A Note on Simple Criteria for Optimal Portfolio Selection

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Source: *The Journal of Finance*, Vol. 43, No. 1 (Mar., 1988), pp. 241-245
Published by: Blackwell Publishing for the American Finance Association
Stable URL: http://www.jstor.org/stable/2328333
Accessed: 26/11/2009 07:35

THE JOURNAL OF FINANCE • VOL. XLIII, NO. 1 • MARCH 1988

# A Note on Simple Criteria for Optimal Portfolio Selection

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USING SHARPE'S [7] SINGLE-INDEX model, Elton, Gruber, and Padberg (EGP) [4, 5] have established some simple ranking criteria for optimal portfolio selection. A further extension has been made by Chen and Brown [2] and Alexander and Resnick [1] to incorporate the risk associated with the estimation of input parameters. This note explores the economic rationale underlying the portfolio formation process based on the single-index model. In so doing, we seek to enhance the intuitive appeal of the simplified portfolio criteria developed in the above studies to practitioners.

To this end, we restate the EGP solution with short sales allowed as

$$Z_i = \frac{\beta_i}{\sigma_{\epsilon i}^2}\left(\frac{\bar{R}_i - R_f}{\beta_i} - \phi\right), \quad \text{for} \quad i = 1,2,\cdots,n, \tag{1}$$

where $\phi = \sigma_m^2 \sum_{j=1}^n \beta_j Z_j$ and $Z_i = X_i(\bar{R}_p - R_f)/\sigma_p^2$. Here, $\beta_i$ is the beta coefficient of security $i$, $\sigma_{\epsilon i}^2$ is the residual variance of security $i$, $\bar{R}_i$ is the expected return on security $i$, $R_f$ is the risk-free rate, $\sigma_m^2$ is the variance of the index $I_m$, $X_i$ is the fraction of funds invested in security $i$, $\bar{R}_p$ is the expected return on the optimal portfolio, and $\sigma_p^2$ is the variance of returns of the optimal portfolio.

In the case where short sales are disallowed, EGP have shown that the Kuhn-Tucker conditions are necessary and sufficient for optimality. The first Kuhn-Tucker condition differs from equation (1) by an additive term $\nu_i/\sigma_{\epsilon i}^2$ on the right-hand side, $\nu_i$ being a Lagrange multiplier. The complementarity conditions require that $\nu_i = 0$ for $Z_i > 0$ and $\nu_i \geq 0$ for $Z_i = 0$. To reach the optimal portfolio disallowing short sales, EGP rank securities according to their excess-return-to-beta ratios. The optimal portfolio is formed by adding securities successively to feasible portfolios following the ranking hierarchy. The procedure stops when the excess-return-to-beta ratio of a security falls below the corresponding cutoff rate $\phi$.

The lack of an explicit role for the index in selecting securities for the optimal portfolio in the above procedure is somewhat counter-intuitive since the index is the driving force underlying security returns and hence underlying the returns on the optimal portfolio. It will be shown below that the cutoff rate used in the EGP approach is indeed related to the index. This relationship enables us to see intuitively that securities are accepted because of their ability to capture the driving force in the optimization of the risk-return tradeoff. Further, when and why the above iterative procedure is terminated will become more obvious.

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To establish exactly what role the index plays, let us first rank and label securities following the EGP convention. Specifically, for a portfolio problem involving only positive beta securities, they are labeled in such a way that $(\bar{R}_i - R_f)/\beta_i \geq (\bar{R}_{i+1} - R_f)/\beta_{i+1}$ for all $i$. Now a portfolio $k$ allowing short sales based on securities 1, 2, $\cdots$, $k$ is formed using equation (1). The covariance between the portfolio return $R_k$ and $I_m$ is $\sigma_{km} = \beta_k \sigma_m^2$. It follows from $\beta_k = \sum_{j=1}^k X_j \beta_j = \sum_{j=1}^k Z_j \beta_j / \sum_{j=1}^k Z_j$ and $\sum_{j=1}^k Z_j = (\bar{R}_k - R_f)/\sigma_k^2$ that

$$\rho(k) = \frac{\phi(k)}{\sigma_m \theta(k)}, \tag{2}$$

where $\rho(k) = \sigma_{km}/(\sigma_k \sigma_m)$ is the correlation between $R_k$ and $I_m$, $\theta(k) = (\bar{R}_k - R_f)/\sigma_k$ measures the portfolio performance, and $\phi(k) = \sigma_m^2 \sum_{j=1}^k Z_j \beta_j = \sigma_m^2 \sum_{j=1}^k [\beta_j (\bar{R}_j - R_f)/\sigma_{\epsilon j}^2]/[1 + \sigma_m^2 \sum_{j=1}^k \beta_j^2/\sigma_{\epsilon j}^2]$ provides the cutoff rate for the portfolio based on the $k$ highest ranking securities.

