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Tax Arbitrage and the Existence of Equilibrium Prices for Financial Assets

ROBERT M. DAMMON and RICHARD C. GREEN*

ABSTRACT

In models where both investors and securities are subject to differential taxation, there may be no set of prices that rule out infinite gains to trade, or "tax arbitrage." This paper characterizes the joint restrictions on financial-asset returns and investors' tax schedules that preclude tax arbitrage in the absence of short-sale constraints. The authors show that, if there exists any configuration of marginal tax rates on investors' tax schedules that rule out infinite gains to trade, then "no-tax-arbitrage" prices will exist. They also show that the existence of "no-tax-arbitrage" prices ensures the existence of equilibrium prices.

THE EFFECTS OF DIFFERENTIAL taxation on the equilibrium prices of financial assets have attracted much attention from financial economists in recent years. Otherwise identical securities that contribute to taxable income to different degrees will, in general, be valued differently by taxable investors. As a result, tax considerations have been useful in helping explain the effect of dividend yield on stock returns,[1] the effect of coupon levels and term to maturity on bond prices,[2] the timing of investors' portfolio transactions,[3] and the observed capital structures of firms.[4]

While a rich set of observed behaviors can be better understood by reference to differential taxation, there are well-known difficulties in dealing with taxes in a general-equilibrium setting. To clear markets, relative prices must reflect the marginal rates of substitution of all agents simultaneously. When tax rates differ across investors, however, this condition can be impossible to achieve. To illustrate, consider a world of perfect certainty with two assets: a tax-exempt municipal bond and a taxable government bond. To equate marginal rates of substitution, the rate of return on the government bond, r_g, must equal that on the municipal, r_m, "grossed up" by one minus the investor's marginal tax rate, t_i; that is, $r_g = r_m/(1 - t_i)$. If there are investors in more than one tax bracket, this condition will obviously be impossible to satisfy for all of them simultaneously.

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[1] See Litzenberger and Ramaswamy [7], Miller and Scholes [11], and Elton and Gruber [5].

[2] See Litzenberger and Rolfo [9].

[3] See Constantinides [1] and Constantinides and Ingersoll [2].

[4] See Miller [10], DeAngelo and Masulis [4], and Dammon [3].

Thus, for *any* set of market prices, there will be infinite gains to trade, or "tax arbitrage", for at least one investor. As a result, an equilibrium will fail to exist unless short-sale restrictions are imposed that prevent investors from exploiting such arbitrage opportunities.

The purpose of this paper is to delineate those situations where an equilibrium will not exist without short-sale constraints. In particular, we characterize the joint restrictions on financial-asset returns and investors' tax schedules that are necessary and sufficient to preclude tax arbitrage in the absence of short-sale constraints. The conditions have the qualitative implications one might expect. If "no-tax-arbitrage" prices are to exist, the differences in investors' tax rates must get smaller as assets become closer substitutes on a pretax basis. Making precise the magnitudes involved yields qualitative insights concerning how these factors interact to determine whether a given market structure and tax environment are consistent with otherwise frictionless trading. This is important because, when short-sale constraints are present, they can have a significant effect on equilibrium prices and allocations. For example, Dammon [3] examines a model in which investors face progressive taxation but no short-sale constraints and finds, unlike Miller [10], that tax considerations alone lead to interior optimal capital structures for firms.

Our results indicate that the environments where tax arbitrage necessitates short-sale constraints are not as widespread as one might suspect. We first establish the necessary and sufficient conditions for the existence of prices that rule out tax arbitrage when tax schedules are linear and tax rates are constant. We then generalize these conditions to progressive (i.e., nonlinear) tax schedules and show that, if there exists *any* configuration of marginal tax rates across investors that precludes linear tax-arbitrage opportunities, then, by our definitions, no-tax-arbitrage prices will exist. Furthermore, we are able to show that the existence of no-tax-arbitrage prices is precisely what is needed to ensure the existence of equilibrium prices. This result holds even in situations where taxable income, and hence the set of feasible allocations for the economy, depends upon prices (e.g., capital-gains taxation).

Schaefer [15] studies tax arbitrage in a setting similar to ours and concludes that, under "fairly standard assumptions", an equilibrium in which investors occupy different tax brackets will not exist without restrictions on short sales. Among his "fairly standard assumptions", however, is the existence of differentially taxed assets that have perfectly correlated pretax (and after-tax) payoffs. Under certainty, Schaefer shows that, with frictionless markets, investors either will arbitrage to the same marginal tax bracket in equilibrium if they face progressive tax schedules or will have opportunities for infinite gains to trade if no common tax bracket exists. The restrictions we obtain on asset returns and tax schedules include these as special cases but are more general. First, redundant assets are not required to generate tax arbitrage if tax schedules differ sufficiently across investors. Second, marginal tax rates can differ across investors without admitting arbitrage if markets are incomplete. Our characterization of tax arbitrage allows for these considerations.

The role of incomplete markets in limiting tax arbitrage has also been recognized by Litzenberger and Rolfo [8] in their analysis of tax effects in the pricing of government bonds. They focus on environments where marginal tax rates are fixed and show that investors in different tax brackets may have positive holdings of the same securities when markets are incomplete. Incompleteness per se, however, does not guarantee the absence of tax arbitrage when tax schedules are sufficiently heterogeneous. Our analysis provides a full characterization of the joint restrictions on market structure and tax schedules required for no-tax-arbitrage and equilibrium prices to exist.

In a recent paper, Ross [14] examines the duality between arbitrage and positive implicit state prices in a model with nonlinear tax schedules. Both the structure and, more importantly, the purpose of his model differ from ours, however. Ross [14] explores conditions on asset prices that preclude "local" arbitrage for a particular investor holding a particular portfolio. Although such opportunities must obviously be fully exploited at an equilibrium, the existence of "local" arbitrage does not preclude the *existence* of an equilibrium. Similarly, gains to trade in a no-tax economy, while they too must be fully exploited at an equilibrium, do not prevent one from ever being achieved. Furthermore, to verify whether prices rule out "local" arbitrage, one must know the portfolio position and tax status of each investor in the economy. In contrast, the absence of tax arbitrage as we describe it is a necessary condition for the *existence* of an equilibrium. It is also a property that can be verified or contradicted with knowledge only of tax schedules and asset payoffs, without reference to endowments, market clearing, or the feasibility of particular allocations.

It is not obvious, however, that the existence of prices that preclude tax arbitrage will ensure the existence of an equilibrium. It may be that market-clearing considerations can force individuals to points on their tax schedules that render their marginal rates of substitution mutually inconsistent. Therefore, we also examine the relationship between the existence of no-tax-arbitrage prices and the existence of equilibrium prices. We are able to show that, if there is *any* configuration of marginal tax rates on investors' tax schedules that preclude tax arbitrage, then equilibrium prices for financial assets will exist. The complications that arise when an asset's contribution to taxable income depends upon prices, as with capital-gains taxation, are dealt with by confining our attention to a sensible subset of the no-tax-arbitrage prices—those that assign positive prices to assets with strictly positive pretax payoffs. Analogous conditions characterize the existence of such prices, and, again, their existence is sufficient to ensure the existence of equilibrium prices.

The paper is organized as follows. Section I describes the basic model and outlines our major assumptions. In Section II, some simple examples are used to help illustrate the basic concepts developed in the paper. Section III formally defines tax arbitrage and no-tax-arbitrage prices, first under linear taxation, and then generalizes these definitions and the associated results to progressive tax schedules. In Section IV, we show that the absence of tax arbitrage, along with our other standard general-equilibrium assumptions, is sufficient to ensure the existence of an equilibrium. In Section V, we modify the model to allow an asset's contribution to taxable income to depend upon its initial price. This modification accommodates situations in which assets are subject to capital-gains taxation or in which distinctions between "principal" and "interest" depend upon prices.

Section VI offers some concluding remarks. All proofs appear in the Appendix.

I. The Model

Consider an economy with two dates, where all trading takes place at time 0 and all uncertainty is resolved at time 1. There are m mutually exclusive states of nature, indexed by s, that may occur at time 1. Investors trade n financial assets, indexed by j, which are characterized by their state-contingent, pretax payoffs X_{js}, $s = 1, \cdots, m$. We assume that these payoffs are in units of a single, numeraire commodity and that there are at least as many states as assets, $m \geq n$. The financial assets have prices P_j, $j = 1, \cdots, n$, which are determined through trading in frictionless markets at time 0. The n-vector of these prices is denoted P, and the $(n \times m)$-matrix of asset payoffs in the m different states is denoted X, with column vectors X_s and row vectors X_j.

