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Arbitrage, Continuous Trading, and Margin Requirements

DAVID C. HEATH and ROBERT A. JARROW*

ABSTRACT

This paper studies the impact that margin requirements have on both the existence of arbitrage opportunities and the valuation of call options. In the context of the Black-Scholes economy, margin restrictions are shown to exclude continuous-trading arbitrage opportunities and, with two additional hypotheses, still to allow the Black-Scholes call model to apply. The Black-Scholes economy consists of a continuously traded stock with a price process that follows a geometric Brownian motion and a continuously traded bond with a price process that is deterministic.

CONTINUOUS-TRADING SECURITY MARKET models constitute a significant area of continuing reeseach in financial economics. These models form the basis of the option-pricing methodology (see Merton [16] for a review) and current intertemporal asset-pricing models. (See Merton [15], Breeden [3], and Cox, Ingersoll, and Ross [5].) Given the importance of these models, it is of no surprise that there has been significant academic concern (see Dybvig [9], Cox, Ingersoll, and Ross [4, p. 341], and Ingersoll [13, pp. 13–25]) over the discovery of a continuous-trading strategy by Harrison and Kreps [10], called the *doubling* strategy, that generates a sure dollar from a zero investment. The existence of this doubling strategy violates a basic premise of the option-pricing methodology, the premise of no arbitrage opportunities. Given this violation, the question naturally arises as to whether the option-pricing methodology is internally consistent. Fortunately, Harrison and Pliska [11] showed that the exclusion of arbitrage opportunities (and the doubling strategy) does not invalidate the familiar derivation of the Black-Scholes call option formula.

The exogenous exclusion of these “doubling” strategies is somewhat arbitrary, however. Types of constraints that exclude continuous-trading arbitrage opportunities were studied by Kreps [14], Harrison and Kreps [10], and Dybvig [9]. These papers provide constraints that are unsatisfactory from an institutional point of view since they have no market analogue. Margin requirements, on the other hand, have a market analogue. One purpose of this paper is to show that margin requirements also exclude continuous-trading arbitrage opportunities. To make this argument self-contained, we first provide a unified framework that synthesizes the previously stated results. Our assertion concerning margin re-

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quirements then follows as a simple corollary. To concentrate on the concepts, rather than the technical details, we focus our attention on the well-known Black-Scholes [2] economy.

Unfortunately, margin requirements severely restrict the class of continuous-trading strategies available to investors. In fact, they exclude the standard hedging portfolio used to derive the Black-Scholes option model. Nonetheless, using a dominance argument in the context of the above model, we prove that the Black-Scholes formula still correctly prices calls. This is a comforting result since margin requirements exist, yet the Black-Scholes model provides reasonable approximations to actual market prices. (See Whaley [18] and Rubinstein [17].) Our derivation is consistent with this empirical evidence and further supports the robustness of the Black-Scholes model.

In summary, the following results will be derived below:

1. A demonstration that margin requirements exclude arbitrage opportunities.
2. A proof that the Black-Scholes call option model still applies under margin requirements.

As stated earlier, the analysis in this paper is in the context of the simple Black-Scholes economy. None of the preceding conclusions, however, depends on this limited scenario. All of the stated conclusions are robust to generalizations in both the number of securities trading and the types of stochastic processes. For example, the above results can be extended to the more complicated economy of Harrison and Pliska [11].

An outline of this paper is as follows. Section I briefly reviews the derivation of the Black-Scholes model using the probabilistic approach of Harrison and Kreps [10]. Section II presents the analysis of trading restrictions, Section III derives the Black-Scholes model under margin requirements, and Section IV completes the paper.

I. The Model

Following the publication of Harrison and Kreps' [10] seminal paper, financial economists have been concerned about the internal consistency of the original derivations of the Black-Scholes call option model; see Dybvig [9] and Ingersoll [13, pp. 13–25]. The concern revolves around the existence of a continuous-trading strategy (called the doubling strategy), which generates a sure dollar from a zero investment. This doubling strategy is the continuous-trading version of the familiar doubling strategy used to win a dollar for sure by betting on the flip of a fair coin.¹ Since the Black-Scholes call option model's derivation requires

¹ To win \$1 for sure by betting on the flip of a fair coin, proceed as follows.

- (i) Bet \$1 on heads at toss number 1. If win, stop. Otherwise, bet \$2 on heads at toss number 2.
- (ii) At toss number n , if all previous bets are lost, bet $\$(2)^{n-1}$ on heads. If win, stop. Winnings will cover previous losses plus \$1. If lose, bet $\$(2)^n$ on head in next flip.

It is easy to see that $\text{Prob}(\text{winning } \$1) =$

$$1 - \text{Prob}(\text{lose all bets}) = 1 - \lim_{n \rightarrow \infty} (1/2)^n = 1.$$

The details of the continuous-trading strategy version can be found in Dybvig [9].

both continuous trading and the absence of arbitrage opportunities (see Black-Scholes [2] or Merton [16]), the question arises as to whether the model is internally consistent. The purpose of this section is to review the argument of Harrison and Pliska [11], which shows that the original derivation is sound. This argument is reviewed because it provides the appropriate framework and motivation for studying margin requirements.

