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# On the Optimal Hedge of a Nontraded Cash Position

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**ABSTRACT**

In this paper, we focus on the optimal demand for futures contracts by an investor with a logarithmic utility function who attempts to hedge a nontraded cash position. When the analysis is conducted in the "cash-commodity-price" space, we show that the value function associated with the Bernoulli investor program is not additively separable, thus suggesting that this investor hedges against shifts in the opportunity set as represented by the commodity price. By establishing the equivalence between the cash formulation of the problem and the wealth formulation, we are able to analyze the problem in the "wealth-commodity-price" space. In this space, we show the additive separability of the value function when the futures settlement price process is perfectly locally correlated with the commodity price process. The demand for futures in this instance is composed of (a) a mean-variance term and (b) a minimum-variance component that is a classic feature of models with nontraded assets. Since the first-best (nonmyopic) optimum is attained, however, the deviation from a mean-variance demand should not be interpreted as the expression of a nonmyopic behavior but rather as an attempt to restore a first-best optimum. On the other hand, when the correlation between the futures price and the underlying commodity price is imperfect, in general, the value function does not separate additively, the first-best solution cannot be attained, and the optimal futures trading strategy involves a hedging term against shifts in the opportunity set.

IN THE CLASSIC DYNAMIC portfolio problem, the investor selects a trading strategy in risky and riskless assets so as to maximize his or her expected lifetime utility. In this environment, where all assets can be freely traded, the logarithmic investor behaves myopically to the extent that he or she ignores shifts in the opportunity set when making current choices (Samuelson [27]; Mossin [24]; Hakansson [16]; Merton [22]). In particular, in models with continuous trading, it turns out that the value function corresponding to the investor's program is additively separable in wealth and state variables. The implication is that his or her behavior will be myopic, i.e., that his or her demand functions for assets will be of the pure mean-variance efficiency type and in particular will not include any dynamic hedging components that are characteristic of financial markets with shifting opportunities. In this paper, we investigate the structure of the Bernoulli investor's demand functions in the situation where one asset cannot be freely traded in the market.

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The assumption that some assets are nontradeable is a key characteristic of the traditional hedging model (THM) that considers the problem of hedging a fixed position with a futures contract or some other derivative security correlated with the underlying asset. Two formulations of this hedging problem are possible. In the first, adopted by Ho [18], the investor selects an optimal trading strategy based on a dynamic cash-budget constraint. The effect of the nontraded position on the solution, in this case, arises through the boundary condition that stipulates that the value function at the terminal date be equal to the utility of terminal wealth, composed of the cash accumulated through trades plus the cash generated by the nontraded position at maturity. The optimal futures strategy appears to have the same structure as in the classic portfolio problem without nontraded assets; it is comprised of (a) a mean-variance efficiency component and (b) a Merton/Breeden (dynamic) hedging component. In the second formulation of the problem, the dynamic budget constraint is a wealth constraint involving the cash position augmented by the present value of the nontraded position. This specification, similar to that of Stulz [28], leads to demand functions consisting of three terms: (a) a mean-variance efficiency term, (b) a Merton/Breeden dynamic hedging component, and (c) a minimum-variance component characteristic of one-period models with nontraded assets (Mayers [21]). The apparent difference between the hedging demands is illusory. In Section II, we show that they are equivalent. It follows from this demonstration that the minimum-variance component can be interpreted as an intertemporal hedge "fund" in the sense of Merton.

The remainder of the paper mainly focuses on the demand for futures by an investor with a logarithmic utility function in an economic context where the futures settlement price process is perfectly locally correlated with the commodity price process. In Section III, we show that the value function associated with the Bernoulli investor's program is nonseparable in the space of cash balances and the state variable (the price of the nontraded commodity). A conceivable interpretation of this result is that the Bernoulli investor behaves nonmyopically in this instance. In Section IV, however, we show that the value function is nonetheless separable in the space of wealth and the state variable. As a consequence, the demand for futures by the logarithmic investor is composed of (a) a mean-variance efficiency component and (b) a pure minimum-variance component. The second term, which is preference free, represents an attempt by the investor to hedge against the constraint represented by the nontraded position. With perfect correlation, full insurance can be achieved and, as we show, the first-best optimum is attained. (The investor enjoys the same welfare as in the unconstrained portfolio choice case.) As a consequence of this demonstration, it appears that the Bernoulli investor in fact behaves myopically in this environment. The distortion in his or her demands for assets merely characterizes his or her attempt to attain the (myopic) first-best optimum. On the other hand, when the correlation between the futures price and the underlying commodity price is imperfect, in general, the value function does not separate additively, the first-best solution cannot be attained, and the optimal futures trading strategy involves a hedging term against shifts in the opportunity set.