Under "differential taxation", different assets contribute to taxable income to different degrees. Initially, we assume that these differences take a particularly simple form. We assume that the jth asset contributes a positive fraction, γ_{js}, $0 \leq \gamma_{js} \leq 1$, of its gross payments in state s to taxable income.[5] This formulation captures the basic idea that different securities contribute differently to taxable income and, since γ_{js} is state dependent, that these differences may be contingent on events unknown when the assets are purchased. Nevertheless, it abstracts from the actual tax code in two important respects. First, it does not accommodate situations where the contribution to taxable income depends upon prices. A capital-gains tax clearly has this property. So does the taxation of bonds selling at a premium or discount. While these situations can be dealt with through alternative assumptions, they complicate the arguments sufficiently that we have chosen to pursue them separately in Section V as an extension to our basic model. Second, the tax treatment of a particular asset is not allowed to depend in nonlinear ways on the investor's portfolio composition. This rules out, for example, limitations on interest deductions that depend upon the particular set of assets held in one's portfolio. Modeling this would require that taxes be described as some general function of the investor's portfolio holdings. If the tax function remains convex, much of the mathematical structure of our analysis would survive, but at considerable cost in intuition. Since the tax function would no longer be defined over a single argument, the whole idea of a marginal tax rate would not be well defined. Clearly, however, these additional features would work to further limit opportunities for tax arbitrage, affecting the necessity but not the sufficiency of the conditions we provide that rule out tax arbitrage.

There are l consumers in the economy, indexed by i. Their utility functions are denoted u^i and are defined over non-negative consumption vectors $c^i = (c_1^i, \cdots, c_m^i)$, where $c_s^i \geq 0$ is individual i's consumption in state s. Let u_s^i be the derivative of u^i with respect to c_s^i. We assume that these derivatives exist and that u^i is concave and strictly increasing. Consumer i's endowment consists

[5] The model is general enough to allow γ_{js} to differ across investors as well. This would be an important consideration, for example, if investors had different tax bases for the same asset. Such a situation is difficult to motivate here, however, because of the single-period nature of the model.

of initial security holdings, $\bar{w}^i \in R^n$, which yield $\bar{w}^i P$ in initial wealth.[6] This initial wealth is used to purchase a final portfolio, $w^i \in R^n$, which yields a pretax income of $X_s' w^i$ in state s at time 1. Taxable income in state s is given by $\bar{X}_s' w^i$, where $\bar{X}_s'$ is the n-vector of asset contributions to taxable income in state s, with typical element $\gamma_{js} X_{js}$. Given the security holdings, w^i, the m-vector of consumer i's taxable income across states is $\bar{X}' w^i$, where $\bar{X}'$ is an $(m \times n)$-matrix of asset contributions to taxable income in the m different states.

Once taxable income in state s has been determined, consumer i's total tax liability is computed according to the tax schedule $T_i(\cdot)$. This schedule need not be linear or the same across consumers. We will, however, assume the following.

(A1) $T_i(\cdot)$ is a nondecreasing, continuous, convex, and differentiable function of taxable income, with $T_i(0) = 0$.

The convexity of the tax schedule implies nondecreasing marginal tax rates. The assumption of differentiability is made simply for expository convenience.[7] The marginal rates are assumed to satisfy the following.

(A2) $\bar{t}_i \equiv \lim_{y \to +\infty} T_i'(y)$ exists and is bounded away from unity. $\underline{t}_i \equiv \lim_{y \to -\infty} T_i'(y)$ exists and is greater than or equal to zero.

These bounds on the marginal rates, along with the fact that they are nondecreasing, imply that the government can never expropriate all of or more than the last dollar of taxable income. Similarly, greater losses cannot lead to higher taxes. Along with the assumption that $T_i(0) = 0$, this implies that positive taxable income always leads to non-negative taxes and that losses yield tax rebates or tax forgiveness. To allow for the feasibility of autarky, we will assume the following.

(A3) The initial endowments of asset holdings, $\bar{w}^i$, $i = 1, \cdots, l$, provide positive payoffs in every state that are larger than taxable income. That is, $X' \bar{w}^i \gg 0$ and $X' \bar{w}^i > \bar{X}' \bar{w}^i$ for all $i = 1, \cdots, l$.

Since $\bar{t}_i < 1$, assumption (A3) implies that $\bar{w}^i$ will also provide positive after-tax income in every state for any marginal rate on the consumer's tax schedule. That is, $(X - \bar{X} t_i)' \bar{w}^i \gg 0$ for all $t_i \in [\underline{t}_i, \bar{t}_i]$. Finally, we need to ensure that the absence of arbitrage rules out a zero price for at least one asset. This is accomplished through the following assumption.

(A4) There exists at least one asset with strictly positive pretax payoffs in every state.

[6] Opportunities to consume at the initial date, or state-contingent endowments that are not related to security holdings, could be incorporatd into the model with only notational complications.

[7] Since any continuous, convex function is subdifferentiable (see Rockafellar [12, Chapter 23]), one could replace all of our subsequent statements about the marginal rates with statements about the right- or left-hand derivatives, or "elements of the subdifferential". While this would formally allow for piecewise linear tax schedules, it would not change the intuitive content of the results.

II. Some Examples

The purpose of this section is to use some particularly simple examples to illustrate the basic concepts that are subsequently presented in more generality. In these examples, the relationship between tax arbitrage and the existence of equilibrium prices can be depicted graphically with a simple Edgeworth box. Most of the mathematical structure that ensures the paper's results can be illustrated in this setting. Specifically, the examples help to clarify the following:

(i) the motivation for our definition of "tax arbitrage" and the difference between it and the concept of "local tax arbitrage" as defined by Ross [14],
(ii) why tax arbitrage, as we define it, precludes the existence of an equilibrium while "local tax arbitrage" does not,
(iii) the role of progressive tax schedules in dramatically reducing the opportunities for tax arbitrage,
(iv) that characterizations of tax arbitrage with linear tax rates can be generalized in a simple way to situations where tax schedules are progressive and marginal rates are endogenous, and
(v) that, if there is *any* configuration of marginal tax rates across investors at which arbitrage is impossible, then equilibrium prices will exist.

To begin, suppose that there are two riskless securities available for trade. Asset 1 pays one tax-exempt "dollar", and asset 2 pays one taxable "dollar". Each is outstanding in unit supply. There are two consumers, h and i, who we assume have identical endowments. Since all taxable income is due to holdings of the second asset, the taxes paid by h and i are $T_h(w_2^h)$ and $T_i(w_2^i)$, respectively.

Figure 1 illustrates three different tax scenarios for the two consumers. In Case A, both consumers have linear tax schedules and, therefore, constant marginal rates. Figure 2 illustrates the well-known result that, in this situation, the tax rates must be equal for an equilibrium to exist without short-sale constraints. The Edgeworth box is drawn as the unit cube, but the reader should keep in mind that, with the opportunity to sell short, any point in the plane is feasible. At points outside the boundaries of the box, one agent or the other has a short position in one or both of the assets.

The point $\bar{w}$ represents the traders' endowments. For each unit of the tax-exempt security that consumer h surrenders, he or she requires $1/(1 - t_h)$ additional units of the taxable security to maintain the same after-tax income. Thus, the line through $\bar{w}$ with slope $1/(1 - t_h)$ represents combinations of the two securities that hold consumer h's after-tax income constant. Since the two securities are perfect substitutes on an after-tax basis, this line can be interpreted as an indifference curve. At points in the half-plane to the northeast of this line, consumer h is better off. Similarly, at points to the southwest of the line with slope $1/(1 - t_i)$, consumer i is better off. The intersection between these two half-planes will be a cone, and, as long as the tax rates are not equal, this cone will have an interior consisting of net trade vectors, such as $\hat{w}$, that make both consumers better off. Furthermore, any such vector can be extended arbitrarily

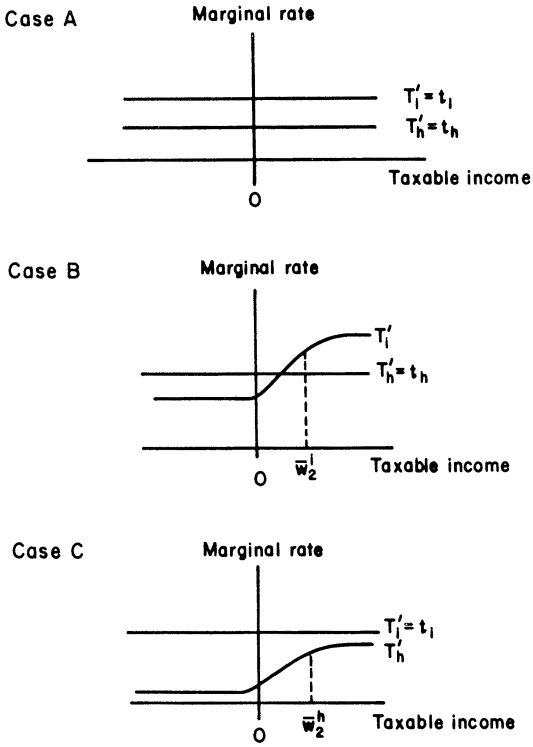

Figure 1. Three Different Tax Scenarios for Consumers h and i

far in the plane.[8] In the parlance of convex analysis, these are "directions of recession". As the tax rates for the two consumers approach each other, the cone of arbitrage trades shrinks. However, as long as the preferred-to sets have an open intersection, no price line can separate them and no equilibrium will exist. Thus, the only case that admits an equilibrium is the one in which all consumers have identical marginal tax rates so that the cone of arbitrage trades has no interior. Then relative prices for the securities of $1/(1 - t)$, where $t = t_i = t_h$, will rule out arbitrage for both agents. Note also that these prices support autarky as an equilibrium. The condition that the cone containing $\hat{w}$ have an empty interior is necessary for no-tax-arbitrage prices to exist. It is also sufficient to ensure the existence of equilibrium prices.