The Black-Scholes economy consists of two traded assets, a stock and a riskless bond. The markets are characterized as frictionless with continuous trading.

(A1) The stock and bond trade continuously in frictionless markets.

By frictionless, we mean that there are no transactions costs, no short-sale restrictions, and no taxes and that asset shares are infinitely divisible.

Let $\{W(t): t \geq 0\}$ be a Brownian motion defined on the probability space $(\Omega, \mathcal{F}, Q)$, where Ω is the state space, $\mathcal{F}$ is the collection of events, Q is a probability measure, and $(\mathcal{F}_t: t \geq 0)$ are the information sets revealed by $W(t)$ prior to and including time t .²

The stock price $\{S(t): t \geq 0\}$ is described by geometric Brownian motion and has no dividends (cash flows) over $[0, T]$.

$$(A2) \quad dS(t) = [\mu dt + \sigma dW(t)]S(t) \quad \text{for } t \geq 0, \quad (1)$$

where $\mu, \sigma > 0$, $S(0) > 0$ are constants and $S(t)$ has no cash flows over $[0, T]$.

The bond's price is assumed to satisfy:

$$(A3) \quad dB(t) = rB(t) dt \quad \text{for } t \geq 0, \quad (2)$$

where $r > 0$ is constant and $B(0) = 1$.

By this assumption, there is a constant risk-free rate r .

The final assumption concerns the meaning and nonexistence of arbitrage opportunities in the above economy. We define the class of *self-financing trading strategies* to be the stochastic processes $\{N_1(t), N_2(t): t \in [0, T]\}$ such that

- (i) $N_1(t), N_2(t)$ are adapted to $(\mathcal{F}_t: t \geq 0)$,³
- (ii) $\int_0^T N_1(t)^2 S(t)^2 dt < +\infty$, $\int_0^T N_2(t)^2 dt < +\infty$ a.e. Q , and
- (iii) if $V(t) = N_1(t)S(t) + N_2(t)B(t)$ represents the value of the portfolio at time $t \in [0, T]$, then

$$V(t) = V(0) + \int_0^t N_1(y) dS(y) + \int_0^t N_2(y) dB(y), \quad (3)$$

where $N_1(t)$ (respectively $N_2(t)$) denote the number of shares of the stock (respectively bond) in the portfolio at time t .

Conditions (i) and (ii) are technical in nature. The economic content of condition (i) is that, at time t , the trading strategy $N_1(t), N_2(t)$ should depend

² Formally, $\mathcal{F}$ is a σ -algebra and $(\mathcal{F}_t: t \geq 0)$ is the Brownian filtration, which is right continuous and increasing, such that $\bigcup_{t \geq 0} \mathcal{F}_t = \mathcal{F}$, where $\mathcal{F}_0$ is the trivial σ -algebra augmented to include all subsets of Q null sets. See Durrett [8, p. 19] for the relevant definitions.

³ For a definition of an adapted process, see Durrett [8, p. 49].

only on information available prior to and at time t . Condition (ii) has little apparent economic content. It is imposed in order to ensure that the stochastic integrals in expression (3) are well defined. Finally, condition (iii) is a *self-financing* condition. It states that the portfolio $V(t)$, once constructed at time 0, has no cash inflows or outflows over $(0, T)$. Any changes in the composition of the portfolio are financed internally, i.e., by buying/selling the stock and bond. This restriction is without loss of generality since any cash flow requirement due to the stock can be financed/invested in the bond and held until time T . For an elaboration of this self-financing condition, see Harrison and Pliska [11, p. 237]. We now state our fourth assumption.

(A4) An investor can only employ self-financing trading strategies as defined in conditions (i) to (iii) above.

This assumption is consistent with assumption (A1) and implicitly employed by Black and Scholes [2] and Merton [16] in their original derivations.

An *arbitrage opportunity* over $[0, T]$ is defined to be a self-financing trading strategy $\{N_1, N_2: t \in [0, T]\}$ such that:

$V(T) \geq 0$ with probability one,

$V(T) > 0$ with strictly positive probability, and

$V(0) \leq 0$, where

$$V(t) = N_1(t)S(t) + N_2(t)B(t) \quad \text{for } t \in [0, T]. \quad (4)$$

This definition corresponds to the standard interpretation of an arbitrage opportunity. It represents a trading strategy requiring no initial cash outflow, no cash flows over $(0, T)$, and a positive probability of a positive cash inflow at time T but zero probability of a cash outflow.

The following lemma is the fundamental result in the probabilistic approach to pricing contingent claims. It is not stated in its most general form; nonetheless, the lemma as stated is sufficient for our needs. Expectation with respect to the probability Q is denoted by $E(\cdot)$.

LEMMA: *Given assumptions (A1) to (A4), consider the class of self-financing trading strategies $\{(N_1(t), N_2(t)): t \in [0, T]\}$ such that $E(V(T)^2) < +\infty$, where $V(t) = N_1(t)S(t) + N_2(t)B(t)$ for $t \in [0, T]$.*

There are no arbitrage opportunities in this restricted class of strategies if and only if $\{V(t)/B(t)\}$ is a $\tilde{Q}$ -martingale, where $\tilde{Q}$ is a probability defined on (Ω, F) by

$$d\tilde{Q}/dQ = \exp\{-(\mu - r)W(T)/\sigma - (\mu - r)^2 T/2\sigma^2\} > 0. \quad (5)$$

Proof: See the appendix. Q.E.D.