## I. The Model

(A1) The investor is endowed with a sure quantity $Q$ of a specified commodity to be received at time $T$.[1] The spot price of this commodity is exogenously specified and satisfies the stochastic differential equation:

$$dP_t = \mu_p(P, t)dt + \sigma_p(P, t)dz(t), \tag{1}$$

where the process $\{z(t); 0 \leqq t \leqq T\}$ is a Standard Brownian Motion Process (BMP). The parameter $\mu_p(P, t)$ represents the expected instantaneous change in the spot price $P_t$, whereas $\sigma_p^2(P, t)$ is the local variance of this process.

(A2) (i) The investor has free and unlimited access to a financial market in which a futures contract written on this commodity is traded. The settlement price of this contract, $F_t$, satisfies

$$dF_t = \mu_f(P, t)dt + \sigma_f^*(P, t)dz_f(t), \tag{2}$$

where $\{z_f(t); 0 \leqq t \leqq T\}$ is a Standard BMP. Given the spot process of assumption (A1) and absent any convenience yield derived from the holding of an inventory of the commodity, the payoffs on the futures contract can be duplicated by an appropriate investment strategy in the commodity and the riskless asset. (See (ii) below.) The absence of arbitrage opportunities then implies that the settlement price process $\{F_t\}$ is perfectly correlated with the spot price process $\{P_t\}$ (i.e., $z_f(t) = z(t)$). The perfect local correlation between these two processes is instrumental in the derivation of Proposition 2.
(ii) An instantaneously riskless bond is available. Its rate of return, $r$, is constant over the period $[0, T]$.

(A3) The investor chooses a trading strategy, $\{x_t\}$, in the futures contract adapted to his or her information set, so as to maximize the expected utility of his or her terminal wealth. The terminal utility function, $u(\cdot, T)$, is twice continuously differentiable, concave, increasing, and such that $u'(0, T) = +\infty$. Let $W_T$ denote the cash position at date $T$ that results from the trading strategy $\{x_t\}$ and $P_TQ$ the unique cash flow generated by the nontraded position. Thus, the investor solves $\max_{\{x_t\}} \mathrm{E}u[W_T + P_TQ, T]$ subject to his or her dynamic budget.

(A4) We restrict the space of admissible trading strategies to those that yield a non-negative terminal wealth position with probability one. In instances where we are led to consider the structure of trading strategies for arbitrary preference specifications, we will not deal with the bankruptcy

[1] In assumption (A1), we have supposed that the quantity of the commodity, $Q$, is fixed. The equivalence result (Proposition 1) and the nonseparability of the value function (Lemma 1), however, also apply when quantity is random and satisfies a stochastic differential equation similar to equation (1). A result akin to Proposition 2 can also be retrieved in this case if an asset that is perfectly correlated with the quantity risk is also traded in the capital market.

problem explicitly. That is, we implicitly assume that the set of utility functions under consideration does not produce such a difficulty.

## II. The Investor Program: An Equivalence Result

The problem of the investor is to maximize his or her expected terminal utility subject to a dynamic budget constraint and to the price process given by equation (1). Two formulations of this budget constraint are as follows. Let $W_t$ represent the cash balance at time $t$:

$$dW_t = rW_t dt + xdF_t. \tag{4}$$

Alternatively, let $Y_t = W_t + P_tQ$ represent total wealth at time $t$:

$$dY_t = r(Y_t - QP_t)dt + xdF_t + QdP_t. \tag{5}$$

In formulation (4), adopted by Ho [18], the investor selects an investment strategy so as to control the level of his or her cash balance position. Modifications of the trading strategy due to the nontraded position are taken into account through the boundary condition that stipulates that the value function at the maturity date $T$ be equal to the utility of the cash position plus the single cash flow generated by the nontraded position.

In (5), $P_tQ$ is the present value of $P_TQ$. Form (5) therefore appears different from form (4). However, Proposition 1 will show that they are equivalent from the point of view of the investor. Consequently, it is not correct to interpret (5) as implying that the investor modifies his or her current trading strategies taking the present value of $P_TQ$ into account while in (4) he or she does not.

PROPOSITION 1: *The optimal futures strategy and the expected utility of the investor based on the cash budget constraint* (4) *are the same as the ones based on the wealth budget constraint* (5).

