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Portfolio Selection in the Mean-Variance Model: A Note

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A LITTLE-RECOGNIZED FEATURE of the mean-variance portfolio-selection model is that induced preferences for asset holdings are not necessarily monotone; more of an asset (or portfolio) is not necessarily better, even if the asset (or portfolio) has positive expected return. The model predicts that an investor typically wants only a limited number of shares of an asset, and beyond that the increase in mean return from costlessly getting more of an asset is not sufficient to compensate for the increased risk. This is reasonable if the model describes a market for contracts with unlimited liability where total returns may be negative (e.g., they may be normally distributed). It is unreasonable, however, if the model is used to describe a stock market with limited liability. In that case, it is desirable to impose conditions that ensure that there is no upper limit to how much of an asset (or portfolio) the investor wants. The sharpest possible conditions are identified in Section I.

Since portfolio preferences are not monotone, global satiation is possible. Again, this is unrealistic when liability is limited. Satiation is bothersome also because it may lead to nonexistence of equilibrium in Black's [4] CAPM without a riskless asset, as demonstrated in Nielsen [9]. (Uniqueness of equilibrium in CAPM is investigated in Nielsen [10].) Section I identifies the exact conditions under which global satiation does or does not occur. The conditions compare the maximum ratio of mean to standard deviation of total return available from the assets with the investor's limiting degree of risk aversion (as measured by the slope of his or her indifference curves) at high levels of risk.

An optimal portfolio may or may not exist in the portfolio-selection problem, depending on the asymptotic slopes of the efficient frontier and the investor's indifference curves. Section II identifies the exact conditions.

Mean-variance behavior can arise from expected utility maximization based on normal distributions. In that case, the conditions for nonsatiation and for existence of optimal portfolios can be stated in terms of the von Neumann-Morgenstern utility function. Section III relates the relevant properties of the utility function for standard deviation and mean to properties of the von Neumann-Morgenstern utility function.

I. Directional and Global Satiation

The investor's utility $U(\sigma, \mu)$ is a function of the standard deviation σ and mean μ of total portfolio return. It is assumed that U is strictly increasing in μ,

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nonincreasing in σ, and concave and that the indifference curves in (σ, μ)-space are strictly convex. These assumptions hold if U is derived from expected utility based on normal distributions (unless the von Neumann-Morgenstern utility function is risk neutral). When U is concave, all indifference curves in (σ, μ)-space have the same asymptotic slope $\bar{\alpha}$ at large values of σ or μ. This follows from results in convex analysis; cf. Rockafellar [11]. The limiting slope $\bar{\alpha}$ is positive or plus infinity.

A *portfolio* is a k-dimensional vector; the jth coordinate indicates the number of shares of the jth asset held (not the fractions of wealth invested). The total returns per share have mean vector $\bar{R}$ and covariance matrix Ω. The standard deviation and mean of total returns to portfolio x are $\sigma(x) = (x'\Omega x)^{1/2}$ and $\bar{R}x$. The investor's induced utility function for portfolios is $V(x) = U(\sigma(x), \bar{R}x)$, a concave function of x. If e is a portfolio with $\bar{R}e > 0$, let $\alpha(e)$ denote its ratio of mean to standard deviation of return: $\alpha(e) = \bar{R}e/\sigma(e)$ if $\sigma(e) > 0$ and $\alpha(e) = \infty$ if $\sigma(e) = 0$.

Suppose the investor starts out with a portfolio x and considers adding a multiple λ of a portfolio e (with $\bar{R}e > 0$). Is more of e always better? I.e., is $V(x + \lambda e)$ a strictly increasing function of $\lambda \geq 0$, or is there an optimal value of λ? In the latter case, say that there is *directional satiation* in the direction of the portfolio e.

The answer relies on a comparison of the ratio of mean to standard deviation $\alpha(e)$ available from the fund e with the asymptotic slope $\bar{\alpha}$ of the indifference curves in (σ, μ)-space. The reason for this is that the (possibly degenerate) hyperbola $(\sigma(x + \lambda e), \bar{R}(x + \lambda e))$ traced out in (σ, μ)-space by the portfolios $x + \lambda e$ has limiting slope $\alpha(e)$ at large values of λ. (When λ is restricted to be non-negative, only the part above $(\sigma(x), \bar{R}x)$ is traced out.)

