathrm{E}(\tilde{U}) = \textstyle\sum_{i=1}^{n} u_i\pi_i,$$

where π_i is the probability of price relative u_i occurring. Let $\hat{u}_j$ be the conditional expected return given that the price relative does not exceed u_j. Then

$$\hat{u}_j = \mathrm{E}(\tilde{U} \mid \tilde{U} \le u_j) = \textstyle\sum_{i=1}^{j} u_i\pi_i / \sum_{i=1}^{j}\pi_i.$$

Clearly, $\hat{u}_1 = u_1$ and $\hat{u}_n = \hat{u}$. Similarly, define $\hat{s}_{ij}$ ($\hat{c}_{ij}$) as the expected stock price (call price) in period i given that the price relative in period i does not exceed u_j.

II. Single-Period Option-Pricing Bounds in an Incomplete Market

Under the assumption that no revisions occur prior to expiration (i.e., $K = 1$), bounds can be obtained by solving the following two linear-programming problems (Ritchken [8]).

$$\underset{\{e_{kj}\}}{\text{maximize (minimize)}}\ C_0 = \textstyle\sum_{j=1}^{n_k} c_{kj}e_{kj}$$

$$\begin{aligned} \text{subject to}\quad & \textstyle\sum_{j=1}^{n_k} s_{kj}e_{kj} = S_0, \\ & \textstyle\sum_{j=1}^{n_k} e_{kj} = r^{-1}, \\ & e_{kj} \ge 0, \qquad j = 1, 2, \cdots, n_k. \end{aligned} \tag{P1}$$

If the stock is a positive-beta security, with higher prices occurring in states of higher aggregate wealth, and if there exists a set of ex ante probabilities across the states, then the option-pricing bounds can be established by solving the following two linear-programming problems (Ritchken [8]).

$$\underset{\{d_{kj}\}}{\text{maximize (minimize)}}\ C_0 = \textstyle\sum_{j=1}^{n_k} c_{kj}\pi_{kj}d_{kj}$$

$$\begin{aligned} \text{subject to}\quad & \textstyle\sum_{j=1}^{n_k} s_{kj}\pi_{kj}d_{kj} = S_0, \\ & \textstyle\sum_{j=1}^{n_k} \pi_{kj}d_{kj} = r^{-1}, \\ & d_{k1} \ge d_{k2} \ge \cdots \ge d_{kn_k} \ge 0, \end{aligned} \tag{P2}$$

where π_{kj} is the ex ante probability of state j occurring in period K and d_{kj} is the discount factor associated with state j in period K. Let $\overline{C}_0$ and $\underline{C}_0$ be the upper and lower bounds on the option price, respectively. Proposition 1 follows by solving the maximization and minimization problems of (P1) for $K = 1$.

Proposition 1: (i) *Under the assumptions of* (P1), *the single-period option bounds in an incomplete market are*

$$\overline{C}_0 = r^{-1}[\theta c_{1n} + (1 - \theta)c_{11}] \tag{1}$$

and

$$\underline{C}_0 = r^{-1}[\alpha c_{1,h+1} + (1 - \alpha)c_{1h}], \tag{2}$$

where $\theta = (r - u_1)/(u_n - u_1)$, $\alpha = (r - u_h)/(u_{h+1} - u_h)$, *and* h *is chosen such that* $u_h \leq r < u_{h+1}$.

(ii) *Under the assumptions of* (P2), *the single-period option bounds in a risk-averse incomplete market are*

$$\bar{C}_0 = r^{-1}[\hat{\theta}c_{1n} + (1 - \hat{\theta})c_{11}] \tag{3}$$

and

$$\underline{C}_0 = r^{-1}[\hat{\alpha}\hat{c}_{1,h+1} + (1 - \hat{\alpha})\hat{c}_{1h}], \tag{4}$$

where $\hat{\theta} = (r - \hat{u}_1)/(\hat{u}_n - \hat{u}_1)$, $\hat{\alpha} = (r - \hat{u}_h)/(\hat{u}_{h+1} - \hat{u}_h)$, *and* h *is chosen such that* $\hat{u}_h \leq r < \hat{u}_{h+1}$.[2]

As a direct corollary to this proposition, if the stock price follows a bounded continuous-state process, then the bounds reduce to

$$\bar{C}_0 = r^{-1}[\bar{\theta}\mathrm{E}(\tilde{C}_1) + (1 - \bar{\theta})c_{11}], \tag{5}$$

$$\underline{C}_0 = r^{-1}\mathrm{E}[\tilde{C}_1 \mid S_1/S_0 \leq u^*], \tag{6}$$

where $\bar{\theta} = (r - u_1)/(\hat{u} - u_1)$ and u^* is chosen such that $\mathrm{E}(\tilde{U} \mid \tilde{U} \leq u^*) = r$. Notice that, while the upper bound (5) is precisely Perrakis's upper bound (expression (9) in Perrakis [5]), the lower bound given in (6) is tighter.[3] Notice too that, if the multinomial process is binomial, then the bounds (1) and (2) (as well as (3) and (4)) converge and the resulting price is the single-period binomial option price.