The proof of this statement appears in the Appendix. There are two elements contributing to the equivalence between the two formulations of the hedging problem. First of all, the two Markovian systems $(W, P)$ and $(Y, P)$ are informationally equivalent; wealth, $Y_t$, can be constructed from the knowledge of the cash balances accumulated, $W_t$, and the spot price $P_t$ and conversely $W_t$ can be inferred from $Y_t$ and $P_t$. This informational equivalence implies that the space of "admissible" strategies is identical under the two formulations of the problem. The second element is that the terminal consumption $c_t$ that is generated by a given trading strategy $\{x_t\}$ in futures contracts is identical under the two formulations. Indeed, $c_T = Y_T = W_T + QP_T$ for a given investment strategy $\{x_t; 0 \leqq t \leqq T\}$. As a result of this equivalence, we are free to use either of these two specifications to analyze the allocation problem of the investor.

## III. The Demand for Futures in the "Cash Balance-Price" Space

It is a well-known result that the Bernoulli investor behaves myopically in discrete-time economies without nontraded assets (Samuelson [27]; Mossin [24]; Hakansson [16]). A well-known result in the continuous-time financial economics literature (Merton [22, 23]; Breeden [3]) is that the investor seeks to insure himself or herself against adverse fluctuations in the opportunity set. In a "smooth" (absent jumps) economic environment with a stochastic opportunity set, the demands for financial assets are comprised of two terms, (a) a mean-variance efficiency component and (b) a (dynamic) hedging component. The second component is an informationally based demand for financial assets motivated by the dependence of future returns on state variables. Since this dependence is the cause of shifts in the opportunity set, the second demand component can be interpreted as a hedge against unfavorable shifts in the opportunity set. For a given economic structure, the magnitude of the hedging demand depends on the cross-partial derivative $J_{WS} \equiv \partial J_W/\partial S$, which represents the effect of state variables on the marginal value of wealth. When the cross-partial derivative of the value function, $J_{WS}$, vanishes, the hedging demand disappears and the investor behaves as if he or she would ignore the link between asset returns and state variables. This behavior can be interpreted as myopic.[2]

In the classic frictionless, unconstrained, and "smooth" portfolio problem with complete information, it turns out that the value function associated with the logarithmic utility is additively separable in wealth (cash) and state variables ($J[W, S, t] = J[W, t] + H[S, t]$), implying that the hedging demand vanishes. The Bernoulli investor thus displays myopia. Conversely, Ingersoll [20] shows that logarithmic utility is a necessary condition for myopia. These results naturally carry over to general-equilibrium representative-investor models that are cast in this structure (absence of friction, etc.). Recent work by Feldman [14] in incomplete-information economies has, furthermore, underlined the sensitivity of the concept of myopia to the representation of the economy. In his model with a Gaussian information structure, the demand for assets by the Bernoulli investor does not depend on the covariance between returns and the estimate of the state variable (the conditional mean) but does depend on the instantaneous covariance between realized output and unobservable state variables as a result of the inference carried out by the agent.[3] Thus, what appears to be myopic in one space seems to be nonmyopic in the other. A utility function, in this case, is said to exhibit generalized myopia if there exists a representation of the space of state variables under which asset demands are independent of the covariance between changes in realized return and changes in the economic state variables (Feldman [14], p. 16). The logarithmic utility can then be said to induce generalized myopia in an incomplete-information economy with a Gaussian information structure.

In the economic environment of this paper (constrained portfolio problem), the investor faces a second-best problem in that he or she cannot freely adjust one position in his or her portfolio (nontraded position). Suppose that we adopt the first formulation of the problem (space $(W, P)$), which is also the natural

[2] When the processes generating the evolution of the economy include discontinuous processes such as jump processes, the linearity of the first-order conditions collapses and classic results in finance are invalid (Merton [22], Ahn and Thompson [1]).

[3] Other studies that have shown myopia of the Bernoulli demands in the space of "substitute" state variables (conditional mean) are those of Detemple [11] and Gennotte [15]. The separability of the value function for the logarithmic utility was used to various ends in Detemple [12] and Dothan and Feldman [13].

formulation of the unconstrained portfolio problem. In this space, it turns out that the value function is not additively separable ($J[W, P, t] \neq J[W, t] + H[P, t]$). That is, the Bernoulli investor will care about the covariance between asset returns and changes in the state variables (the price of the nontraded asset in the environment described here).