These properties generalize to more complex environments. With linear tax schedules, the set of "arbitrage trades" is always a cone, and this cone shrinks as tax rates get closer together. The condition that the cone have no interior points can be characterized using linear duality theory (i.e., Farkas' Lemma), which is also used to characterize the absence of arbitrage in a world without taxes. Ruling out tax-arbitrage trades in this way is necessary for equilibrium prices to exist, and, with other standard assumptions, it is also sufficient.

[8] Note that, while the wealth of both consumers is increasing along any arbitrage trade vector, such as $\hat{w}$, the size of the Edgeworth box remains fixed because the net aggregate supply of securities is fixed.

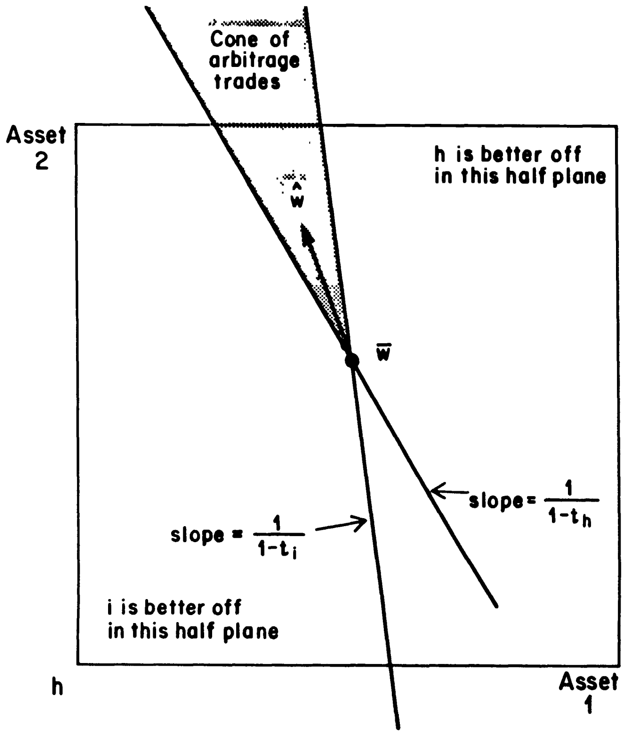

Figure 2. Arbitrage Trades with Linear Tax Schedules and Perfectly Correlated, Differentially Taxed Assets

When assets are less than perfect substitutes, tax rates need not be identical across consumers to rule out arbitrage. Figure 3 illustrates such a case with two investors, two assets, and two states of the world. The first asset pays one tax-exempt dollar in each state. The second is fully taxable and pays one dollar in the first state and two dollars in the second. Assume that investor h has a constant tax rate of zero. The striped cone in each panel of Figure 3 shows the vectors of net trades that have non-negative payoffs for investor h in both states. These vectors solve the system of inequalities: $\hat{w}_1 + \hat{w}_2 \geq 0$ and $\hat{w}_1 + 2\hat{w}_2 \geq 0$. The net trades that are feasible and generate positive payoffs for the taxable investor, i, taking the origin to be the lower left-hand corner, solve the system: $-\hat{w}_1 - (1 - t_i)\hat{w}_2 \geq 0$ and $-\hat{w}_1 - 2(1 - t_i)\hat{w}_2 \geq 0$. Panels A, B, and C show these vectors of net trades for tax rates of $t_i = 0.25$, 0.5, and 0.75, respectively, as the lightly shaded cones. In Panel A, investors' tax rates are not identical, but they are also not far enough apart for the two cones to intersect and generate arbitrage. At $t_i = 0.5$ (Panel B), the cone of arbitrage trades consists of a single vector, $\hat{w}$. Trades along this vector pay exactly zero to each investor in one of the two states, but $\hat{w}$ makes positive payments to consumer h in state 2 and positive payments to consumer i in state 1. Fifty percent is the lowest tax rate for which the tax-exempt asset dominates the taxed asset for investor i. Any equilibrium, therefore, would require $P_1 > P_2$ in this case. However, for the tax-exempt investor, h, asset 2 strictly dominates asset 1, implying that $P_2 > P_1$ in equilibrium. Obviously these restrictions are mutually inconsistent, and, therefore, no equilibrium will exist. When the tax rates differ sufficiently, the cone of arbitrage trades will have an interior, as shown in Panel C, for a tax rate of $t_l = 0.75$.

When tax schedules are linear, any gains to trade due to differential taxation will provide infinite tax-arbitrage opportunities. When they are present, equilibrium prices will fail to exist. In the presence of nonlinear tax schedules, however, gains to trade can be one of two types. There will typically be gains to trade that, because marginal tax rates adjust as they are exploited, cannot be extended to arbitrary magnitudes. Ross [14] refers to these as "local-tax-arbitrage" opportunities. As later sections will show, "local tax arbitrage" does not preclude the existence of an equilibrium. There may also be gains to trade that would keep the economy from attaining an equilibrium (in the absence of short-sale constraints) because they lead to infinite demands at all prices. One of the most fundamental results in convex analysis is that convex sets can be unbounded only in directions that form its "recession cone". Thus, the vectors of net trades that provide infinite gains to trade in the nonlinear case also form a cone, just as in the linear tax case. Because of this, differential taxation will cause the economy to "blow up" only in situations that can be characterized analytically (or graphically) through the types of duality arguments that were employed in the linear case.

The paper's main result shows that, if there is *any* set of marginal tax rates across investors for which the cone of arbitrage trades is empty, then equilibrium

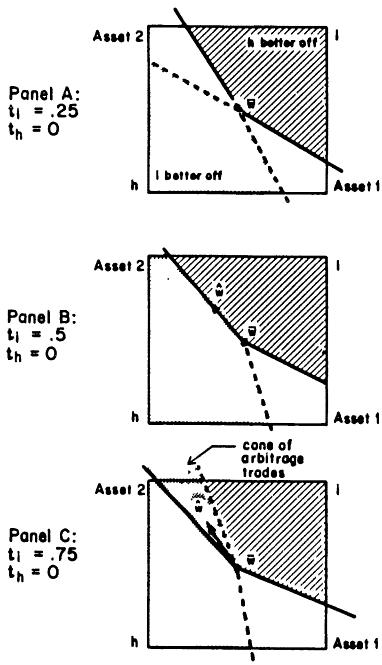

Figure 3. Arbitrage Trades with Linear Tax Schedules and Differentially Taxed Assets That Are Less Than Perfectly Correlated

prices will exist. In the context of the certainty example, this condition is particularly simple. From Figure 2, we know that, with constant marginal rates, the cone of arbitrage trades is empty only if all consumers have identical tax rates. Thus, to preclude tax arbitrage in the nonlinear case, investors' tax schedules must simply have some marginal tax rate in common, as in Case B of Figure 1. Figure 4 shows that, in this situation, there are gains to trade, or "local tax arbitrage", but no infinite tax-arbitrage opportunities. At the endowment point, i's marginal tax rate is above h's, so i's indifference curve is steeper. As i's holdings of asset 2 decrease, his or her tax rate falls, and, at the point where the marginal rates intersect, i's indifference curve will be parallel to h's. At lower taxable incomes, i's indifference curve is less steep than h's, and, since the tax rates are monotonic, the curves must eventually intersect again. Accordingly, any vector of net trades, such as w, that makes both consumers better off cannot be infinitely extended without eventually crossing i's indifference curve. Thus, there are no infinite tax-arbitrage opportunities, and, furthermore, an equilibrium clearly exists. Here, h's indifference curve serves as an equilibrium price line, and all the gains to trade, at the equilibrium allocation, accrue to consumer i.

In Case C of Figure 1, the tax schedules do not overlap, and, as Figure 5 shows, there is arbitrage at any and all prices when the assets are perfect substitutes on an after-tax basis. At the endowment point, h's marginal tax rate, and hence the slope of his or her indifference curve, is less steep than i's. It remains less steep as h's holdings of the taxable asset increase. This leaves a cone of net trades, such as $\hat{w}$, that provides more after-tax income to both individuals and that can be infinitely extended. The left-hand boundary of this cone is a dashed half-line, emanating from the endowment, with a slope of $1/(1 - \bar{t}_h)$, where $\bar{t}_h$ is consumer h's maximum tax rate. This is the limiting slope of h's indifference curve as well. If consumer h had a linear tax schedule with a slope equal to his or her maximum rate, then this dashed line would be h's indifference curve and $\hat{w}$ would be an arbitrage trade vector. Indeed, if h had a linear tax schedule with a slope equal to *any* of the marginal rates on his or her schedule, $t_h \in [\underline{t}_h, \bar{t}_h]$, $\hat{w}$ would still represent an arbitrage opportunity. However, if the two assets were less-than-perfect substitutes on an after-tax basis, the absence of arbitrage opportunities would not require tax schedules to overlap (as Figure 3 demonstrates). This motivates our general definition of tax arbitrage; there is tax-arbitrage at all prices if and only if there is a vector, such as $\hat{w}$, that provides infinite gains to trade at *all* possible configurations of marginal tax rates across investors.