III. Multiperiod Option-Pricing Bounds in an Incomplete Market

To derive option bounds for call options for the multiperiod problem, we first consider a two-period ($K = 2$) trinomial ($n = 3$) process and then generalize the results in Proposition 2.

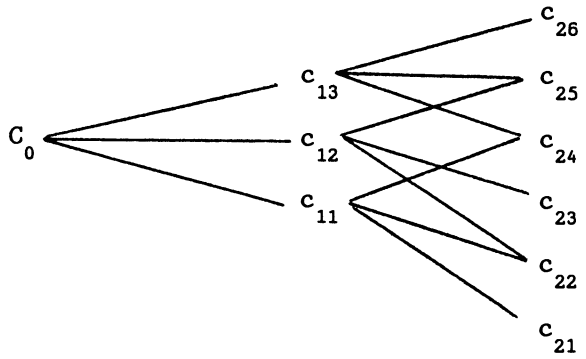
The option prices corresponding to the stock price of Figure 1 are shown in Figure 2.

To obtain bounds for C_0, we proceed by using backward recursion. Let $\bar{c}_{1j}$ ($\underline{c}_{1j}$) represent the upper (lower) bound on the call price in period 1 when the stock price is s_{1j} ($j = 1, 2, 3$). Assuming that $u_1 < u_2 < r < u_3$, then, using result (i) of Proposition 1, bounds on the option prices in period 1 can be obtained. Let $\bar{c}_{1j}$ and $\underline{c}_{1j}$ represent the upper and lower bounds. Now consider the initial period. The optimization problems that must be solved are as follows.

$$\underset{\{e_{ij}\}}{\text{maximize (minimize)}}\ C_0 = \textstyle\sum_{j=1}^{3} c_{1j}e_{1j}$$

[2] The proofs of (i) and (ii) are obtained by applying Propositions 2 and 3 of Ritchken [8] to the case of multinomial stock price.

[3] Substituting $\tilde{C}_1 = \max(\tilde{S}_1 - X, 0)$ into (6) and rewriting the equation, we obtain $\underline{C}_0 = \max(0, S_0 - Xr^{-1} + r^{-1}\mathrm{E}(\tilde{P}_1)/\mathrm{prob}(\tilde{U} \leq u^*))$, where $\tilde{P}_1 = \max(X - \tilde{S}_1, 0)$. Comparing this formula to that of Perrakis yields the result.



where $c_{2j} = \mathrm{Max}(0,\ s_{2j} - X)$.

Figure 2. A Sample Call Price Process

subject to $\sum_{j=1}^{3} s_{1j}e_{1j} = S_0$,

$\sum_{j=1}^{3} e_{1j} = r^{-1}$,

$$e_{1j} \geq 0, \qquad j = 1, 2, 3. \tag{P3}$$

Let $C_{0,\max}$ and $C_{0,\min}$ be the optimal solutions to the two problems. Unfortunately, since the call prices in period 1 are unknown, Proposition 1 is not directly applicable. Notice, however, that the call prices do not enter the constraints. Moreover, since $\bar{c}_{1j} \geq c_{1j}$ ($j = 1, 2, 3$), it follows that

$$\bar{C}_0 \equiv \max \sum_{j=1}^{3} \bar{c}_{1j}e_{1j} \geq \max \sum_{j=1}^{3} c_{1j}e_{1j} = C_{0,\max}.$$

Similarly,

$$\underline{C}_0 \equiv \min \sum_{j=1}^{3} \underline{c}_{1j}e_{1j} \leq \min \sum_{j=1}^{3} c_{1j}e_{1j} = C_{0,\min}.$$

Hence, bounds on the call price, at the current time, can be obtained by solving (P3), with $\bar{c}_{1j}$ ($\underline{c}_{1j}$) replacing c_{1j} for the maximization (minimization) problem. Clearly, $\underline{C}_0 \leq C_0 \leq \bar{C}_0$.