If $\alpha(e) > \bar{\alpha}$, then the curve $(\sigma(x + \lambda e), \bar{R}(x + \lambda e))$ cuts across higher and higher indifference curves as λ increases. See Figure 1 (where the curve is labeled h). The function $V(x + \lambda e)$ is strictly increasing; there is no optimal value of λ, and there is not directional satiation. If $\alpha(e) < \bar{\alpha}$, then the function $V(x + \lambda e)$ is eventually strictly decreasing. There is an optimal value of λ, which is unique, so in this case there is directional satiation.

Note that the initial portfolio x does not matter for whether there is directional satiation. This is not immediately obvious because, in general, the fund e will be correlated with the initial portfolio x, and the mean and standard deviation of the combined portfolio will depend on the correlation and, hence, on the initial portfolio.

We can use the observations about directional satiation to determine whether there is global satiation. Global satiation means that there is a portfolio that the investor prefers to all other portfolios (regardless of whether he or she can afford them). Formally, a *global satiation point* is a portfolio x such that $V(x) \geq V(y)$ for all portfolios y. The possibility of global satiation in this model was recognized by Bierwag and Grove [3] and has been discussed by Tsiang [12], Bierwag [2], Jones and Roley [7], and Friedman and Roley [5].

If there is a riskless asset or portfolio with nonzero return, then obviously there is no satiation point. So suppose there is no riskless asset, i.e., Ω is regular.

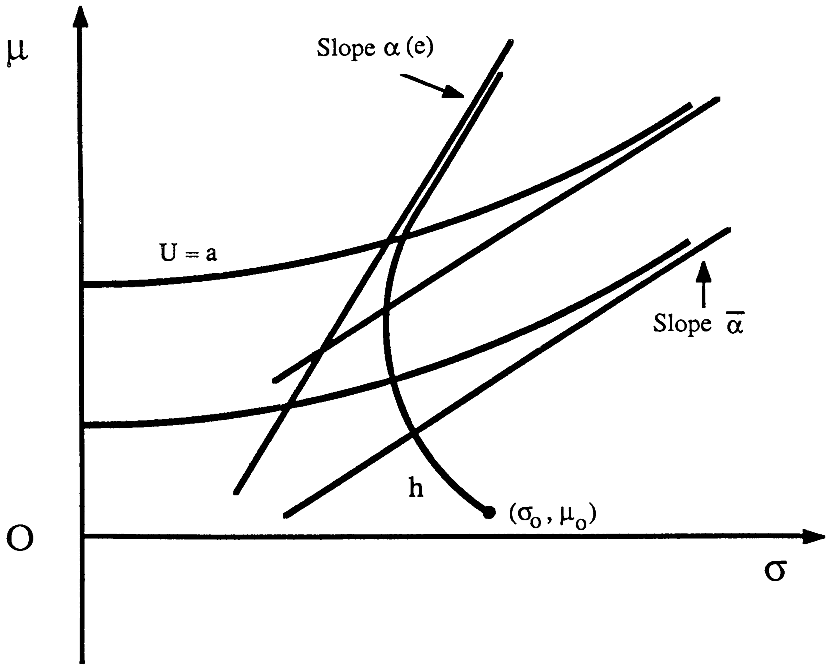

Figure 1. $\alpha(e) > \bar{\alpha}$

Across all portfolios, the ratio of mean to standard deviation $\alpha(e)$ is maximized by (positive multiples of) the portfolio $e = \Omega^{-1}\bar{R}$. Existence of a global satiation point is equivalent to directional satiation in the direction $e = \Omega^{-1}\bar{R}$. The maximal ratio of mean to standard deviation is $\alpha(e) = (\bar{R}'\Omega^{-1}\bar{R})^{1/2}$. So, there is a global satiation point if and only if $(\bar{R}'\Omega^{-1}\bar{R})^{1/2} < \bar{\alpha}$. For example, if $\bar{\alpha} = \infty$, then there has to be a satiation point. When mean-variance behavior is derived from normal distributions, it will typically be the case that $\bar{\alpha} = \infty$. (See Section III.) Unfortunately, this implies that there typically will be satiation.

The issues of directional and global satiation have been analyzed using total returns, although it is usually preferable to express the mean-variance model in rates of return. The rates-of-return formulation involves fewer variables (only rates of return instead of both total returns and prices) and makes some formulas simpler, and rates of return are easier to compare across assets and over time than total returns. However, directional and global satiation cannot be analyzed using rates of return, essentially because these questions do not involve asset prices or a fixed budget or wealth constraint. When portfolio value and wealth vary, the utility function for rates of return changes, and so does the indifference map in mean-standard-deviation space for rates of return.