As demonstrated in Kwan [6] and Cheung, Kwan, and Yip [3], the function $\phi(k)$, where $k = 1, \cdots, n$, reaches its single maximum, which corresponds to the cutoff rate for the solution where short sales are disallowed. It is shown in the Appendix that $\rho(k)$ displays the same functional behavior. In other words, if one forms portfolios by adding securities successively from the highest rank to the lowest rank, the optimal portfolio is reached when the correlation of the portfolio returns and the index is at its maximum. Since the expected return on the index must be positive if the investor is to invest in stocks, an objective of the investor using the single-index model to establish the risk-return tradeoff is to pick securities that benefit the most from a market upswing. Here the role of the index in the selection of securities is made explicit.

Given that $\rho(k) = \sigma_{km}/(\sigma_k \sigma_m)$, $\sigma_{km} = \beta_k \sigma_m^2$, and $\sigma_k^2 = \beta_k^2 \sigma_m^2 + \sigma_{\epsilon k}^2$, we also have

$$\rho(k) = \sqrt{\beta_k^2 \sigma_m^2 / \sigma_k^2} = \sqrt{1 - (\sigma_{\epsilon k}^2/\sigma_k^2)}, \tag{3}$$

indicating that $\rho(k)$ measures directly the systematic risk of the portfolio as a fraction of its total risk. Therefore, the maximization of $\rho(k)$, which captures the market momentum suggested above, actually corresponds to minimizing the idiosyncrasies of individual securities. This rendition of the simple criteria for optimal portfolio selection is consistent with the practice of some practitioners to search for the market leadership.

Table I provides a numerical example illustrating the above features underlying the portfolio formation process. In this example, we used a subset of stocks that make up the Dow Jones Industrial Index. Monthly returns from January 1980 to December 1984 for both the equally weighted index and twenty Dow Jones stocks were taken from the CRSP tape to estimate betas, residual risks, and average returns. Here, the stocks in Table I have been ranked according to excess-return-to-beta ratios. Notice that the optimal portfolio disallowing short sales occurs when $\phi(k)$ reaches its single maximum of 0.01034 at $k = 7$. The same behavior can likewise be observed for $\rho(k)$, which has its maximum of 0.82696 at $k = 7$. Also, the residual risk as a fraction of the total risk is minimized when optimality is reached at $k = 7$. The above pattern is consistent with our analytical results.

**Table I**

An Illustrative Example Based on Actual Data Showing the Behavior of $\phi(k)$, $\rho(k)$, and $\sigma_{\epsilon k}^2/\sigma_k^2$ (Input Data: $\sigma_m^2 = 0.00207$, $R_f = 0.00718$)