LEMMA 1: *For the model with a nontraded position in the cash balance-state variable space, the value function for the logarithmic utility is not additively separable. The demand for futures $x^*$ can be written as*

$$x^* = -\frac{J_w}{J_{ww}} (\mu_f/\sigma_f^2) - \frac{J_{wp}}{J_{ww}} \sigma_{pf}/\sigma_f^2, \tag{6}$$

*where $J[W, P, t]$ denotes the value function associated with the Bernoulli investor's program.*

The demand for futures contract (equation (6)) is composed of a mean-variance component ($\mu_f/\sigma_f^2$) and a hedging term ($\sigma_{pf}/\sigma_f^2$). The latter, which vanishes in the classic unconstrained portfolio problem but not here, is a direct consequence of the imperfection introduced in the model (nontradedness assumption). Since the investor cannot adjust his or her commodity position, he or she is forced to experience the fluctuations in the value of this position. This constrained exposure is what generates the hedging demand for futures that appears in equation (6) above. This deviation from a pure mean-variance demand, furthermore, could conceivably be interpreted as the expression of nonmyopic behavior on his or her part. A more appropriate interpretation, however, is provided by the examination of the second formulation of the problem.

## IV. The Demand for Futures in the "Wealth-Price" Space

In this section, we suppose without loss of generality that the futures contract available is a Breeden-type contract with instantaneous maturity so that $F_t = P_t$. Lemma 1 above constitutes a negative result (nonseparability of the value function) but does not provide an explicit solution for the Bernoulli investor's problem. In Proposition 2, we go further and explicitly solve for the demand function for futures contracts. The transformation of the state space is not instrumental in producing a solution though it provides some intuition about the structure of the solution to the Bellman equation. Yet this reformulation of the state space arises naturally since it is suggested by the boundary condition and, more important, it has the advantage of separating the effect of the nontraded position on portfolio demands from informationally based effects. For an arbitrary expected utility maximizer the demand for futures takes the form:

$$x^* = -\frac{J_y}{J_{yy}} (\mu_f/\sigma_f^2) - \frac{J_{yp}}{J_{yy}} (\sigma_{pf}/\sigma_f^2) - Q(\sigma_{pf}/\sigma_f^2), \tag{7}$$

where the first component is a mean-variance component, the second term a Merton/Breeden informationally based hedging component, and the last term a pure minimum-variance component. The third component of the demand for futures ($Q\sigma_{pf}/\sigma_f^2$) represents the effect on asset demands of the constraint placed on the commodity position. It therefore represents a hedging demand against the local effect of nontradedness, clearly distinct from an attempt to hedge against shifts in the opportunity set (informationally based hedging). Given this decomposition, one could argue that the investor exhibits myopic behavior if the second term in equation (7) vanishes, that is if the cross-partial $J_{yp}$ equals zero. We now turn to the analysis of the Bernoulli investor's problem.

PROPOSITION 2: *In the wealth-price space, the Bernoulli investor has a separable value function, $J[Y_t, P_t, t] = \log[Y_t] + H(P_t, t)$, and his or her optimal demand for futures contracts is equal to*

$$x^* = Y(\mu_f/\sigma_f^2) - Q(\sigma_{pf}/\sigma_f^2), \tag{8}$$

*where $Y$ (wealth) is the absolute risk tolerance of the investor's value function.*

When the correlation between the futures settlement price process and the commodity price process is perfect, the futures contract provides complete insurance against fluctuations in the nontraded position. The Bernoulli investor will use such a contract to offset variations in his or her nontraded position. The hedge ratio that provides perfect insurance is $Q\sigma_{pf}/\sigma_f^2 = Q$ and constitutes the second term on the right-hand side of expression (8). The first term is a demand for mean-variance efficiency motives. Thus, the logarithmic investor acts, in the case where the spot price and the futures settlement price are perfectly correlated, so as (perfectly) to insure his or her position. Once this objective is achieved, he or she does not care about the information conveyed by the current price level $P$ about future moments of commodity and futures price changes (i.e., the cross-partial $J_{yp} = 0$).

It is also noteworthy that the mean-variance demand for the futures is inflated in the presence of the nontraded position. This follows since $Y = W + PQ$. That is, the prospect of receiving a cash flow $QP_T$ at the date $T$ causes the logarithmic investor to increase his or her speculative position in the futures market by the present value $QP_t$ of the terminal cash flow multiplied by the position per unit of wealth ($\mu_f/\sigma_f^2$).