Condition (5), although technical in nature, implies that the probability Q assigns positive probability to an event if and only if the probability $\tilde{Q}$ does.⁴

⁴By definition, condition (5) states that $\tilde{Q}(A) = \int_A \exp\{-(\mu - r)W(T)/\sigma - (\mu - r)^2 T/2\sigma^2\} dQ$, where $A \in F$.

Hence, an arbitrage opportunity with respect to Q is an arbitrage opportunity with respect to $\tilde{Q}$, and vice versa. We will denote expectation with respect to $\tilde{Q}$ by $\tilde{E}(\cdot)$.

An interpretation of this lemma corresponds to the risk-neutrality argument utilized by Cox and Ross [6]. Consider an economy identical to that developed in this paper with the exception that it is populated by identical risk-neutral investors whose probability beliefs are $\tilde{Q}$ instead of Q . In this economy, it is easy to see that there are no arbitrage opportunities if and only if any portfolio is valued according to its discounted expected value, i.e., $V(t) = \tilde{E}(V(T) | F_t)B(t)/B(T)$. However, this is true if and only if $V(t)/B(t)$ is a $\tilde{Q}$ -martingale. The lemma, therefore, gives a formalization of the risk-neutrality argument and identifies the risk-neutral probability $\tilde{Q}$.

The above lemma provides the key step in the derivation of the Black-Scholes call option model. Although in the above economy the call option does not trade, we can nonetheless construct a synthetic call utilizing a self-financing trading strategy. We value this trading strategy using the lemma. Consider the payoff to a European call option on the stock with an exercise price of K and a maturity date T . The cash flow to the call at time T is

$$\max(S(T) - K, 0). \quad (6)$$

By assumption (A1), $E(\max(S(T) - K, 0))^2 < +\infty$. To apply the lemma, we need to guarantee the existence of a self-financing trading strategy $\{N_1(t)^*, N_2(t)^*: t \in [0, T]\}$ such that $V^*(T) = \max(S(T) - K, 0)$. We can do so by using either Merton's [16] original argument or by applying the martingale representation theorem.⁵ Given the assumption of no arbitrage opportunities, the lemma implies that $V^*(T)/B(T)$ is a $\tilde{Q}$ -martingale. This means that

$$V^*(0) = \tilde{E}(V^*(T)/B(T)) = \tilde{E}(\max(S(T) - K, 0)/B(T)), \quad (7)$$

where expression (7) is the Black-Scholes formula.⁶

Expression (7) completes the derivation of the Black-Scholes formula⁷ under the assumptions (A1) to (A4) and the assumption of no arbitrage opportunities. These assumptions are consistent; in fact, the nonexistence of arbitrage opportunities is equivalent to restricting the class of trading strategies in assumption (A4) to a subset. Indeed, under the probability $\tilde{Q}$, the relative price $V(t)/B(t)$

⁵ See the proof of the lemma for the statement of this theorem.

⁶ To prove this assertion, define

$$\tilde{W}(t) = W(t) + (\mu - r)t/\sigma. \quad (A)$$

By Levy's theorem (Durrett [8, p. 75]), $\tilde{W}(t)$ is a Brownian motion with respect to $\tilde{Q}$. Substitution of (A) into (1) yields

$$dS(t) = [rdt + \sigma d\tilde{W}(t)]S(t).$$

Consider $\log S(t)$, by Itô's Lemma,

$$S(t) = S(0)\exp\{(r - \sigma^2/2)t + \sigma\tilde{W}(t)\}. \quad (B)$$

Substitution of (B) into (7) gives the Black-Scholes formula.

⁷ To prove that a traded call option must satisfy expression (7), one needs to expand the previous economy to allow the call to trade. This expansion is discussed in Section III below.

can be written as a stochastic integral with no drift term;⁸ that is, $V(t)/B(t)$ is what is called a *local martingale* with respect to $\tilde{Q}$. For a local martingale, $\sup\{t \in [0, T]: \tilde{E}[|V(t)/B(t)|] = +\infty\}$ is possible, while, for a martingale, $\sup\{t \in [0, T]: \tilde{E}[|V(t)/B(t)|] < +\infty\}$ must be true. The assumption of no arbitrage opportunities excludes from consideration all those trading strategies that are not $\tilde{Q}$ -martingales. This excluded set contains, among others, the Harrison and Kreps [10] doubling strategy. The trading strategies remaining have been called the class of *admissible trading* strategies by Harrison and Pliska [11].

There is no logical flaw in restricting consideration to the class of strategies that exclude arbitrage opportunities. This exclusion is consistent with the notion of a well-functioning capital market. The next section of the paper explores other assumptions, institutional in nature, that would imply that all feasible trading strategies are admissible.