LEMMA 1: *Under the assumptions of* (P1), *the option bounds in a two-period trinomial problem are*

$$\bar{C}_0 = r^{-2}[\theta^2 c_{26} + 2\theta(1 - \theta)c_{24} + (1 - \theta)^2 c_{21}] \tag{7}$$

and

$$\underline{C}_0 = r^{-2}[\alpha^2 c_{26} + 2\alpha(1 - \alpha)c_{25} + (1 - \alpha)^2 c_{23}]. \tag{8}$$

Proof: To obtain an upper bound, we solve (P3) using the objective function: $C_0 = \sum_{j=1}^{3} \bar{c}_{1j}e_{1j}$. Since the piece-wise linear function connecting adjacent (s_{1j}, $\bar{c}_{1j}$) coordinates is convex, the conditions of Proposition 1 of Ritchken [8] are satisfied. Hence, the form of the optimal solution is given by

$$\bar{C}_0 = r^{-1}[\theta\bar{c}_{13} + (1 - \theta)\bar{c}_{11}]. \tag{9}$$

Substituting expressions for $\bar{c}_{13}$ and $\bar{c}_{11}$ from (5) into (9) yields the result. The proof of the lower bound follows in a similar way. Q.E.D.

It is interesting to note that, for the two-period trinomial model, the upper bound does not depend on the value u_2. Indeed, the maximum is fully determined by u_3 and u_1 and by the states that can be reached by u_3 and u_1 price movements. More generally, for an n-state multinomial model, the upper bound is fully determined by those states that can be reached only from u_n and u_1 price movements. Similarly, the lower bound is fully determined by u_{h+1} and u_h and by the terminal call prices obtained by these price movements alone. In essence, the maximum value for the option is based on the maximum volatility of the underlying security, while the minimum value is based on the minimum volatility of the security around the risk-free rate. Based on this observation, the upper bound can be obtained by considering the binomial tree of prices generated by the u_n and u_1 price relatives. Similarly, the lower bound can be obtained by considering the binomial tree of prices generated by u_{h+1} and u_h price-relative movements. Recursively, using Proposition 1, the results obtained in the two-period trinomial-state problem can be generalized to the following proposition.

PROPOSITION 2: *For a K-period multinomial process, with revision opportunities available at the end of each period, bounds on the call price are given by*

$$\bar{C}_0 = r^{-K}\left\{\sum_{j=0}^{K} \frac{K!}{j!(K-j)!}\,\theta^j(1-\theta)^{K-j}\max(0,\, u_n^j u_1^{K-j} S_0 - X)\right\} \tag{10}$$

and

$$\underline{C}_0 = r^{-K}\left\{\sum_{j=0}^{K} \frac{K!}{j!(K-j)!}\,\alpha^j(1-\alpha)^{K-j}\max(0,\, u_{h+1}^j u_h^{K-j} S_0 - X)\right\}, \tag{11}$$

where $\theta = (r-u_1)/(u_n-u_1)$, $\alpha = (r-u_h)/(u_{h+1}-u_h)$, *and h is chosen such that* $u_h \le r < u_{h+1}$.

IV. Multiperiod Option-Pricing Bounds under Risk Aversion

By considering a two-period multinomial problem and then a multiperiod multinomial problem, bounds on option prices can be obtained under the assumptions that investors in the economy are risk averse and the conditions of (P2) are satisfied. The resulting bounds are summarized in Proposition 3.

PROPOSITION 3: *Under the assumptions of* (P2), *for a K-period multinomial process with revision opportunities available at the end of each period, bounds on the call price are given by*

$$\bar{C}_0 = r^{-K}\left\{\sum_{\{j_\mu\}} \frac{K!}{j_1!j_2!\cdots j_n!}\,p_1^{j_1}p_2^{j_2}\cdots p_n^{j_n}\max(0,\, u_1^{j_1}u_2^{j_2}\cdots u_n^{j_n}S_0 - X)\right\} \tag{12}$$

and

$$\underline{C}_0 = r^{-K}\left\{\sum_{\{j_\mu\}} \frac{K!}{j_1!j_2!\cdots j_{h+1}!}\,q_1^{j_1}q_2^{j_2}\cdots q_{h+1}^{j_{h+1}}\max(0,\, u_1^{j_1}u_2^{j_2}\cdots u_{h+1}^{j_{h+1}}S_0 - X)\right\}, \tag{13}$$

where

$$p_1 = \hat{\theta}\pi_1 + (1 - \hat{\theta}), \qquad p_j = \hat{\theta}\pi_j, \quad \textit{for} \quad j = 2, 3, \cdots, n,$$