These two features of the demand for futures are easily understood if we compare the second-best program of the investor (the constrained problem) with the first-best (the unconstrained portfolio choice problem).

COROLLARY 1: *Given the model of this section, the first-best optimum is attained by the futures strategy $x^* \equiv \{x_t^*, 0 \leqq t \leqq T\}$ given in Proposition 1.*

This corollary asserts that the investor attains the same welfare when he or she holds a nontraded asset that he or she hedges with a perfectly correlated futures contract as when he or she freely and continuously adjusts his or her position in the underlying commodity. The capital market in this instance enables him or her (effectively) to remove the nontradedness constraint. Once the first-best optimum is attained, the logarithmic utility investor does not care about the fluctuations in the opportunity set and will not hedge against them in the sense of Merton and Breeden. Thus, in view of the corollary, one can reasonably make a case that this investor behaves myopically.[4] Our last result reinforces this interpretation.

**Proposition** 3: *In economies where the futures settlement price is imperfectly correlated with the spot price, the Bernoulli investor, in general, behaves in a nonmyopic fashion. That is, his or her hedging demand for futures becomes*

$$x^* = -\frac{J_y}{J_{yy}}(\mu_f/\sigma_f^2) - \frac{J_{yp}}{J_{yy}}(\sigma_{pf}/\sigma_f^2) - Q(\sigma_{pf}/\sigma_f^2)$$

*and deviates from the minimum-variance hedge by an amount equal to* $(J_{yp}/J_{yy})(\sigma_{pf}/\sigma_f^2)$.

When the futures return is imperfectly correlated with the return on the nontraded asset or security, a perfect hedge cannot be attained.[5] The minimum-variance hedge, consequently, does not provide complete coverage against fluctuations in the value of the nontraded position, and the investor is subject to residual (post-hedge) uncertainty if he or she selects the minimum-variance hedge. As the proof of the proposition demonstrates, even the Bernoulli investor cares about this residual uncertainty and its effect on the moments of future changes in wealth (the cross-partial $J_{yp}$ does not vanish); he or she modifies accordingly the quantity of futures contracts purchased and consequently displays nonmyopic behavior.

One can also appraise the result from a different perspective. When the investor holds a nontraded position, he or she basically faces a constrained problem and therefore will use the existing menu of assets to reach a level of utility as close as possible to the first-best solution (i.e., the solution of the unconstrained problem). When a futures exists that is spanned by the riskless asset and the

[4] In the classic portfolio problem without nontraded assets, the logarithmic utility function constitutes a separating utility in the class of utilities with a wealth-independent measure of relative compensating variation—WIRCV—(Breeden [4]). Specifically for this class of utility functions, it can be shown that $-[J_{py}/J_{yy}] = Y(1 - T^*)\gamma_p$, where $T^* \equiv -[J_y/YJ_{yy}]$ is the investor's relative risk tolerance and $\gamma_p = -[J_p/YJ_y]$ is the measure of relative compensating variation. It follows that, in a pure portfolio setting, the Bernoulli investor does not hedge (since $T^* = 1$) and separates overhedgers ($T^* \lesseqgtr 1$ when $\gamma_p \gtreqless 0$) from reverse hedgers ($T^* \gtreqless 1$ when $\gamma_p \gtreqless 0$). For the constrained portfolio problem studied in this paper, it is easy to show that the logarithmic as well as the power utility functions belong to the WIRCV class when the futures settlement price is perfectly correlated with the spot price. For this class of utility functions (WIRCV), the deviation from the mean-variance efficiency demand can be written as $-[(J_{yp}/J_{yy}) + Q] = [Y(1 - T^*)\gamma_p - Q]$. Thus, the minimum-variance hedge appears as a natural benchmark that constitutes a dividing line between overhedgers and reverse hedgers. With perfect correlation, the logarithmic utility function hedges at the benchmark and hence, again, separates different classes of behavior. In the absence of perfect correlation, our results (Proposition 3) suggest that the log-utility may not belong to this class anymore and thus may cease to function as a benchmark separating different types of behavior.

[5] One instance where the correlation is imperfect is when the interest rate or the convenience yield is locally stochastic and imperfectly correlated with the underlying asset. Another example is when the traded security used for hedging purposes is written on an asset distinct from, and imperfectly correlated with, the nontraded position in the investor's portfolio (cross-hedging problem). The results pertaining to Proposition 3 should be interpreted with such a context in mind.