III. Tax Arbitrage

The purpose of this section is to formally introduce our definition of "tax arbitrage" and to characterize the joint restrictions on tax schedules and asset payoffs that will guarantee the existence of no-tax-arbitrage prices. The existence of prices that rule out tax arbitrage is obviously a necessary condition for the existence of equilibrium prices. However, as Section IV will show, the existence of no-tax-arbitrage prices, along with our other standard assumptions, is also sufficient to guarantee the existence of equilibrium prices.

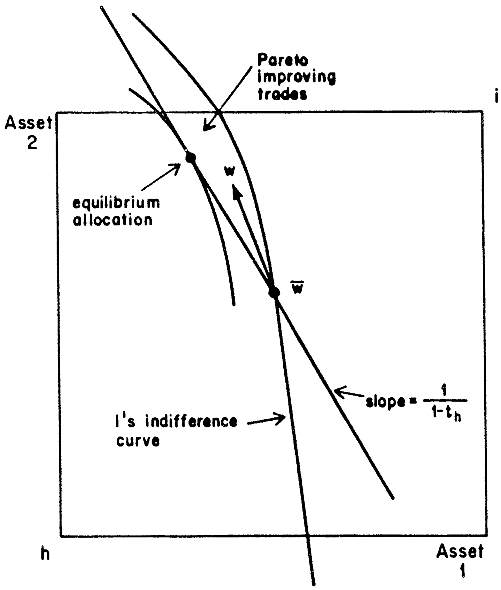

Figure 4. "Local" Tax Arbitrage with Progressive Tax Schedules and Perfectly Correlated, Differentially Taxed Assets

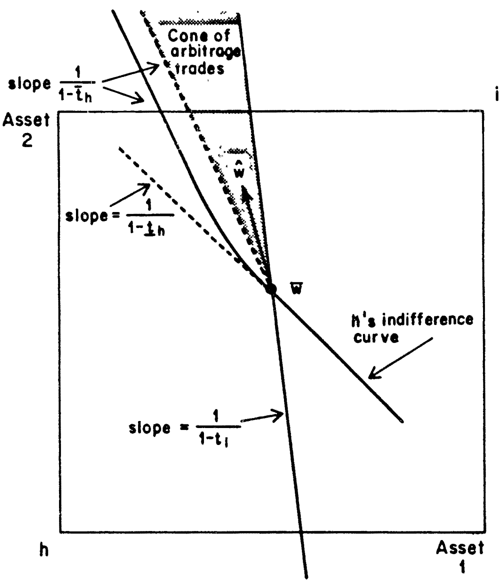

Figure 5. Arbitrage Trades with Progressive Tax Schedules and Perfectly Correlated, Differentially Taxed Assets

The question of the existence of no-arbitrage prices does not arise in models without taxes. In the presence of differing tax schedules across consumers and differential taxation of assets, however, infinite gains to trade may exist for some consumers at *any* set of prices. We wish to characterize these situations. The presence of nonlinear tax schedules complicates matters, however, since the existence of tax arbitrage at some marginal rates on the investor's tax schedule does not necessarily imply the existence of arbitrage at all rates.

Our strategy, therefore, will be as follows. We first define and characterize tax arbitrage, assuming that all tax schedules are linear. This allows us to derive characterizations of no-tax arbitrage using the types of duality results (i.e., Farkas' Lemma) that are standard in arbitrage theory. We generalize them to the case of nonlinear tax schedules in a straightforward way by requiring that these conditions hold for *all* possible configurations of marginal rates. At first, this generalization may seem to be an unnecessarily strong notion of tax arbitrage (and a weak notion of no-tax arbitrage), but it turns out to be precisely what is required. Section IV will show that, if there is *any* configuration of marginal tax rates across investors that rule out tax arbitrage, it is in fact attainable in general equilibrium, and, hence, equilibrium prices will exist.

A. Linear Tax Regimes

Let t_i denote the marginal tax rate for individual i, $i = 1, \cdots, l$, which we assume for the moment is constant across all taxable incomes. The after-tax income generated by asset j in state s, using consumer i's tax rate, will thus be $(1 - \gamma_{js} t_i) X_{js}$. Let $(X - \bar{X} t_i)$ denote the $(n \times m)$-matrix of these after-tax, state-contingent payoffs. The case where the after-tax payoffs from one asset are linearly dependent on the other assets is of some interest. For instance, Schaefer's [15] results specifically assume linear dependence, or "spanning", as did the illustrative examples of the previous section. Thus, we allow for the possibility that $(X - \bar{X} t_i)$ has less than full row rank.

Definition 1: The price vector $P \in R^n$ is a "no-tax-arbitrage" price vector for investor i if and only if there is no solution, $\hat{w}^i \in R^n$, to the following system of inequalities:[9]

$$(X - \bar{X} t_i)' \hat{w}^i > 0, \qquad \text{(P1)}$$

$$P' \hat{w}^i \leq 0.$$

In writing the above system of inequalities, we adopt the convention that, for any vector z, $z > 0$ means that some component of z is positive and all are non-negative. We will use $z \gg 0$ to imply strict posivitity. Thus, according to the above definition, an arbitrage portfolio, $\hat{w}^i$, is one that provides non-negative after-tax payoffs in all states and positive after-tax payoffs in some, and that has a nonpositive cost. It is a well-known result that there is no solution to P1 if and only if there are positive implicit state prices that correctly value the after-tax

[9] Note that this definition does not explicitly rule out prices that assign a nonzero cost to any portfolio with zero after-tax payoffs. Implicitly, however, such prices are ruled out by the existence of an asset with strictly positive pretax (and after-tax) payoffs. (See assumption (A4).) To avoid arbitrage, such an asset must command a strictly positive price. Consequently, unless all portfolios with zero after-tax payoffs have a zero price, there will exist solutions to P1—portfolios that provide strictly positive after-tax payoffs and have a nonpositive price.

cash flows.[10] That is, the following system is dual to P1:

$$P = (X - \bar{X}t_i)\rho_i, \tag{D1}$$

$$\rho_i \gg 0,$$

where ρ_i is a vector in R^m.

For investor i, the set of price vectors for which there is a solution to D1 is a relatively open, convex cone in R^n.[11] By assumption (A4), it will not contain the origin. If there exists a price vector $P \in R^n$ such that D1 has a solution for every i, $i = 1, \cdots, l$, then these l cones have a nonempty intersection. We will refer to such a price vector as a "no-tax-arbitrage" price vector. Because the after-tax payoffs, $(X - \bar{X}t_i)$, differ across tax rates, however, it may be that no such price vector exists. The following theorem shows that the existence of prices that rule out tax arbitrage simultaneously for all investors is equivalent to having a cone of tax-arbitrage allocations that is empty.

THEOREM 1: *There exist "no-tax-arbitrage" prices for an economy with constant marginal tax rates, $T_i'(\cdot) = t_i$, $i = 1, \cdots, l$, if and only if there is no set of asset holdings $\hat{w}^i \in R^n$, $i = 1, \cdots, l$, solving the following system of inequalities*:

$$(X - \bar{X}t_i)'\hat{w}^i \geq 0, \quad i = 1, \cdots, l, \tag{D2}$$

$$\textstyle\sum_{i=1}^{l} \hat{w}^i = 0.$$

where, for at least one i, the above inequalities are semipositive.

Solutions to D2, if they exist, represent offsetting transfers of securities between investors. Because these transfers net out to zero, they cost the financial sector nothing. However, because they make at least one person better off without making anyone else worse off (except, of course, the government), they are "Pareto improving." The resulting allocation is clearly feasible, and, with linear taxes, $\hat{w}$ can be scaled up to arbitrary magnitudes. Thus, a solution to the Pareto Planner's problem will not exist and obviously cannot be supported or decentralized as a competitive equilibrium. Consequently, we will refer to problem D2 as the *tax-arbitrage problem* and solutions to it as *tax-arbitrage allocations*. In the presence of such opportunities, *any* set of asset prices will admit tax arbitrage for at least one consumer.

Tax-arbitrage allocations, if they exist, form a cone as in Figure 2. As assets become more alike in terms of their pretax cash flow patterns and/or as tax rates become more divergent across investors, the cone of tax-arbitrage allocations expands. In the special case where there are two perfectly correlated, differentially taxed assets, say j and k, with $\gamma_{js} = \gamma_j > \gamma_k = \gamma_{ks}$ for all s, the tax-arbitrage problem D2 will have a solution unless all investors have identical tax rates. This

[10] See Ross [13] for the classic development. If $(X - \bar{X}t_i)$ has less than full row rank, the redundant assets can be deleted and priced as a linear combination of the remaining securities.

[11] If $(X - \bar{X}t_i)$ is of less than full rank, there will exist (nonzero) portfolios that generate after-tax payoffs of zero in every state. The absence of arbitrage then requires that these portfolios be assigned a price of zero. The no-tax-arbitrage prices will then be a cone in the lower dimensional subspace orthogonal to these portfolios. This cone will not be open in R^n but will be open relative to the subspace in which it is contained.

result is essentially that obtained by Schaefer [15]. Under more general conditions, however, tax arbitrage may exist at any set of prices if tax rates are sufficiently far apart. Consequently, an equilibrium may not exist without short-sale constraints even in situations where perfectly correlated, differentially taxed assets do not exist.