Before this analysis, however, we need to record an observation that is needed in Section III below. In the above derivation of the Black-Scholes formula (expression (7)), the duplicating strategy was not explicitly characterized. This characterization can be obtained by applying Itô's Lemma to expression (7). This process yields that the duplicating strategy $\{N_1^*(t), N_2^*(t): t \in [0, T]\}$ satisfies

$$N_1^*(t) = \Phi(d), \quad N_2^*(t) = -K\Phi(d - \sigma\sqrt{T}), \quad (8)$$

where $\Phi(x)$ is the cumulative normal distribution function and

$$d = [\log(S(t)/K) - (r - \sigma^2/2)T]/\sigma\sqrt{T}.$$

The number of shares in the stock at time t is $N_1^*(t)$, and the number of shares in the bond at time t is $N_2^*(t)$. This strategy is admissible and self-financing.

II. Market Constraints on Trading Strategies

In the previous section of this paper, we reviewed the original derivation of the Black-Scholes model. In this derivation, we showed that the assumption of no arbitrage opportunities corresponds to the exclusion of those strategies that are local martingales but not martingales with respect to $\tilde{Q}$. The restriction was exogenously imposed. From an economic perspective, it is important to understand (given continuous trading) what types of trading constraints⁹ could be imposed to exclude these strategies and guarantee that no arbitrage opportunities

⁸ To prove this, note that $V(t)/B(t) = N_1(t)S(t)/B(t) + N_2(t)$, so $d(V(t)/B(t)) = N_1(t)d[S(t)/B(t)]$. By Itô's Lemma and footnote 6,

$$d(V(t)/B(t)) = [\sigma S(t)/B(t)]d\tilde{W}(t).$$

Substitution yields

$$d(V(t)/B(t)) = [\sigma N_1(t)S(t)/B(t)] d\tilde{W}(t).$$

By the definition of a stochastic integral, this is a local martingale with respect to $\tilde{Q}$. A local martingale is a stochastic process that, when stopped (by a sequence of stopping times approaching T), becomes a martingale; see Durrett [8, p. 50].

⁹ We restrict our attention to market constraints (i.e., relaxations of assumption (A1)) with respect to trading strategies. We do not consider taxes or transactions costs in any detail. It can be shown, however, that fixed and variable transactions costs will not alter the substance of our analysis and conclusions.

exist. The purpose of this section is to review those constraints already studied in the literature and to show that margin requirements are a subset of the positive wealth constraints studied by Dybvig [9] and, therefore, exclude arbitrage opportunities. We pay special attention to determining whether the Black-Scholes formula remains valid under the differing trading constraints.

The first constraint (proposed by Kreps [14]) that excludes arbitrage opportunities is a uniform bound on the maximum number of shares held either short or long. Although a natural candidate for the bound would be the total number of shares outstanding, as pointed out by Kreps [14], this bound does not incorporate existing short sales and it is inconsistent with asset shares being infinitely divisible. Under this constraint, however, the Black-Scholes formula still applies since, by condition (8), the duplicating strategy $\{N_1^*, N_2^*\}$ is not excluded if the uniform bound exceeds the exercise price K .

A second type of constraint, considered by Harrison and Kreps [10], is to allow only trading strategies that can be altered a finite number of times over $[0, T]$, where the adjustment dates must be predetermined at time 0. These simple trading strategies do not include any arbitrage opportunities; however, they have no actual market counterpart, and the call option duplicating strategy (condition (8)) does not lie in this class. Unless additional structure is added (e.g., convex and suitably continuous preferences), the Black-Scholes formula need not hold.

Dybvig [9] investigated positive wealth constraints. His constraint (slightly generalized) requires that $\{N_1(t), N_2(t): t \in [0, T]\}$ must satisfy

$$V(t) \geq -LB(t) \text{ a.e. for all } t \in [0, T], \quad (9)$$

where $V(t) = N_1(t)S(t) + N_2(t)B(t)$ and $L \geq 0$ is a constant. The value of wealth in the portfolio at any time t , $V(t)$, must exceed some constant (which changes by the risk-free rate). A natural value for L is 0, and, under this restriction, condition (9) is a positive wealth constraint. The proof that this constraint excludes arbitrage opportunities is contained in the Appendix.

Condition (9), to be effective, must be applied to the investor's *entire* portfolio. To duplicate a call option, therefore, an investor must consider the *addition* of the duplicating strategy (8) to his or her existing portfolio and guarantee that the wealth constraint (9) is not violated. This is possible, and the details of this argument are postponed until the next section, where it is discussed again for a special case of condition (9).

The positive wealth constraint has a clear economic interpretation, but more stringent constraints appear to be imposed on actual security markets. These tighter constraints take the form of margin requirements, which we now characterize. Consider those trading strategies $\{N_1(t), N_2(t): t \in [0, T]\}$ that satisfy

$$\max(|N_1(t)S(t)|, |N_2(t)B(t)|) \leq a(V(t) + LB(t)) \quad (10)$$

for all $t \in [0, T]$, where $a \geq 1$ and $L \geq 0$ are constants. We call these strategies *margin-restricted trading strategies*. A slight generalization of condition (10) can be obtained by letting both a and L be bounded by deterministic functions of time. We use this slight generalization of condition (10) below.

First, since the left-hand side of expression (10) is non-negative, condition (10) implies that $V(t) \geq -LB(t)$ for all t . Hence, these strategies are a subset of the

positive wealth constraint strategies and therefore exclude arbitrage opportunities.