II. Portfolio Selection

This section shows how the existence of an optimal portfolio subject to a budget constraint depends on market parameters and risk aversion via the asymptotic slopes of the efficient frontier and the indifference curves.

First, suppose there is no riskless asset. (Ω is positive definite.) The efficient frontier in terms of total returns can be derived by arguments like those for rates of return in Merton [8]. Let $p \neq 0$ be the asset price vector. Set $A = \bar{R}'\Omega^{-1}p$, $B = \bar{R}'\Omega^{-1}\bar{R}$, $C = p'\Omega^{-1}p$, and $D = BC - A^2$. Assume that $[\bar{R}\, p]$ has rank 2. Then, $C > 0$ and $D > 0$. The efficient portfolios with cost w trace out the upper branch of a hyperbola in (σ, μ)-space:

$$\mu = \bar{\mu} + \rho(\sigma^2 - \bar{\sigma}^2)^{1/2}, \quad \sigma \geq \bar{\sigma},$$

where $\bar{\mu} = Aw/C$, $\bar{\sigma} = |w| C^{-1/2}$, and $\rho = (D/C)^{1/2} > 0$. To each $\mu \geq \bar{\mu}$ corresponds a unique efficient portfolio x with cost w, given by $x = \mu g + wh$, where $g = (C/D)\Omega^{-1}\bar{R} - (A/D)\Omega^{-1}p$ and $h = -(A/D)\Omega^{-1}\bar{R} + (B/D)\Omega^{-1}p$.

Note that $pg = 0$, $\bar{R}g = 1$, $ph = 1$, and $\bar{R}h = 0$. To construct an efficient portfolio with cost w and mean μ, invest all your wealth in the portfolio h that costs a dollar per unit and gives zero expected return, and then adjust the mean (and variance) by adding some of portfolio g, which is free but has an expected return of one per unit.

A simple computation shows that $\alpha(g) = \rho$, where ρ is the slope of the asymptote to the efficient frontier. There is an optimal portfolio (optimal subject to $px = w$) if an only if there is directional satiation in the direction g. In other words, an optimal portfolio x exists if and only if $\rho < \bar{\alpha}$.

Next, consider portfolio selection when there is a riskless asset. Let $R_f > 0$ be the total return per share of the riskless asset, and let $\bar{R}$ and Ω refer to risky assets only. Assume that Ω is positive definite. Normalize the price of a share of the riskless asset to be one, and let p be the vector of prices of risky assets.

Efficient portfolios with cost w are combinations of the riskless asset and the risky fund $\Omega^{-1}(\bar{R} - R_f p)$. The frontier in (σ, μ)-space is a half-line starting at $(0, R_f w)$ with slope $\theta = [(\bar{R} - R_f p)'\Omega^{-1}(\bar{R} - R_f p)]^{1/2}$. Existence of an optimal portfolio among portfolios with cost w is equivalent to directional satiation in the direction of the fund $e = (-p\Omega^{-1}(\bar{R} - R_f p), \Omega^{-1}(\bar{R} - R_f p))$, which is a zero-cost combination of the riskless asset and the risky fund $\Omega^{-1}(\bar{R} - R_f p)$. A simple computation shows that $\alpha(e) = \theta$. An optimal portfolio exists if and only if $\theta < \bar{\alpha}$.

Note the dilemma: If $\bar{\alpha} = \infty$, then optimal portfolios exist but satiation is sure to occur (when all assets are risky). If $\bar{\alpha} < \infty$, then satiation may not occur, but it is also possible that optimal portfolios fail to exist.

Of course, if the portfolio-selection problem is embedded in an equilibrium capital asset pricing model, then optimal portfolios exist for all investors in equilibrium. Asset prices and hence the (limiting) slope ρ or θ of the frontier must adjust to equilibrium in such a way that the conditions identified above hold.

III. Normal Distributions

Conditions for existence of optimal portfolios and for nonsatiation have been expressed in terms of $\bar{\alpha}$. This section computes $\bar{\alpha}$ in terms of the characteristics of the von Neumann-Morgenstern utility function when the utility of standard deviation and mean is derived from expected utility based on normal distributions.