B. *Progressive Tax Regimes*

If tax schedules are not linear, then investors need not occupy the same tax bracket *ex post* under uncertainty even though there may exist differentially taxed assets with perfectly correlated pretax returns. As Dammon [3] points out, if markets are incomplete, this behavior is also not inconsistent with the existence of an equilibrium where investors have unrestricted short-sale opportunities. The possibility that marginal tax rates may be state dependent greatly reduces the opportunities for infinite gains to trade. As the main result of this section shows, if there exists *any* configuration of marginal tax rates across investors that rules out tax-arbitrage allocations, then infinite gains to trade will not exist for the financial sector.

We first extend the notion of no-tax-arbitrage prices to nonlinear (i.e., progressive) tax schedules. To be a no-tax-arbitrage price vector, we simply require that there exist some tax rate on the interior of the investor's tax schedule that precludes arbitrage at those prices.

Definition 2: The price vector $P \in R^n$ is a "no-tax-arbitrage" price vector for investor i if and only if there is no solution $\hat{w}^i \in R^n$ to the following system of inequalities:

$$\begin{bmatrix} (X - \overline{X}\overline{t}_i)' \\ (X - \overline{X}\underline{t}_i)' \end{bmatrix} \hat{w}^i > 0, \qquad \text{(P2)}$$

$$P'\hat{w}^i \leq 0,$$

where $\overline{t}_i$ is investor i's highest marginal tax rate and $\underline{t}_i$ is the lowest.[12]

If $\hat{w}^i$ solves P2, then it must provide non-negative after-tax payoffs at both $\overline{t}_i$ and $\underline{t}_i$ and semipositive after-tax payoffs at at least one of these extreme rates. The values $[\lambda(X - \overline{X}\overline{t}_i)' + (1 - \lambda)(X - \overline{X}\underline{t}_i)']\hat{w}^i$, with $\lambda \in (0, 1)$, are the after-tax payoffs on $\hat{w}^i$ at tax rates that lie between $\overline{t}_i$ and $\underline{t}_i$. Therefore, if $\hat{w}^i$ solves P2, it serves as an arbitrage portfolio at any rate on the interior of investor i's tax schedule.

Notice that this definition involves a very strong notion of tax arbitrage. To be a tax-arbitrage portfolio, $\hat{w}^i$ must generate arbitrage at *all* of investor i's marginal rates. This definition may seem counterintuitive since it is not very restrictive. For example, it does not rule out prices that provide an investor with arbitrage opportunities along most of his or her tax schedule. Nevertheless, as we will show below, the existence of prices that rule out tax arbitrage, as defined above, for all investors simultaneously is precisely what is required to ensure the existence of equilibrium prices.

[12] The observation made in footnote 9 applies here as well.

Solutions to P2, if they exist, can be scaled up to arbitrary magnitudes. That is, if $\hat{w}^i$ solves P2, then $\alpha\hat{w}^i$ will also solve P2 for any $\alpha > 0$. In the presence of such opportunities, there will obviously be no (bounded) solution to an individual consumer's optimization problem as long as more is preferred to less in all states. In the parlance of convex analysis, the tax-arbitrage portfolio, $\hat{w}^i$, is a "direction of recession" for consumer i's problem. The investor's demand for securities will be unbounded in the direction of $\hat{w}^i$. More generally, directions of recession for consumer i's portfolio problem are the nonzero vectors, v^i, with the property that, for any feasible set of asset holdings, w^i, the position $w^i + \alpha v^i$ is feasible and provides investor i with higher utility than w^i for all $\alpha > 0$. If such a direction exists, the investor's portfolio problem will obviously not attain a maximum. According to Rockafellar [12] it is also "a remarkable fact . . . that such behavior is possible *only* if a direction of recession exists" (p. 265, emphasis his).[13] The next result shows that the directions in which investors' demands become unbounded are precisely the solutions to P2. If they exist, they represent portfolios that generate tax arbitrage at all of the investor's marginal rates.

LEMMA 1: *A direction of recession for the investor's portfolio problem exists if and only if there exists a tax-arbitrage portfolio solving* P2.

Intuitively, the proof of the lemma exploits the fact that, for any direction of recession with tax consequences, after-tax payoffs eventually become linear in the portfolio weights as the maximum or minimum marginal tax rate is reached. The after-tax payoffs generated in this way must remain non-negative in every state. Since marginal rates are progressive, the maximum and minimum rates are the least favorable to the taxpayer in states where the portfolio generates positive and negative taxable income, respectively. Thus, a direction of recession must provide semipositive after-tax payoffs, or tax arbitrage, at all interior rates. Further, when prices rule out tax arbitrage, no direction of recession will exist, and, hence, investors' portfolio-optimization problems will have well-behaved, bounded solutions. When, then, will there exist prices that rule out tax arbitrage? The arguments leading to Theorem 1 can be adapted very simply to prove the following theorem:

THEOREM 2: *There exist "no-tax-arbitrage" prices for an economy with progressive taxation if and only if there is no set of asset holdings* $\hat{w}^i \in R^n$, $i = 1, \cdots, l$, *solving the following system of inequalities*:

$$[\lambda_i(X - \bar{X}\bar{t}_i)' + (1 - \lambda_i)(X - \bar{X}\underline{t}_i)']\hat{w}^i \geq 0, \quad i = 1, \cdots, l, \tag{D3}$$

$$\textstyle\sum_{i=1}^{l} \hat{w}^i = 0,$$

for all $\lambda_i \in [0, 1]$, $i = 1, \cdots, l$, *and where, for at least one i, the above inequalities are semipositive at* $\lambda_i = 1$ *or* $\lambda_i = 0$.

Once again, this notion of tax arbitrage is extremely strong. If *any* combination of tax rates across investors precludes tax-arbitrage allocations, then, by our definitions, no-tax-arbitrage prices exist. Alternatively stated, there is no set of

[13] Theorem 27.3 of Rockafellar [12] provides the appropriate proof.

asset prices that precludes infinite tax arbitrage only if there exists a set of asset holdings, $\hat{w}^i$, $i = 1, \cdots, l$, that solves D3 for *all* possible combinations of marginal tax rates across investors. In fact, the existence of tax-arbitrage allocations at each configuration of marginal rates is not enough. Rather, the *same* reallocation of asset holdings must work at *all* configurations of marginal rates.

As an example, if there are assets with perfectly correlated pretax payoffs that contribute a constant but different fraction of their pretax payoffs to taxable income, the existence of no-tax-arbitrage prices requires tax schedules to overlap. (See Figure 4). That is, there must be some marginal tax rate that all investors have in common on their tax schedules. However, because security returns are stochastic, investors need not occupy the same tax bracket *ex post*, nor does Theorem 2 require it.

IV. Existence of Equilibrium Prices

Given the strong nature of tax arbitrage defined above, it may seem surprising that the existence of no-tax-arbitrage prices is sufficient to ensure the existence of equilibrium prices. To see that this is the case, consider the set of portfolio holdings that satisfy the semipositivity condition in P2 for the ith consumer. Call this set K^i. It is a convex cone, closed everywhere but along the subspace containing vectors that generate zero after-tax payoffs in all states at both the maximum and minimum tax rates. Let $\bar{K}^i$ denote its closure. The prices that have a strictly positive inner product with every vector in K^i (the negative of the price vectors that are interior to the dual cone) and a non-negative inner product with every vector in $\bar{K}^i$ will be a relatively open, convex cone. Call this cone C^i, where $C^i = \{P \in R^n \colon P'w^i > 0,\ w^i \in K^i \text{ and } P'w^i \geq 0,\ w^i \in \bar{K}^i\}$. The prices that rule out tax arbitrage simultaneously for all investors will be the intersection of these cones. Theorem 2 gives necessary and sufficient conditions for this intersection to be nonempty, and, as long as it is not empty, it too will be a relatively open cone. Denote this cone as C, where $C = \cap_{i=1}^{l} C^i$, and let $\bar{C}$ be its closure. Prices on the relative boundary of $\bar{C}$ admit tax arbitrage for at least one consumer. Thus, equilibrium prices, if they exist, must be in the relative interior of $\bar{C}$. The next theorem shows that equilibrium prices do exist whenever the conditions of Theorem 2 are satisfied. Our proof (see the Appendix) follows that provided by Geanakoplos and Polemarchakis [6] for the no-tax case. Where taxes add no complications we omit the details of the arguments and refer the interested reader to the original. First, however, we define a competitive financial market equilibrium.

Definition 3: A *competitive financial market equilibrium* is a set of security holdings, $\{w^i\}_{i=1}^{l}$, and a set of security market prices, $P \in R^n$, such that (a) w^i maximizes consumer i's utility subject to the budget constraint $\sum_j P_j(w_j^i - \bar{w}_j^i) = 0$ and (b) all markets clear, i.e., $\sum_i (w_j^i - \bar{w}_j^i) = 0$ for all j.

THEOREM 3: *A competitive financial-market equilibrium exists if and only if no-tax-arbitrage prices exist.*

V. Taxation That Depends upon Prices

The model presented above allows the contribution to taxable income to depend upon the state of the world and the asset. It does not, however, allow an asset's contribution to taxable income to depend upon its price and so fails to capture important features of the tax code that distinguish between "capital gains" and "ordinary income" as well as between "interest" and "return of principal". In this section, we develop an alternative model that allows an asset's contribution to taxable income to depend upon its initial price, and we describe conditions that ensure that analogous results to those in the previous two sections will hold.