| Security | Input Data | | | | Output | | |
|---|---|---|---|---|---|---|---|
| $i$ | $\bar{R}_i - R_f$ | $\beta_i$ | $\frac{\bar{R}_i - R_f}{\beta_i}$ | $\sigma_{\epsilon i}^2$ | $\phi(k)$[a] | $\rho(k)$ | $\frac{\sigma_{\epsilon k}^2}{\sigma_k^2}$ |
| 1. Amer. Brands | 0.00960 | 0.529 | 0.01813 | 0.00310 | 0.00286 | 0.39724 | 0.84220 |
| 2. Woolworth | 0.01139 | 0.656 | 0.01737 | 0.00819 | 0.00408 | 0.47790 | 0.77160 |
| 3. Proc. & Gam. | 0.00821 | 0.475 | 0.01729 | 0.00233 | 0.00585 | 0.57588 | 0.66836 |
| 4. Sears | 0.01891 | 1.327 | 0.01426 | 0.00416 | 0.00896 | 0.75089 | 0.43616 |
| 5. Gen. Elec. | 0.01110 | 0.801 | 0.01386 | 0.00232 | 0.00991 | 0.80028 | 0.35955 |
| 6. Gen. Motor | 0.00846 | 0.672 | 0.01260 | 0.00418 | 0.01010 | 0.81160 | 0.34130 |
| 7. Goodyear | 0.01221 | 1.005 | 0.01214 | 0.00488 | 0.01034 | 0.82696 | 0.31614 |
| 8. Owens III | 0.00944 | 0.954 | 0.00990 | 0.00476 | 0.01030 | 0.82321 | 0.32232 |
| 9. Westinghouse | 0.00961 | 1.224 | 0.00784 | 0.00309 | 0.00981 | 0.77197 | 0.40406 |
| 10. 3M | 0.00456 | 0.690 | 0.00660 | 0.00178 | 0.00949 | 0.73526 | 0.45940 |
| 11. Amer. Express | 0.00886 | 1.528 | 0.00580 | 0.00372 | 0.00879 | 0.65356 | 0.57286 |
| 12. Alum. Co. Am. | 0.00363 | 0.770 | 0.00472 | 0.00663 | 0.00868 | 0.64032 | 0.58999 |
| 13. Du Pont | 0.00196 | 0.964 | 0.00203 | 0.00257 | 0.00804 | 0.55006 | 0.69744 |
| 14. U.S. Steel | 0.00074 | 1.101 | 0.00067 | 0.00454 | 0.00755 | 0.48578 | 0.76402 |
| 15. U. Tech. | 0.00040 | 1.020 | 0.00039 | 0.00244 | 0.00687 | 0.40839 | 0.83322 |
| 16. Union Car. | 0.00032 | 0.932 | 0.00035 | 0.00267 | 0.00642 | 0.36521 | 0.86662 |
| 17. Merck | −0.00087 | 0.519 | −0.00167 | 0.00250 | 0.00625 | 0.34713 | 0.87950 |
| 18. Allied | −0.00258 | 0.901 | −0.00286 | 0.00449 | 0.00592 | 0.31476 | 0.90092 |
| 19. Eastman K. | −0.00154 | 0.435 | −0.00354 | 0.00372 | 0.00583 | 0.30576 | 0.90651 |
| 20. Beth. Steel | −0.00579 | 1.364 | −0.00424 | 0.00650 | 0.00529 | 0.25823 | 0.93332 |

[a] $k$ represents the portfolio based on securities $1, 2, \cdots, k$.

A further insight can be derived for the case when short sales are allowed. It follows directly from equation (2) that

$$\Delta\phi = \sigma_m[\theta + \Delta\theta/2][\rho + \Delta\rho/2]\left[\frac{\Delta\rho}{\rho + \Delta\rho/2} + \frac{\Delta\theta}{\theta + \Delta\theta/2}\right], \tag{4}$$

where the symbols have been abbreviated for simplicity, i.e., $\theta = \theta(k-1)$, $\rho = \rho(k-1)$, $\Delta\theta = \theta(k) - \theta(k-1)$, $\Delta\rho = \rho(k) - \rho(k-1)$, and $\Delta\phi = \phi(k) - \phi(k-1)$. As shown in the Appendix, $\theta(k)$ is an increasing function of $k$, and thus the first bracketed term above is always positive. The second bracketed term above can be thought of as the correlation halfway between the old level and the new level as an extra security is added; this term is always positive. When a security beyond the single maximum of $\phi$ is added to the portfolio, $\Delta\phi$ must be negative, thereby making the last bracketed term above negative. Since $\Delta\theta/(\theta + \Delta\theta/2)$ must be positive, $\Delta\rho/(\rho + \Delta\rho/2)$ must be not only negative but also of greater absolute magnitude. Intuitively, $\Delta\rho/(\rho + \Delta\rho/2)$ is nothing more than the reduction in the systematic risk as a fraction of the total risk as a security is sold short, whereas $\Delta\theta/(\theta + \Delta\theta/2)$ represents the corresponding improvement in portfolio performance. In effect, the major impact of allowing short sales is not so much to improve portfolio performance as to reduce the systematic risk content.

What has been said up to this point also holds in the presence of estimation risk. The cutoff function in the decision rule similar to equation (1), as derived by Chen and Brown [2] and Alexander and Resnick [1], has the same functional form as the case without estimation risk. As a result, the economic intuition derived here applies equally well to the decision rule given in the above two studies.