$$q_{h+1} = \hat{\beta}, \qquad q_j = (1 - \hat{\beta})\pi_j/\textstyle\sum_{i=1}^{h} \pi_i, \quad \textit{for} \quad j = 1, 2, \cdots, h,$$

and $\{j_\mu\}$ *represents all sets of non-negative integers satisfying*

$$\textstyle\sum_{\mu=1}^{n} j_\mu = K \textit{ for } (12) \textit{ and } \sum_{\mu=1}^{h+1} j_\mu = K \textit{ for } (13),$$

where

$$\hat{\theta} = (r - u_1)/(\hat{u} - u_1) \quad \textit{and} \quad \hat{\beta} = (r - \hat{u}_h)/(u_{h+1} - \hat{u}_j).$$

V. Numerical Examples

With the given case parameters, Table I illustrates option bounds derived under the assumption that the stock price process is trinomial and that there are four revision opportunities. The results are compared with bounds derived under the assumption that there are no revision opportunities. Notice that the existence of trading opportunities tightens the bounds.

The second example considers a trinomial stock price process where the frequency of revision periods, K, over the time to expiration, T, is increased such that, in the limiting case, the trinomial process converges to the geometric Wiener process. Let μ and σ represent the instantaneous expected return and standard deviation of the limiting process. Following Cox and Rubinstein [3], the finite-period trinomial-process parameters are chosen as

$$u_1 = e^{-\sigma\sqrt{2T/k}}, \qquad u_2 = 1, \qquad u_3 = u_1^{-1},$$

$$\pi_1 = \frac{1}{4} - \frac{1}{4}\left(\frac{\mu}{\sigma}\right)\sqrt{2T/k}, \qquad \pi_3 = \frac{1}{4} + \frac{1}{4}\left(\frac{\mu}{\sigma}\right)\sqrt{2T/k}, \qquad \pi_2 = 1 - \pi_1 - \pi_3.$$

Table I

Option Bounds in a Risk-Averse Incomplete Market

Strike Price	Merton's Lower Bound	No Revisions ($k = 1$)		Four Revision Opportunities ($k = 5$)	
X	$\max(0, S - X/r)$	$\bar{C}_0$	$\underline{C}_0$	$\bar{C}_0$	$\underline{C}_0$
75.00	8.571	9.135	8.582	8.627	8.584
80.00	3.810	5.603	4.418	4.712	4.509
85.00	0.000	2.855	1.572	2.125	1.846
90.00	0.000	1.161	0.193	0.735	0.588
95.00	0.000	0.414	0.019	0.228	0.170

Case Parameters

$S_0 = 80$, $r = 1.05$,
$u_1 = (0.9)^{1/5}$, $\pi_1 = 0.2$,
$u_2 = 1.0$, $\pi_2 = 0.5$,
$u_3 = (1.4)^{1/5}$, $\pi_3 = 0.3$

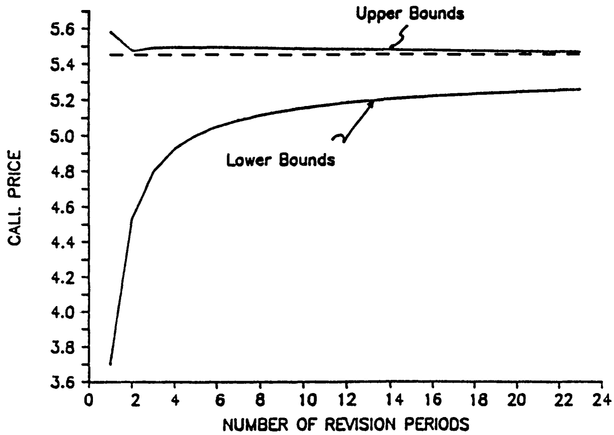


Figure 3. Option Bounds versus Number of Revision Periods

Figure 3 illustrates how the option bounds converge as the frequency of revision opportunities increases. These bounds are compared with the Black-Scholes option price. In this example, the stock price and strike price are fifty, the time to maturity is 0.5 years and the risk-free rate is ten percent. In addition, the instantaneous mean and volatility parameters are 0.2 and 0.3, respectively.

VI. Summary

This article has extended the single-period linear-programming option-bounds model to allow for multiple revisions. Option bounds were derived in an incomplete market and a risk-averse incomplete market. The upper bounds derived were shown to be identical to the bounds of Perrakis, while the lower bounds were generally tighter. The models developed generalize the binomial option-pricing approach and are valuable when the continuous-trading assumption is not appropriate. These models can be extended easily to establish bounds on European put options. Since the approach allows periodic revisions, bounds on American put options may also be obtained.

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