The case of $L = 0$ and $a \geq 2$ is the most relevant for our purposes. We restrict our attention to these parameter values unless otherwise noted. In this case, $V(t) \geq 0$ for all $t \in [0, T]$, and the portfolio must have positive value.

To interpret expression (10) as a margin requirement, we consider four cases. First, if one purchases only bonds, (10) implies no restriction. Similarly, if one purchases only stock, (10) implies no restriction. Next, consider buying a stock and borrowing $\beta > 0$ percent of the stock's value. This is buying the stock on β percent margin. Restriction (10) implies that

$$aV(t) = a(1 - \beta)S(t) \geq \max(S(t), \beta S(t)) \geq 0.$$

Simplification generates the following constraint

$$1 > (a - 1)/a \geq \beta.$$

The total borrowings are restricted by the ratio $(a - 1)/a$. The constraint holds for all $t \in [0, T]$, and, so, it requires that the margin is marked to market movements, i.e., updated continuously. If $a = 2$, this corresponds to a fifty percent margin requirement. If $a = 4$, this corresponds to a twenty-five percent margin requirement. This constraint (with $a(0) = 2$ and $a(t) = 4$ for $t \in (0, T]$) corresponds to actual margin requirements; see Cox and Rubinstein [7, p. 99].

Next, consider shorting the stock and maintaining a margin account of $\beta > 0$ percent of the stock's value in the bond. Restriction (10) implies that

$$aV(t) = a(-1 + \beta)S(t) \geq \max(S(t), \beta S(t)) \geq 0.$$

Simplification generates

$$\beta \geq a/(a - 1) > 1.$$

The margin account is restricted to be greater than $a/(a - 1)$ percent of the stock's value. If $a = 3$, this corresponds to putting the proceeds from the sale of the stock in bonds plus an additional fifty percent of the stock's value. If $a = 1.3/0.3$, then this corresponds to putting the proceeds from the sale in the margin account, plus thirty percent additional margin. This constraint (with $a(0) = 3$ and $a(t) = 1.3/0.3$ for $t \in (0, T]$) corresponds to the actual short-sale margin requirement, with the exception that the proceeds from the short sale do not normally earn interest; see Cox and Rubinstein [7, p. 99].

In summary, the above discussion demonstrates that expression (10) with $L = 0$ and $a \geq 2$ excludes arbitrage opportunities and is a margin requirement as they are employed in actual security markets. This analysis provides a justification for the existence of margin requirements. The alternative justification for their existence, of course, is to avoid defaults on the part of individual investors. However, in a continuous-trading economy, under (A4), defaults would never occur. Rational investors, without margin constraints, would be able to generate a sure dollar from nothing (an infinite number of times). For continuous-trading models, therefore, defaults are never an issue, but the prohibition of arbitrage opportunities is. Numerous institutional constraints can be understood from this point of view.

III. Option Pricing under Margin Requirements

This section of the paper derives the value of a European call option when trading strategies are restricted by the margin requirements of condition (10) with $L = 0$ and $a \geq 2$. The pricing argument contains two steps. First, we analyze whether a call option can be constructed trading in the stock and bond, under condition (10). Unfortunately, since condition (10) with $L = 0$ and $a \geq 2$ prohibits short positions without margin, a call option cannot be constructed. However, we show that the call option can “almost” be constructed. Second, we introduce trading in the call option along with the stock and bond. Under (plausible) additional hypotheses, we prove that the call’s value must be given by the Black-Scholes formula.

To price a European call option with exercise price K and maturity T , we attempt to construct a portfolio that duplicates the cash flow $\max(S(T) - K, 0)$ at time T .

First, consider the Black-Scholes strategy $\{N_1^*(t), N_2^*(t) : t \in [0, T]\}$ represented by expression (8). This is a long position in the “call”. It is easy to construct an example in which this strategy violates the margin restriction.¹⁰ The reason is that, for stocks that are near the money, the call’s value is low, but $|N_2^*(t)B(t)|$ is some substantial fraction of $KB(t)$.

Consider now what happens when we add $\varepsilon > 0$ additional bonds as margin to this portfolio. The margined portfolio is then $\{N_1^*(t), N_2^*(t) + \varepsilon : t \in [0, T]\}$.

Define $V^*(t) = N_1^*(t)S(t) + N_2^*(t)B(t)$ and

$$V(t) = V^*(t) + \varepsilon B(t).$$

The value $V^*(t)$ is the Black-Scholes formula, expression (7). We shall use the following facts without additional reference in the subsequent analysis: (i) $0 \leq N_1^*(t) \leq 1$, (ii) $-K \leq N_2^*(t) \leq 0$, and (iii) $\max(S(t) - KB(t), 0) \leq V^*(t) \leq S(t)$. These facts are well-known implications of the Black-Scholes formula.