## Appendix

We will establish below that the function $\rho(k)$ in equation (2) in the text has a single maximum and occurs concurrently with the single maximum of $\phi(k)$. What needs to be shown is that $\rho_k > \rho_{k-1}$ whenever $\phi_k > \phi_{k-1}$ and $\rho_k < \rho_{k-1}$ whenever $\phi_k < \phi_{k-1}$, where each subscript represents the portfolio for which the corresponding function is evaluated.

*Case* 1: $\phi_k > \phi_{k-1}$. Defining $\Delta\rho^2 = \rho_k^2 - \rho_{k-1}^2$, $\Delta\phi^2 = \phi_k^2 - \phi_{k-1}^2$, and $\Delta\theta^2 = \theta_k^2 - \theta_{k-1}^2$, we obtain from equation (2)

$$\Delta\rho^2 = \frac{\theta_{k-1}^2\Delta\phi^2 - \phi_{k-1}^2\Delta\theta^2}{\sigma_m^2(\theta_{k-1}^2 + \Delta\theta^2)\theta_{k-1}^2}. \tag{A1}$$

To simplify notation, let $A_i = \sigma_m^2\beta_i^2/\sigma_{\epsilon i}^2$, $t_i = (\bar{R}_i - R_f)/\beta_i$, $P_k = \sum_{i=1}^k A_i t_i^2$, $N_k = \sum_{i=1}^k A_i t_i$, and $M_k = 1 + \sum_{i=1}^k A_i$. From $\theta_k^2 = (\bar{R}_k - R_f)^2/\sigma_k^2 = \sum_{i=1}^k (\bar{R}_i - R_f)Z_i$ and equation (1), we have $\sigma_m^2\theta_k^2 = P_k - N_k^2/M_k$, which leads to $\sigma_m^2\Delta\theta^2 = A_k(M_{k-1}t_k - N_{k-1})^2/(M_{k-1}M_k)$. Since $A_k$ and $M_k$ are positive for all $k$, $\Delta\theta^2$ is always positive; i.e., $\theta(k)$ is an increasing function of $k$.

With $\sigma_k^2 = N_k^2/M_k^2$, we have

$$\Delta\phi^2 = \frac{A_k(M_{k-1}t_k - N_{k-1})(A_k t_k M_{k-1} + 2M_{k-1}N_{k-1} + A_k N_{k-1})}{(M_{k-1} + A_k)^2 M_{k-1}^2}. \quad \text{(A2)}$$

Therefore, the numerator on the right-hand side of equation (A1) can be expressed as

$$\theta_{k-1}^2 \Delta\phi^2 - \phi_{k-1}^2 \Delta\theta^2 = \frac{A_k}{\sigma_m^2 M_k^2}(t_k - \phi_{k-1})$$
$$\times [(\textstyle\sum_{i=1}^{k-1} A_i t_i^2 - \sum_{i=1}^{k-1} A_i t_i t_k)N_{k-1} + (\sum_{i=1}^{k-1} A_i t_i^2 - \sum_{i=1}^{k-1} A_i t_i \phi_{k-1})A_k t_k]. \quad \text{(A3)}$$

Given that $\phi_k > \phi_{k-1}$, which implies that $t_1 \geq t_2 \geq \cdots \geq t_k > \phi_k > \phi_{k-1}$ (see Cheung, Kwan, and Yip [3] and Kwan [6]), $\theta_{k-1}^2 \Delta\phi^2 - \phi_{k-1}^2 \Delta\theta^2$ must be positive. It follows from equation (A1) that $\Delta\rho^2 > 0$ or, equivalently, $\rho_k > \rho_{k-1}$, as long as $\phi_k > \phi_{k-1}$.

*Case* 2: $\phi_k < \phi_{k-1}$. Defining $\Delta\rho = \rho_k - \rho_{k-1}$, $\Delta\phi = \phi_k - \phi_{k-1}$, and $\Delta\theta = \theta_k - \Delta_{k-1}$, we obtain from equation (2) $\Delta\rho = (\Delta\phi - \sigma_m \rho_{k-1}\Delta\theta)/(\sigma_m \theta_k)$. Recall that $\theta(k)$ is an increasing function of $k$. This implies that $\Delta\theta > 0$. Given the condition $\phi_k < \phi_{k-1}$, $\Delta\rho$ is negative or, equivalently, $\rho_k < \rho_{k-1}$.

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