The vector R of total returns per share is distributed according to a jointly normal probability distribution π. The investor's *von Neumann-Morgenstern utility function* u is a function of total portfolio return, defined on the entire real line, concave and strictly increasing. Define the utility of standard deviation and mean by

$$U(\sigma, \mu) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} u(\mu + \sigma y)\exp(-y^2/2)\, dy.$$

That is, $U(\sigma, \mu)$ is the expected value of u with respect to a normal distribution with standard deviation σ and mean μ. For the integral to be well defined and finite for all (σ, μ), u must not go too fast to minus infinity as wealth goes to minus infinity. Specifically, it is necessary and sufficient that, for all $\gamma > 0$, there exists y such that $u(x) \geq -\exp(\gamma x^2)$ for $x \leq y$. If V is the utility function for portfolios induced from U, then $V(x) = \mathrm{E}u(xR)$ for all portfolios x; $V(x)$ is the expected utility of the total return to x.

Following Bertsekas [1] and Hart [6], define $S^+ = \inf_t u^-(t)$ and $S^- = \sup_t u^-(t)$, where $u^-(t)$ is the left derivative of u at t. These are the *asymptotic slopes* of u. Clearly, $0 \leq S^+ \leq S^- \leq \infty$, $S^+ < \infty$, and $0 < S^-$. If e is a portfolio, define

$$E^+(e) = \int (eR)1_{\{(eR)>0\}}\pi\ (dR); \quad E^-(e) = \int (eR)1_{\{(eR)<0\}}\pi\ (dR).$$

These are the *incomplete means* of the returns to the portfolio. It follows from Proposition 2 of Bertsekas [1] that $V(\lambda e)$ is a nondecreasing function of $\lambda \geq 0$ if and only if $S^+E^+(e) + S^-E^-(e) \geq 0$.

Say that the investor is *asymptotically risk neutral* if $0 < S^+$ and $S^- < \infty$. If the graph of the utility function approaches a nonhorizontal asymptote at plus infinity and a nonvertical asymptote at minus infinity, then it is asymptotically risk neutral. However, u may be asymptotically risk neutral even if its graph has no asymptotes.

Define a real function m on the real line by $m(\alpha) = \mathrm{E}[(\alpha + Y)1_{(\alpha+Y)>0}]$, where Y is some standard normal variate. Then $m(\alpha)$ is the non-negative incomplete mean of a normal distribution with mean α and unit variance.

PROPOSITION.

1. *If u is not asymptotically risk neutral, then $\bar{\alpha} = \infty$.*
2. *If u is risk neutral, then $\bar{\alpha} = 0$.*
3. *If u is asymptotically risk neutral but not risk neutral, then $\bar{\alpha}$ is the unique solution to $S^+/S^- = \alpha/m(\alpha)$. In particular, $0 < \bar{\alpha} < \infty$.*

Proof: Provisionally redefine $\bar{\alpha}$ by each of the conditions (1), (2), and (3) in turn. Put $\alpha = \alpha(e)$. Then we have to show that $V(\lambda e)$ is nondecreasing in $\lambda \geq 0$ if and only if $\alpha \geq \bar{\alpha}$, i.e., if and only if $S^-E^-(e) + S^+E^+(e) \geq 0$. Since $E^+(e) = \sigma(e)m(\alpha)$ and $E^-(e) = \sigma(e)(\alpha - m(\alpha))$, the latter is equivalent to $S^+m(\alpha) + S^-(\alpha - m(\alpha)) \geq 0$. But this is obviously equivalent to $\alpha \geq \bar{\alpha}$ when $\bar{\alpha}$ is defined by the conditions in each of the cases (1), (2), and (3). Q.E.D.

So, in the case of normal distributions, typically $\bar{\alpha} = \infty$. Optimal portfolios exist, but there is satiation if all assets are risky. The exception is when the investor is asymptotically risk neutral. Then satiation may or may not occur, and optimal portfolios may or may not exist.

IV. Conclusion

There is a conflict between ensuring nonsatiation and ensuring existence of optimal portfolios. If the investor has a high limiting degree of risk aversion at high levels of expected return or risk (or, in the case of a von Neumann-Morgenstern utility function, at extreme (positive or negative) levels of final wealth), typically an optimal portfolio exists, but there is satiation if all assets are risky. When the limiting degree of risk aversion is low, satiation can be avoided, but an optimal portfolio often does not exist.

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