We assume, then, that financial assets are of two types: taxable and tax exempt. In terms of our previous notation, $\gamma_{js} = \gamma_j = 0$ or 1. An asset's end-of-period payoff consists of two parts. First, there is a state-contingent payment, Y_{js}, that is fully taxable if $\gamma_j = 1$ and fully tax exempt if $\gamma_j = 0$. Thus, Y_{js} can be thought of as a dividend or coupon payment. Second, there is a state-dependent terminal value of the asset, P_{js}, that contributes $(P_{js} - P_j)$ to taxable income independently of γ_j. Thus, $(P_{js} - P_j)$ can be thought of as capital-gain income, or interest income on pure discount bonds. Under these assumptions, the state-s taxable income associated with asset holdings w^i is $\sum_j w_j^i(\gamma_j Y_{js} + P_{js} - P_j)$. An investor with a marginal tax rate of t_i in state s would receive after-tax payments of $\sum_j w_j^i[Y_{js}(1 - \gamma_j t_i) + P_{js}(1 - t_i) + t_i P_j]$, where $t_i P_j$ is the tax shield generated by the basis. Analogous to assumption (A4), we assume that at least one asset, which will be denoted asset 1, has strictly positive pretax payoffs in all states. That is, $Y_{1s} + P_{1s} > 0$, $Y_{1s} \geq 0$, and $P_{1s} \geq 0$ for all s.

Investors are assumed to be price takers, so the fact that an asset's price affects investors' taxable income does not alter the properties of their portfolio problems. For investor i, we say that the price vector $P \in R^n$ precludes tax arbitrage if there is no solution, $\hat{w}^i \in R^n$, to the system of inequalities:

$$\begin{aligned} &\textstyle\sum_j \hat{w}_j^i[Y_{js}(1 - \gamma_j \bar{t}_i) + P_{js}(1 - \bar{t}_i) + \bar{t}_i P_j] \geq 0 \quad \forall s, \qquad \text{(P1*)} \\ &\textstyle\sum_j \hat{w}_j^i[Y_{js}(1 - \gamma_j \underline{t}_i) + P_{js}(1 - \underline{t}_i) + \underline{t}_i P_j] \geq 0 \quad \forall s, \\ &\textstyle\sum_j \hat{w}_j^i P_j \leq 0, \end{aligned}$$

with strict positivity in at least one state at either $\bar{t}_i$ or $\underline{t}_i$.[14]

Again, Theorem 27.3 of Rockafellar [12] ensures that investors' portfolio problems have well-behaved solutions unless a direction of recession exists. Analogous to Lemma 1, these directions of recession correspond to tax-arbitrage portfolios, $\hat{w}^i$, solving P1* and providing semipositive after-tax payoffs in every state at all of investor i's marginal tax rates. No-tax-arbitrage prices for the economy are those price vectors for which there is no solution to P1* for any investor.

When we move from partial- to general-equilibrium results, however, we encounter the problem that no-tax-arbitrage prices, if they exist, may not constitute a convex set. In particular, what is required for Theorems 2 and 3 is

[14] The observation made in footnote 9 applies here as well.

that no-tax-arbitrage prices form a convex cone. This problem is evident from examining the system of inequalities given by P1*. Prices enter the first two inequalities in a nonproportional way. Thus, they cannot be jointly scaled up or down by a positive constant without potentially changing the signs of the inequalities.

The particular type of pathology that needs to be ruled out is the existence of an asset with strictly positive pretax payoffs but a negative price. While this would appear to represent an arbitrage opportunity, it may not when taxable income depends upon prices. For example, if the asset's price is a sufficiently large negative number, the after-tax payoffs can be negative (because of the adverse effect a negative price has on an investor's basis) even though the pretax payoffs are strictly positive. Thus, with capital-gains taxation, positive prices are both a bad thing, because they make assets cost more, and a good thing, because they shield income from taxes through the basis. For this reason, one cannot be sure that positive pretax payoffs will command a positive price even though this is the basic premise behind notions of arbitrage and arbitrage pricing.

One way to deal with this problem is to assume that there is at least one tax-exempt investor. The set of no-tax-arbitrage prices for such an investor must lie in the half-space that ascribes a positive price to the asset with strictly positive pretax payoffs. The intersection between this and the set of prices that rule out tax arbitrage for all other investors will have the desired properties, as the results below demonstrate, as long as this intersection is not empty.

More generally, however, the possibility that equilibria may exist in which there are negative prices for assets with strictly positive pretax payoffs, or vice versa, seems rather far removed from behaviors that one observes or that seem intuitively reasonable. Accordingly, we develop results in the remainder of this section that focus on the issue of when no-tax-arbitrage and equilibrium prices will exist that have the characteristic that the asset with strictly positive pretax payoffs has a positive price. The first result shows that, under these circumstances, whenever there is an arbitrage portfolio that adversely affects the investor's basis, there is also one that has higher payoffs and leaves the basis unchanged. Intuitively, if an arbitrage portfolio has a negative cost and the asset with positive payoffs is positively priced, the arbitrageur can increase both his or her basis and the other sources of after-tax income by investing the excess time-zero funds in the asset with positive payoffs. The second result then shows that the no-tax-arbitrage prices that ascribe a positive price to the asset with positive pretax payoffs form a relatively open, convex cone. This follows from the fact that an arbitrage portfolio does not ever need to contribute negatively to an investor's basis, by Lemma 2. Therefore, the prices of the securities can be ignored in all but the last inequality in P1*, which is homogeneous in the prices.

LEMMA 2: *If $\hat{w}^i$ is a tax-arbitrage portfolio that solves* P1* *for some $P \in R^n$, with $P_1 > 0$, then there exists a tax-arbitrage portfolio, $\check{w}^i$, that solves* P1* *at those prices and that has zero cost, $P'\check{w}^i = 0$.*

LEMMA 3: *The set of no-tax-arbitrage prices that ascribe a positive price to the asset with strictly positive pretax payoffs is a relatively open, convex cone.*

The arguments leading to Theorems 1 and 2 can now be applied to prove the following theorem.

THEOREM 4: *There exist "no-tax-arbitrage" prices (that assign a positive price to the asset with strictly positive pretax payoffs) for an economy with progressive taxation, and taxable income that depends upon prices, if and only if there is no set of asset holdings, $\hat{w}^i \in R^n$, $i = 1, \cdots, l$, solving the following system of inequalities*:

$$\textstyle\sum_j \hat{w}^i_j[(Y_{js} + P_{js}) - \lambda_i \bar{t}_i(\gamma_j Y_{js} + P_{js}) - (1 - \lambda_i)\underline{t}_i(\gamma_j Y_{js} + P_{js})] \geq 0 \quad \forall s, \qquad \text{(D1*)}$$

$$\textstyle\sum_i \hat{w}^i_j = 0,$$

for all $\lambda_i \in [0, 1]$, $i = 1, \cdots, l$, *and where, for at least one* i, *the above inequalities are semipositive at* $\lambda_i = 1$ *or* $\lambda_i = 0$.

Theorem 4 gives the necessary and sufficient conditions for the existence of no-tax-arbitrage prices that also assign a positive price to assets with strictly positive pretax payoffs. These conditions are equivalent to the absence of offsetting transfers between investors that would make no one worse off, and some strictly better off, at all configurations of marginal rates across consumers. By Lemma 3, if such prices exist, they will be a relatively open, convex cone. Denote this cone as C and let $\bar{C}$ be its closure. As before, equilibrium prices, if they exist, must be in the interior of $\bar{C}$ since those on the boundary admit arbitrage for at least one consumer. Arguments identical to those leading to Theorem 3 will imply that equilibrium prices exist whenever the conditions of Theorem 4 are met and the interior of $\bar{C}$ is nonempty.

THEOREM 5: *If "no-tax-arbitrage" prices exist that assign a positive price to the asset with strictly positive pretax payoffs, then equilibrium prices exist.*

In interpreting this result, two caveats should be kept in mind. As discussed at the beginning of the section, it is possible that no-tax-arbitrage prices exist, and conceivably equilibrium prices as well, that do not assign a positive price to the asset with strictly positive pretax payoffs. Second, our model has a free normalization of security prices that is exploited in the proofs of both Theorems 3 and 4. It is the same normalization that leads authors to set the riskless rate of interest to zero "without loss of generality". In the present context, however, this normalization has real consequences since the overall level of security prices affects the aggregate level of taxable income and, thus, taxes paid. This may seem arbitrary, but, in the "real world", the overall level of interest rates, as well as their relative magnitudes, do affect the aggregate tax liability. In the "real world", however, there is consumption "today" and consumption "tomorrow", so that this normalization is not free. Similarly, a version of our model that allows for initial consumption would tie down the "riskless rate" and, therefore, the general level of security prices. The results we have obtained, which involve relative prices and rates of return, would still hold in such a model.