Consider the following sequence of inequalities:

$$\begin{aligned} 0 &\leq N_1^*(t)S(t)/V(t) \leq S(t)/V(t) \\ &\leq [V^*(t) + KB(t)]/V(t) \leq 1 + (K - \varepsilon)B(t)/V(t) \\ &\leq 1 + (K - \varepsilon)B(t)/\varepsilon B(t) \leq K/\varepsilon. \end{aligned}$$

Similarly,

$$|N_2^*(t) + \varepsilon|B(t)/V(t) \leq (K + \varepsilon)B(t)/(V^*(t) + \varepsilon B(t)) \leq 1 + K/\varepsilon.$$

Together, these imply that

$$\max(|N_1^*(t)S(t)|, |N_2^*(t) + \varepsilon|B(t)) \leq (1 + K/\varepsilon)V(t). \quad (11)$$

Thus, if the additional margin is chosen such that $1 + K/\varepsilon \leq a$, the margin requirement (10) is satisfied. A long position in the call can therefore be

¹⁰ For $a = 7$, the following typical parameters violate the margin requirement: $S(0) = \$40$, $K = \$40$, $r = 0.12$, $T = 0.33$ years, $\sigma = 0.3$. Here, the Black-Scholes value is $\$3.54$, $N_1^*(0)S(0) = \$24.95$, and $N_2^*(0)B(0) = \$21.41$, but $aV^*(t) = 7(\$3.54) = \24.78 . The value $a = 7$ was chosen to exceed the value implied by typical margin constraints in our economy.

constructed as long as at least $K/(a-1)$ additional bonds are included as margin.

To continue the argument, consider constructing a portfolio to duplicate a short position in the call. The Black-Scholes duplicating strategy (reversed) $\{-N_1^*(t), -N_2^*(t): t \in [0, T]\}$ has a nonpositive value, violating the margin constraint (10). The modified portfolio strategy $\{-N_1^*(t) + 2, -N_2^*(t) + \varepsilon: t \in [0, T]\}$, where $\varepsilon > 0$, adds a long position in two shares of the stock and $\varepsilon > 0$ bonds as additional margin. These additions imply that $V(t) = [-N_1^*(t) + 2]S(t) + [-N_2^*(t) + \varepsilon]B(t) = 2S(t) - V^*(t) + \varepsilon B(t) \geq S(t) + \varepsilon B(t) > 0$.

Consider the following pair of inequalities:

$$\begin{aligned} &|-N_2^*(t) + \varepsilon| B(t)/V(t) \leq |-N_2^*(t) + \varepsilon| B(t)/\varepsilon B(t) \leq 1 + K/\varepsilon \text{ and} \\ &|-N_1^*(t) + 2| S(t)/V(t) \leq 2S(t)/V(t) \leq 2. \end{aligned}$$

Together, these imply that

$$\max(|-N_1^*(t) + 2| S(t), |-N_2^*(t) + \varepsilon| B(t)) \leq \max(1 + K/\varepsilon, 2)V(t). \quad (12)$$

If $\varepsilon \geq K/(a-1)$ and $a \geq 2$, then a short position in the call can be constructed along with two shares long in the stock and ε additional bonds.

The above argument demonstrates three facts. First, a position in the call (either long or short) cannot be constructed in isolation of an investor's alternative portfolio holdings. Second, if an investor has at least $K/(a-1)$ bonds unconstrained by the margin requirement (10), then he or she can duplicate a long position in the call that has the Black-Scholes value. These investors, therefore, will never pay more than the Black-Scholes value to own the call. Third, if an investor has at least $K/(a-1)$ bonds and two shares long in the stock, both unconstrained by the margin requirement (10), then he or she can duplicate a short position in the call that has the Black-Scholes value. These investors, therefore, would be willing to sell the call if the price they received for this package exceeded the Black-Scholes value. Given the existence of these investors, the above logic suggests that traded calls must trade at the Black-Scholes price. To complete this line of argument, we must expand the economy to consider trading in calls.

Consider expanding the economy of Section I to allow trading in the European call option with exercise price K and maturity date T . The value of this call, $\{C(t): t \in [0, T]\}$, is a stochastic process on (Ω, F) . We impose no assumptions on the stochastic behavior of this process.

Next, given trading in calls, we augment the class of trading strategies in the simplest manner that will allow us to prove our result. Trading strategies in the stock and bond are identical to the strategies defined in Section I. Trading in the call is only allowed at time 0; hence, calls must be held until time T . Trading in the call is a buy/sell and hold strategy. Hence, $\{N_1(t), N_2(t), N_3(t): t \in [0, T]\}$ are trading strategies if $\{N_1(t), N_2(t)\}$ satisfy the conditions (i) to (iii) in Section I and $N_3(t) = N_3(0)$ for all $t \in [0, T]$, where $V(t) = N_1(t)S(t) + N_2(t)B(t) + N_3(t)C(t)$. Note that transactions in the stock and bond over time must be self-financing. An arbitrage opportunity is defined in the same way as in condition (4) of Section I, with the exception that $V(t) = N_1(t)S(t) + N_2(t)B(t) + N_3(t)C(t)$ includes the call option.