In a similar vein, it should be pointed out that Proposition 3 does not assert that the cross-partial $J_{yp}$ will never disappear. The result, indeed, might well collapse for some particular structure of the stochastic processes of the commodity price and of the futures price.

underlying asset, the first-best optimum can be attained exactly by an appropriate choice of futures (Proposition 2 and Corollary 1). With a zero-variance hedge, the investor is effectively unconstrained. If now the futures is imperfectly correlated with the asset, a first-best solution is not feasible. The second-best optimum that is reached, although it is constrained optimal, does not satisfy the investor since it forces him or her to experience unwanted fluctuations (in the first-best sense). This constrained exposure to risk and its stochastic evolution through time then becomes a source of hedging activity for the Bernoulli investor.[6]

## V. Conclusion

In this paper, we have shown that, in the presence of a nontraded position, the Bernoulli investor hedges against shifts in the opportunity set as represented by the commodity price. However, whereas the analysis provides a fairly complete characterization of the optimal futures strategy and of the value function for the allocation problem with logarithmic utility when the correlation between the futures settlement price and the commodity price is perfect, it fails to derive an explicit solution when this correlation is imperfect. Such a development would be of importance in view of the large number of results in financial economics that are based on the classic properties of the logarithmic utility model.

## Appendix

*Proof of Proposition* 1: Consider a complete probability space $(\Omega, F, P)$ on which the one-dimensional BMP $z \equiv \{z(t); 0 \leqq t \leqq T\}$ is defined. Let $F^z \equiv \{F^z_t; 0 \leqq t \leqq T\}$ denote the family of $\sigma$-algebras generated by the process $z$. ($F^z$ captures the information revealed by $z$.) Similarly, $F^{Y,P}$ and $F^{W,P}$ denote the information conveyed by the systems $(Y, P)$ and $(W, P)$, respectively.

In the first formulation of the problem, the investor selects an "admissible" trading strategy in futures, $x = \{x(t); 0 \leqq t \leqq T\}$, so as to solve the program (P1), defined as

$$\max_x \mathrm{E}u[W_T + QP_T, T] \tag{A1}$$

subject to

$$dW_t = rW_t dt + x_t dF_t \tag{A2}$$

$$dP_t = \mu_p(P, t)dt + \sigma_p(P, t)dz_t, \tag{A3}$$

where the expectation operator is taken with respect to the probability measure $P(\cdot)$. By admissible strategy, we mean a strategy that is adapted to the information set $F^{W,P}$ and that satisfies certain boundedness conditions. (See Cox and

[6] Note that Hakansson [17] shows that the Bernoulli investor behaves myopically when returns are serially correlated and the "no-easy-money" condition is satisfied. This latter condition in fact is a no-arbitrage condition on equilibrium prices. The reason our results differ from those of Hakansson is that, in addition to his no-arbitrage condition, our model imposes a constraint on the amount of one asset held in the portfolio.

Huang [7], Section 3.) Define the process $Y \equiv \{Y_t \equiv W_t + QP_t;\ 0 \leqq t \leqq T\}$. By Itô's Lemma, $Y$ is a solution to

$$dY_t = r[Y_t - QP_t]dt + x_t dF_t + QdP_t. \tag{A4}$$

The second formulation of the investor's problem involves solving (P2), defined as

$$\max_x \mathrm{E}u[Y_T, T] \tag{A5}$$

subject to (A3), (A4), and the appropriate restrictions on the strategy $x$. (In particular, $x$ is adapted to $F^{Y,P}$.)

Let $I(x)$ (respectively $J(x)$) denote the value of the objective function of program (P1) (respectively (P2)) evaluated at an admissible policy $x$. Let $x^1$ and $x^2$ represent the optimal policies for each program. For an $F^{W,P}$-measurable policy $x$, we have

$$I(x) = J(x) \leqq J(x^2),$$

where the equality follows since (a) $F^{Y,P} = F^{W,P}$ and (b) $Y_T(x) = W_T(x) + QP_T$ (P-a.s.). The inequality follows since $x^2$ is the optimal strategy for (P2). In particular, for $x = x^1$ we get $I(x^1) \leqq J(x^2)$. The converse inequality is shown along the same lines. It then follows that $I(x^1) = J(x^2)$ and that $x^1 = x^2$ (i.e., $x_t^1(\omega) = x_t^2(\omega)$ (P-a.s.)). This completes the proof of Proposition 1. Q.E.D.

All other proofs can be obtained upon request from the authors.

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