VI. Concluding Remarks

We have considered a standard general-equilibrium model of financial markets, with the addition of differential taxation. We show that the duality conditions that characterize no-arbitrage prices can be easily extended to environments where taxable income depends upon prices and tax schedules are nonlinear. In this context, a "no-tax-arbitrage" condition can be appended to standard assumptions to provide both necessary and sufficient conditions for the existence of equilibrium prices for financial assets. In particular, we have shown that, if there is *any* set of marginal tax rates on investors' tax schedules that preclude infinite gains to trade, then equilibrium prices for financial assets will exist. In the special case where there are assets that have perfectly correlated pretax payoffs, but are differentially taxed, the "no-tax-arbitrage" condition simply requires that investors' tax schedules must overlap. However, if markets are incomplete, investors may not occupy the same tax bracket *ex post*. Similar results hold when taxes depend upon prices if attention is restricted to prices that rule out arbitrage while assigning a positive price to the asset with strictly positive pretax payoffs. These results are not only intuitively appealing, but also important in terms of their equilibrium implications for models with differential taxation and unrestricted short sales.

Appendix

Proof of Theorem 1: For each investor, the set of portfolios that provide semipositive after-tax payoffs is a convex cone. It is closed everywhere but along the null space of $(X - \bar{X}t_i)$. Let K^i denote the cone of such portfolios for investor i, and let $\bar{K}^i$ be its closure. Note that $\bar{K}^i$ includes all portfolios that provide zero after-tax payoffs. If $\cup_{i=1}^{l} K^i$ is strictly contained in a half-space, then there is no solution to $\sum_{i=1}^{l} \hat{w}^i = 0$, $\hat{w}^i \in \bar{K}^i$ for all i, $\hat{w}^i \in K^i$ for some i. However, there does exist a vector, $P \in R^n$, orthogonal to the hyperplane defining the half-space, such that, for all i, $P'w^i \geq 0$ for all $w^i \in \bar{K}^i$ and $P'w^i > 0$ for all $w^i \in K^i$. This is a no-tax-arbitrage price vector. If $\cup_{i=1}^{l} K^i$ is not contained in a half-space, then $\sum_{i=1}^{l} \hat{w}^i = 0$, $\hat{w}^i \in \bar{K}^i$ for all i and $\hat{w}^i \in K^i$ for some i, has a solution. However, then there is no vector, $P \in R^n$, such that, for all i, $P'w^i > 0$ for all $w^i \in K^i$ and $P'w^i \geq 0$ for all $w^i \in \bar{K}^i$. Therefore, a no-tax-arbitrage price vector does not exist. Q.E.D.

Proof of Lemma 1: Any solution to P1 is obviously a direction of recession for consumer i's problem since it provides semipositive after-tax payoffs at all marginal rates on consumer i's tax schedule and has a nonpositive cost. Therefore, it remains for us to show that any direction of recession, v^i, for consumer i's problem is a tax-arbitrage portfolio, $\hat{w}^i$

Let v^i be a direction of recession for consumer i's portfolio problem. The budget set for this problem is given by

$$c_s^i = X_s'(w^i + \alpha v^i) - T_i(\bar{X}_s'(w^i + \alpha v^i)) \geq 0, \quad \forall s, \tag{A1}$$

$$P'(w^i + \alpha v^i) \leq P'\bar{w}^i. \tag{A2}$$

If (A2) is to hold as $\alpha \to \infty$, then it must be that $P'v^i \le 0$. If (A1) is to hold, then c_s^i must remain non-negative as $\alpha \to \infty$, for all s. Using L'Hôpital's rule, we have

$$\lim_{\alpha\to\infty} X_s'(w^i + \alpha v^i) - T_i(\bar{X}_s'(w^i + \alpha v^i)) = \begin{cases} (X_s - \bar{X}_s\bar{t}_i)'v^i, & \text{if } \bar{X}_s'v^i \ge 0, \\ (X_s - \bar{X}_s\underline{t}_i)'v^i, & \text{if } \bar{X}_s'v^i \le 0. \end{cases} \tag{A3}$$

If $\bar{X}_s'v^i > 0$, then, for all $t_i < \bar{t}_i$,

$$(X_s - \bar{X}_s t_i)'v^i > (X_s - \bar{X}_s\bar{t}_i)'v^i \ge 0. \tag{A4}$$

Similarly, if $\bar{X}_s'v^i < 0$, then, for all $t_i > \underline{t}_i$,

$$(X_s - \bar{X}_s t_i)'v^i > (X_s - \bar{X}_s\underline{t}_i)v^i \ge 0. \tag{A5}$$

Therefore, it follows that, for any $\lambda_i \in [0, 1]$, we have

$$[\lambda_i(X - \bar{X}\bar{t}_i)' + (1 - \lambda_i)(X - \bar{X}\underline{t}_i)']v^i \ge 0, \tag{A6}$$

with the inequality strict in some state at all interior tax rates (i.e., $\lambda_i \in (0, 1)$) and at least one of the extreme rates $\bar{t}_i$ or $\underline{t}_i$, unless $\bar{t}_i = \underline{t}_i$ or unless v^i is tax neutral in all states (i.e., $\bar{X}'v^i = 0$). If $\bar{t}_i = \underline{t}_i$ or $\bar{X}'v^i = 0$, then, by (A3), either $(X - \bar{X}\bar{t}_i)'v^i = (X - \bar{X}\underline{t}_i)'v^i$ is semipositive and (A6) holds or v^i solves $(X - \bar{X}t_i)'v^i = 0$ for all $t_i \in [\underline{t}_i, \bar{t}_i]$. In the latter case, however, v^i has no effect on investor i's utility and thus contradicts the assumption that v^i is a direction of recession for consumer i's problem. Therefore, v^i must be a tax-arbitrage portfolio. Q.E.D.

Proof of Theorem 2: For each λ_i, the portfolios generating semipositive after-tax payoffs for investor i form a cone in R^n. Those that do so for all $\lambda_i \in [0, 1]$ will be the intersection of these cones and, hence, a cone. Call this cone K^i and its closure $\bar{K}^i$. As in the proof of Theorem 1, if $\cup_{i=1}^{l} K^i$ is strictly contained in a half-space, then (a) there is no solution to $\sum_{i=1}^{l} \hat{w}^i = 0$, $\hat{w}^i \in \bar{K}^i$ for all i and $\hat{w}^i \in K^i$ for some i, and (b) there exists a vector, $P \in R^n$, orthogonal to the hyperplane defining the half-space, such that, for all i, $P'w^i > 0$ for all $w^i \in K^i$ and $P'w^i \ge 0$ for all $w^i \in \bar{K}^i$. This is a no-tax-arbitrage price vector. If $\cup_{i=1}^{l} K^i$ is not contained in a half-space, then (a) there is a solution to $\sum_{i=1}^{l} \hat{w}^i = 0$, $\hat{w}^i \in \bar{K}^i$ for all i and $\hat{w}^i \in K^i$ for some i, and (b) there is no vector, $P \in R^n$, such that, for all i, $P'w^i \ge 0$ for all $w^i \in \bar{K}^i$ and $P'w^i > 0$ for all $w^i \in K^i$. Q.E.D.

Proof of Theorem 3: The "only if" part is immediate. If $\{w^i\}_{i=1}^{l}$ is an equilibrium allocation, then there can be no other allocation that is budget feasible and provides higher utility. Therefore, a tax-arbitrage portfolio (which is costless and provides semipositive after-tax payoffs at all marginal rates on the investor's tax schedule) cannot exist at the equilibrium prices.

To prove the converse, assume that no-tax-arbitrage prices exist so that the interior of $\bar{C}$ is nonempty. Since $\bar{C}$ is a pointed cone, it can be cut with a hyperplane H. The set $\bar{S} = H \cap \bar{C}$ is a compact, convex set in R^n and represents a normalization of the price vectors in $\bar{C}$ that leaves consumers' budget sets unchanged. By Lemma 1, and Theorem 27.3 of Rockafellar [12], investors' portfolio problems have well-behaved solutions for $P \in \text{rel int}(\bar{S}) \subset \text{rel int}(C)$. On the boundary of $\bar{S}$, however, portfolio demands are unbounded in some direction of recession for at least one consumer. Therefore, consider a sequence of economies where the kth economy constrains the portfolio demands of each consumer i, $i = 1, \cdots, l$, to lie within the cube defined by $M_k = \{w^i \in R^n: |w_j^i| < k, j = 1, \cdots, n\}$. For $P \in \bar{S}$, define $w_k(P) = \sum_{i=1}^{l} w_k^i(P)$ to be the vector of aggregate excess demands for the n securities in the kth economy. Because demands are bounded and utility functions are continuous, $w_k(P)$ will be upper hemicontinuous and convex valued. Therefore, the mapping that maximizes the value of excess demand,

$$\Psi_k(P) = \text{argmax}_{q \in \bar{S}}\, q' w_k(P),$$

and carries $\bar{S}$ into itself will have a fixed point, P_k^*. By Walras' Law, $P_k^{*\prime} w_k(P_k^*) = 0$. Moreover, for large enough k, $P_k^* \in \text{rel int}(\bar{S})$. If it were not, then some investor would have a tax-arbitrage opportunity and, as k increases, the investor's demands would become increasingly large. By Theorem 2 and the fact that the interior of $\bar{C}$ is nonempty, the cone of portfolio demands that simultaneously provide semipositive after-tax payoffs for all consumers is strictly contained in a half-space. Therefore, there are no offsetting transfers between investors that make no one worse off and some strictly better off and that leave aggregate demand unchanged. Consequently, the tax-arbitrage position or positions must eventually dominate aggregate excess demand, $w_k(P_k^*)$. Since any price $\hat{P} \in \text{rel int}(\bar{S})$ ascribes a positive value to portfolios with strictly positive after-tax payoffs, $\hat{P}' w_k(P_k^*) > P_k^{*\prime} w_k(P_k^*) = 0$ for large enough k, which contradicts P_k^* being the price that maximizes the value of excess demand.