Finally, we need to augment the margin-requirement condition (10) to include calls. We use the margin requirements in existence; see Cox and Rubinstein [7, p. 104]. For a long position in the call, there is no margin. For a short position in the call, an uncovered position requires margin (in bonds) that is marked to the market. Alternatively, a long position in the stock can be used instead to cover the call. This later alternative is easier to apply since the margin (the stock) need not be explicitly adjusted for movements in the call's price. Hence, the margin constraint becomes

$$\max(|N_1(t)S(t)|, |N_2(t)B(t)|) \leq a(V(t) - N_3(t)C(t))$$

for all $t \in [0, T]$ if $N_3(t) \geq 0$, (13a)

$$\max(|N_1(t) + N_3(t)|S(t), |N_2(t)B(t)|)$$

$\leq a(V(t) - N_3(t)C(t) + N_3(t)S(t))$

for all $t \in [0, T]$ if $N_3(t) < 0$. (13b)

In essence, the margin requirement is decomposable. First, positions in the calls are considered. Next, positions in the bond and the stock are considered.

To prove the assertion that the Black-Scholes formula corresponds to the value of the call, we add two assumptions. The first of these assumptions excludes dominated portfolios from the economy. If markets are complete, this would be identical to the assumption of no arbitrage opportunities. However, in incomplete markets, it is a stronger assumption. More precisely, consider two trading strategies $\{N_1(t), N_2(t), N_3(t): t \in [0, T]\}$ and $\{\bar{N}_1(t), \bar{N}_2(t), \bar{N}_3(t): t \in [0, T]\}$, where

$$V(t) = N_1(t)S(t) + N_2(t)B(t) + N_3(t)C(t)$$

$$\bar{V}(t) = \bar{N}_1(t)S(t) + \bar{N}_2(t)B(t) + \bar{N}_3(t)C(t).$$

Portfolio $V(t)$ is said to *strictly dominate* portfolio $\bar{V}(t)$ if

- (i) $V(T) \geq \bar{V}(T)$ with probability one,
- (ii) $V(T) > \bar{V}(T)$ with strictly positive probability, and
- (iii) $V(0) \leq \bar{V}(0)$. (14)

This is similar to the definition of an arbitrage opportunity, with the exception that two nontrivial portfolios are compared, instead of a nontrivial portfolio to the trivial (or zero) portfolio.

(A5) There are no strictly dominated portfolios in the economy.

The second assumption formalizes the argument given before:

- (A6) There is at least one investor in the economy who has two shares of stock and $K/(a - 1)$ bonds unconstrained by the margin-requirement condition (13).

The investor could be a mutual fund or professional institution. We now state the proposition.

Under assumptions (A1) to (A6), given the margin requirement (13), the Black-Scholes formula holds; i.e., $C(0) = V^*(0)$, where $\{N_1^*(t), N_2^*(t): t \in [0, T]\}$ is the Black-Scholes trading strategy, expression (8), and $V^*(0)$ satisfies condition (7).

The proof of this proposition is by contradiction. It consists of two steps. First, suppose that $C(0) > V^*(0)$. To take advantage of this situation, the investor in assumption (A6) forms the portfolio:

$$\{(N_1^*(t) + 1), (N_2^*(t) + \varepsilon), -1: t \in [0, T]\} \quad \text{for } \varepsilon = K/(a - 1).$$

This is long the Black-Scholes strategy (plus margin) and short the call (plus “margined” stock). By condition (11), it satisfies the margin requirement (13b). The portfolio strictly dominates $\{1, +\varepsilon, 0: t \in [0, T]\}$, which contradicts assumption (A5).

Next, suppose that $C(0) < V^*(0)$. To take advantage of this situation, the investor of assumption (A6) forms the following portfolio:

$$\{-N_1^*(t) + 2, -N_2^*(t) + \varepsilon, 1: t \in [0, T]\} \quad \text{for } \varepsilon = K/(a - 1).$$

This is short the Black-Scholes strategy (plus margin) and long the call. By condition (12), it satisfies the margin requirement (13a). The portfolio strictly dominates $\{2, \varepsilon, 0: t \in [0, T]\}$, contradicting assumption (A5). This completes the proof of the proposition.

In fact, assumption (A6) can be weakened even further. Under the frictionless market assumption (A1), asset shares are infinitely divisible. The investor in our proof, therefore, need not buy/short an entire call option, but merely some fractional part of one, call it $\alpha \in (0, 1)$. As long as the investor has some small but nonzero dollar amount of both stocks and bonds unrestricted by the margin-requirement condition (13), then the argument still follows (for some suitably chosen α).

IV. Conclusion

This paper studied the relationship between arbitrage opportunities and continuous-trading strategies. In this analysis, the following results were obtained:

1. A demonstration that margin requirements exclude arbitrage opportunities.
2. A proof that the Black-Scholes call option model still applies under margin requirements.

These results were derived in the Black-Scholes economy. This simplification was employed to facilitate the presentation of the concepts. The preceding results all generalize in the more complicated economy of Harrison and Pliska [11].

Appendix

Proof of the Lemma:

Step 1: Suppose that, for every strategy, $V(t)/B(t)$ is a $\tilde{Q}$ -martingale. Then $V(0) = \tilde{\mathbb{E}}(V(T)/B(T)) < +\infty$. Consider a strategy such that $Q(V(T) \geq 0) = 1$, $Q(V(T) > 0) > 0$. This implies that $\tilde{Q}(V(T) \geq 0) = 1$, $\tilde{Q}(V(T) > 0) > 0$, by (5).

Hence, $\tilde{E}(V(T)/B(T)) > V(0) > 0$. There are no arbitrage opportunities. Note that, for this step in the proof, we do not use the fact that $E(V(T)^2) < +\infty$.

Step 2: This is proved by contradiction. Suppose that there are no arbitrage opportunities in this class of strategies, yet there is a strategy $\{N_1, N_2\}$ such that $V(t)/B(t)$ is not a $\tilde{Q}$ -martingale. This implies that $\tilde{E}(V(T)/B(T)) \neq V(0)/B(0)$. Without loss of generality, let $\tilde{E}(V(T)/B(T)) < V(0)/B(0)$.

Define $Y_t \equiv \tilde{E}(V(T)/B(T)|F_t)$. Given that $\tilde{E}(|V(T)/B(T)|) \leq E((V(T)/B(T))^2)^{1/2} E((d\tilde{Q}/dQ)^2)^{1/2} < +\infty$, Y_t is a martingale. It is continuous since F_t is a Brownian filtration; see Durrett [8, p. 86]. By the martingale-representation theorem (Ikeda and Watanabe [12, p. 80]), there exists a predictable process $\{M(t): t \in [0, T]\}$ such that

$$Y_t = Y_0 + \int_0^t M d\tilde{W}(t),$$

where

$$\int_0^T M^2 dt < +\infty \text{ a.e. } \tilde{Q}.$$

Next, we find a trading strategy that duplicates Y_t , i.e., find $\{\bar{N}_1, \bar{N}_2: t \in [0, T]\}$ such that $\bar{V}(t)/B(t) = Y_t$, where $\bar{V}(t) = \bar{N}_1(t)S(t) + \bar{N}_2(t)B(t)$ and $d\bar{V}(t) = \bar{N}_1(t)dS(t) + \bar{N}_2(t)dB(t)$.

Define $\bar{N}_1(t) = B(t)M(t)/\sigma S(t)$ and $\bar{N}_2(t) = Y_t - M(t)/\sigma$. This strategy has $\bar{V}(t)/B(t) = Y_t$; hence, $\bar{V}(t)/B(t)$ is a $\tilde{Q}$ -martingale.

Finally, we now show that there exists an arbitrage opportunity.

Consider the strategy:

$$N_1^*(t) = \bar{N}_1(t) - N_1(t)$$

$$N_2^*(t) = \bar{N}_2(t) - N_2(t) - (\tilde{E}(Y_T) - V(0)).$$

By construction,

$$\begin{aligned} V^*(0) &= N_1^*(0)S(0) + N_2^*(0)B(0) \\ &= (\tilde{E}(Y_T) - V(0)) - (\tilde{E}(Y_T) - V(0)) = 0. \end{aligned}$$

There are no cash flows over $(0, T)$, and, at time T ,

$$N_1^*(T)S(T) + N_2^*(T)B(T) = -(\tilde{E}(Y_T) - V(0))B(T) > 0.$$

This is an arbitrage opportunity—a contradiction. Q.E.D.

Proof that the Wealth Constraint (9) Excludes Continuous-Trading Arbitrage Opportunities

By footnote 8,

$$V(t)/B(t) + L = \int_0^t (\sigma N_1(t)S(t)/B(t)) d\tilde{W}(t) + V(0) + L.$$

Define the stopping times $\tau_n = \inf\{t \in [0, T]: \tilde{\mathbb{E}}(\int_0^t \sigma^2 N_1^2 S(t)^2/B(t)^2 dt) \geq n\}$. Note that, as $n \rightarrow \infty$, $\tau_n \rightarrow T$. By definition of the stochastic integral, $V(t\wedge\tau_n)/B(t\wedge\tau_n) + L$ is a martingale, where $(t\wedge\tau_n) \equiv \inf(t, \tau_n)$, so $\tilde{\mathbb{E}}(V(t\wedge\tau_n)/B(t\wedge\tau_n) + L) = V(0) + L$ for all $t \in [0, T]$. As $n \rightarrow \infty$, $V(t\wedge\tau_n)/B(t\wedge\tau_n) \rightarrow V(t)/B(t)$ a.e. $\tilde{\mathbb{Q}}$. By Fatou's Lemma (Bartle [1, p. 49]), since, by constraint (10), $V(t)/B(t) + L \geq 0$ a.e. for all $t \in [0, T]$, $\tilde{\mathbb{E}}(|V(t)/B(t) + L|) = \tilde{\mathbb{E}}(\lim_{n \rightarrow \infty} |V(t\wedge\tau_n)/B(t\wedge\tau_n) + L|) \leq \liminf_{n \rightarrow \infty} \tilde{\mathbb{E}}(V(t\wedge\tau_n)/B(t\wedge\tau_n)) + L = V(0) + L$.

Hence, $\tilde{\mathbb{E}}(|V(t)/B(t)|) \leq \tilde{\mathbb{E}}(|V(t)/B(t) + L|) + |L| \leq V(0) + 2L$ for all $t \in [0, T]$. So, $\sup_{t \in [0, T]} \tilde{\mathbb{E}}(|V(t)/B(t)|) < +\infty$; $V(t)/B(t)$ is a $\tilde{\mathbb{Q}}$ -martingale. $\mathbb{Q.E.D.}$

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