Having established that $P_k^* \in \text{rel int}(\bar{S})$, it now follows that, for large enough k, $w_k(P_k^*) = 0$ and markets clear. If excess demand were not identically zero, one could choose a price, $\hat{P} \in \bar{S}$, close to P_k^* but outside the space orthogonal to $w_k(P_k^*)$ such that $\hat{P}' w_k(P_k^*) > 0$ since $P_k^* \in \text{rel int}(\bar{S})$. If the after-tax payoff matrix has less than full rank, the cone of no-tax-arbitrage prices will not have full dimension. In this case, there may be no price vector, $\hat{P} \in \bar{S}$, such that $\hat{P}' w_k(P_k^*) > 0$ whenever $w_k(P_k^*) \neq 0$. However, the directions orthogonal to the entire cone of no-tax-arbitrage prices will be precisely those associated with portfolios that have zero after-tax payoffs for some investor. Thus, one can always offset any excess aggregate demand or supply by adjusting this investor's portfolio holdings. These transactions will have zero cost since they involve directions orthogonal to P_k^* and are thus budget feasible for all involved. They will not change any agent's after-tax payoffs or utility and so will be individually optimal. Thus, markets will clear at P_k^*. Geanakoplos and Polemarchakis [6] provide the arguments that allow k to become unbounded so that investors' demands are unrestricted. This completes the proof. Q.E.D.

Proof of Lemma 2: Suppose that $\hat{w}^i$ solves P1* for some $P \in R^n$. If $P'\hat{w}^i = 0$, we are done. If not, then it must be the case that $P'\hat{w}^i < 0$. By assumption, the asset with strictly positive pretax payoffs (asset 1) has a positive price, $P_1 > 0$, and, therefore, this asset has strictly positive after-tax payoffs. Consider now the portfolio $\check{w}^i = \hat{w}^i - (P'\hat{w}^i/P_1)e_1$, where e_1 is an n-vector with a one in the first position and zeros everywhere else. The portfolio $\check{w}^i$ is simply the portfolio $\hat{w}^i$

augmented to include a positive amount of the asset with strictly positive payoffs. The cost of this portfolio is $P'\check{w}^i = P'\hat{w}^i + P'(P'\hat{w}^i/P_1)e_1 = 0$. The payoffs on this portfolio are higher in every state than those for $\hat{w}^i$ (since asset 1 has strictly positive after-tax payoffs). Therefore, $\check{w}^i$ is an arbitrage portfolio with zero cost. Q.E.D.

Proof of Lemma 3: Assume that no solution exists for P1* for any investor at prices $P_a \in R^n$ and $P_b \in R^n$. Then, P_a and P_b are no-tax-arbitrage price vectors. Let $P_c = \lambda_a P_a + \lambda_b P_b$, where $\lambda_a \geq 0$, $\lambda_b \geq 0$, and $\lambda_a + \lambda_b > 0$. If P_c allows arbitrage for some investor i, then, by Lemma 2, there is a solution, $\check{w}^i$, to P1* with $P_c'\check{w}^i = 0$. That is, $\check{w}^i$ solves

$$\textstyle\sum_j \check{w}_j^i[Y_{js}(1 - \gamma_j \bar{t}_i) + P_{js}(1 - \bar{t}_i)] \geq 0, \quad \forall s, \tag{A7}$$

$$\textstyle\sum_j \check{w}_j^i[Y_{js}(1 - \gamma_j \underline{t}_i) + P_{js}(1 - \underline{t}_i)] \geq 0, \quad \forall s, \tag{A8}$$

$$P_c'\check{w}^i = (\lambda_a P_a + \lambda_b P_b)'\check{w}^i = 0, \tag{A9}$$

with positive after-tax payoffs in at least one state at either $\bar{t}_i$ or $\underline{t}_i$. Since $P_c'\check{w}^i = 0$, either $P_a'\check{w}^i \leq 0$ or $P_b'\check{w}^i \leq 0$. If $P_a'\check{w}^i = 0$ or $P_b'\check{w}^i = 0$, then $\check{w}^i$ solves P1* for either P_a or P_b, which contradicts the assumption that they are no-tax-arbitrage price vectors. If either $P_a'\check{w}^i < 0$ or $P_b'\check{w}^i < 0$, some amount of the asset with strictly positive pretax payoffs (asset 1) can be added to $\check{w}^i$ to construct a portfolio, $\hat{w}^i$, that satisfies the non-negativity constraints, (A7) and (A8), and has zero cost. However, then $\hat{w}^i$ solves P1* for either P_a (if $P_a'\check{w}^i < 0$) or P_b (if $P_b'\check{w}^i < 0$), which contradicts the assumption that P_a and P_b are no-tax-arbitrage price vectors. Therefore, $P_c'\check{w}^i > 0$, and P_c is a no-tax-arbitrage price vector. This proves that the set of no-tax-arbitrage price vectors is a convex cone. It is relatively open by the earlier arguments of Section IV. Q.E.D.

Proof of Theorem 4: Consider the cone, K^i, of portfolios solving (A7) and (A8), and let $\bar{K}^i$ be its closure. It will include a portfolio consisting solely of the asset with strictly positive pretax payoffs. Following the arguments in Theorem 2, if $\cup_{i=1}^{l} \bar{K}^i$ is contained in a half-space, there will clearly be no solution to D1*, but there will be a price vector that assigns positive prices to all $w^i \in K^i$. By Lemma 2, therefore, there will be no solution to P1* for any investor. If $\cup_{i=1}^{l} \bar{K}^i$ is not contained in a half-space, then there is a solution to D1*, and there will also be no $P \in R^n$ such that, for all i, $P'w_i > 0$ for all $w^i \in K^i$ and $P'w^i \geq 0$ for all $w^i \in \bar{K}^i$. If the asset with strictly positive pretax payoffs has a positive price, a solution, $\hat{w}^i$, to (A7) and (A8) with $P'\hat{w}_i < 0$ implies a solution, $\check{w}^i$, with $P'\check{w}^i = 0$, and the latter will also solve P1*. Q.E.D.

Proof of Theorem 5: Given Lemma 3, the arguments leading to the proof of Theorem 3 can be applied directly to show that no-tax-arbitrage prices ensure the existence of equilibrium prices. Q.E.D.

REFERENCES

1. G. Constantinides. "Capital Market Equilibrium with Personal Tax." *Econometrica* 51 (May 1983), 611–36.

2. ——— and J. Ingersoll. "Optimal Bond Trading with Personal Taxes." *Journal of Financial Economics* 13 (September 1984), 299–335.
3. R. Dammon. "A Security Market and Capital Structure Equilibrium under Uncertainty with Progressive Personal Taxes." In A. Chen (ed.), *Research in Finance*, vol. 7, Greenwich, CT: JAI Press, 1987.
4. H. DeAngelo and R. Masulis. "Optimal Capital Structure under Corporate and Personal Taxation." *Journal of Financial Economics* 8 (March 1980), 3–29.
5. E. Elton and M. Gruber. "Marginal Stockholder Tax Rates and the Clientele Effect." *Review of Economics and Statistics* 52 (February 1970), 68–74.
6. J. Geanakoplos and H. Polemarchakis. "Existence, Regularity, and Constrained Suboptimality of Competitive Portfolio Allocations When the Asset Market is Incomplete." In W. Heller (ed.), *Uncertainty, Information, and Communication: Essays in Honor of Kenneth J. Arrow*, vol. 3. Cambridge, MA: Cambridge University Press, 1987.
7. R. Litzenberger and K. Ramaswamy. "The Effects of Personal Taxes and Dividends on Capital Asset Prices: Theory and Empirical Evidence." *Journal of Financial Economics* 7 (June 1979), 163–95.
8. R. Litzenberger and J. Rolfo. "An International Study of Tax Effects on Government Bonds." *Journal of Finance* 39 (March 1984), 1–22.
9. ———. "Arbitrage Pricing, Transaction Costs and Taxation of Capital Gains: A Study of Government Bonds with the Same Maturity Date." *Journal of Financial Economics* 13 (September 1984), 337–52.
10. M. Miller. "Debt and Taxes." *Journal of Finance* 32 (May 1977), 261–76.
11. ——— and M. Scholes. "Dividends and Taxes." *Journal of Financial Economics* 6 (December 1978), 333–64.
12. R. Rockafellar. *Convex Analysis*. Princeton, NJ: Princeton University Press, 1972.
13. S. Ross. "Risk, Return, and Arbitrage." In I. Friend and J. Bicksler (eds.), *Risk and Return in Finance*. Cambridge, MA: Ballinger, 1976.
14. ———. "Arbitrage and Martingales with Taxation." *Journal of Political Economy* 95 (April 1987), 371–93.
15. S. Schaefer. "Taxes and Security Market Equilibrium." In W. Sharpe and C. Cootner (eds.), *Financial Economics: Essays in Honor of Paul Cootner*. Englewood Cliffs, NJ: Prentice-Hall, 1982.