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A Model of Intertemporal Discount Rates in the Presence of Real and Inflationary Autocorrelations

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ABSTRACT

This paper discusses the pricing of assets in an intertemporal rational-expectations model when real production and inflation evolve according to first-order autocorrelated processes. The focus is on the structure of the various intertemporal discount rates (yields) exhibited by this economy. Yield curves are identified for consumption claims, indexed bonds, and nominally riskless bonds and can be extended to any claim that can be approximated by a (finite) linear combination of such securities. The model demonstrates that, if the average term structure for nominally riskless securities is upward sloping, then the yield curve for consumption (market) claims is downward sloping, suggesting that conventional methods for computing long-term discount rates err by not accounting for maturity factors. The paper also explores the relationship between the intertemporal equilibrium and its embedded single-period equilibria. The single-period risk measures in this economy are derived and shown to be (generally) functions of maturity. A model of nominal bond betas is constructed along these lines. It is shown that bond betas that are increasing functions of maturity do not necessarily imply an upward-sloping term structure.

THIS PAPER HAS TWO objectives. The first is to utilize the rational-expectations model to obtain a general perspective on pricing, rather than restricting its focus only to the term structure of nominally riskless assets. The second objective is to investigate the implications of autocorrelations in the basic processes of the economy. In a recent working paper, Fama and French [11] report evidence of temporary components for portfolio returns that resemble negative autocorrelation. One interpretation of this phenomenon is market inefficiency, but Fama and French are careful to note that autocorrelations may not be inconsistent with more sophisticated efficient-market models such as that of Cox, Ingersoll, and Ross [6].

This paper models a rational-expectations economy in which production and inflation are generated by first-order autocorrelated processes. By construction, the maket is efficient even though return series display some level of autocorrelation. The hypothesized processes are sufficiently rich so as to generate a wide variety of yield curves for nominally riskless bonds, indexed bonds, and market claims. The relationship of these yield curves is then examined. It is shown that the (average) term structures of market yields and (zero-coupon) nominal bonds exhibit a symmetry about a horizontal line. The negative autocorrelation in

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nominal market returns observed by Fama and French would imply that market yields are downward sloping (out to about three to five years). The model then implies that the nominal term structure should be upward sloping out to that point. This must be verified directly from the term structure because Section III demonstrates that holding-period term premia may not exhibit a (monotonically) related structure.

The model identifies two factors that can lead to an upward-sloping (average) term structure: negative autocorrelation of real returns and/or negative real-inflation covariance. This result is to be contrasted with that of Benninga and Protopapadakis [2], who find that an upward-sloping (real) term structure can be explained by the presence of concave production technologies. These technologies must, however, form a complete set of state-specific production (output in one state) and exhibit decreasing returns to scale. They also show that negative autocorrelation in the real series will lead to an upward-sloping real term structure, but, because their model does not allow for inflation, it applies only to indexed bonds and cannot be translated directly into nominal implications. The model presented here incorporates the effect of inflation in a nonstationary sense and provides the nominal yield curve for all maturities. (Benninga and Protopapadakis consider only a three-date model.) In contrast to Cox, Ingersoll, and Ross [6, 7], this paper gives explicit attention to the effect of autocorrelations in the basic processes of the economy and their implications for the pricing of alternative securities.

A number of related aspects are also considered in this paper. The form of inflation adjustments in this economy is studied. This adjustment is not the Fisher Effect, except for indexed bonds. Benninga and Protopapadakis [1] have indicated that the Fisher Effect cannot hold in general for nominally riskless bonds because of risk adjustments related to inflation. The model of this paper provides a specific equilibrium example of this phenomenon and generalizes it to the other assets studied. The model demonstrates the origins of single-period risk in a multiperiod equilibrium. In particular, the betas for bonds are shown to be nonstochastic functions of maturity, with sign depending on the nature of the basic autocorrelations, and positive bond betas do not always imply an upward-sloping yield-to-maturity curve.

The paper proceeds as follows: Section I decribes the discrete-time, rational-expectations model of the economy used in this paper. Section II describes the equilibrium price and yield characteristics of a wide class of assets, identifies the structural relationships among yields, and investigates the single-period equilibria embedded in the model. Section III investigates the implications of the model for the pricing of nominally riskless assets. Section IV discusses the relationship of the model to some of the recent empirical work on the term structure. Finally, Section V contains the summary of the paper.

I. The Model

This section describes the details of the economy to be examined. The model to be presented is similar to that of Rubinstein [18]. It is a single-good model of a frictionless intertemporal economy. Its specialization lies in the fact that it incorporates single-period production for which returns are determined by a real and an inflationary factor, each of which displays first-order autocorrelation.

A. Real Production

In this model, all future consumption is provided by the investment activities of firms. The production process considered is single period in nature and subject to constant returns to scale, and investment is reversible. Firms are assumed to behave competitively, and production equilibrium is characterized by value equal to investment for each producing firm at each date. These assumptions are sufficient to identify the stochastic distribution of real wealth accumulation as that of the aggregate physical rate of return.[1]

Fama and Gibbons [12] provide empirical evidence of nonstationarity in the real rate of return on T-bills. Fama and French [11] document long-run temporary components of security prices that, they argue, represent negative autocorrelations in prices. In the constant relative risk-aversion model, real interest rates are nonstochastic if the real return on the market portfolio is stationary and thus free of autocorrelation.[2] Neither would market returns be autocorrelated. To generate mean nonstationarity in the model, it is assumed that the real (aggregate) physical rate of return on capital, and therefore the rate of return on the market portfolio, r_{mt}, is autocorrelated. The process to be studied is

$$\ln(1 + r_{mt}) = \lambda_1[\ln(1 + r_{mt-1}) - \mu_\varepsilon] + \varepsilon_t, \qquad (1)$$

where ε_t is i.i.d. and

$$\varepsilon_t \sim N(\mu_\varepsilon, \sigma_\varepsilon^2).$$

The parameter λ_1 allows for the possibility of autocorrelation in sequential real rates of return to capital, and it is assumed that $|\lambda_1| < 1$ so that the process is long-run stationary. It should be remarked that (1) explicitly indicates that inflation is not relevant to the future real opportunity set. An inflationary process will be introduced, but its scope is limited to providing purchasing-power uncertainty with respect to nominal claims.

B. Consumers

To identify the endogenous asset-price distributions in this model, a representative consumer with an infinite lifetime whose utility of consumption is linear in the log of each period's real consumption, c_t, is assumed:

$$U(c_0, c_1, \cdots) = \textstyle\sum_{t=0}^{\infty} \rho^{-t} \ln c_t, \qquad (2)$$

where ρ represents the consumer's subjective time-discount factor. An important property of this utility function is that the fraction of wealth invested at each

[1] The market portfolio includes *all* assets and is not limited to equity. By virtue of the assumptions, $V_{mt} = K_{mt}$ for each t after firm investment, where V_{mt} is the value of the market portfolio at time t and K_{mt} is the contemporaneous aggregate investment. At each date, V_{mt+1}, prepayout, is equal to $K_{mt}(1 + r_{mt+1})$.

[2] This result is due to Rubinstein [18] (Theorem IV, p. 413).

date, a_t, is independent of the opportunity set and equal to ρ^{-1}, even in the presence of capital-market nonstationarity.[3]

In a representative consumer framework, the portfolio held must be the market portfolio; therefore, the intertemporal consumption schedule is given by

$$c_t = \rho^{-1} c_{t-1}(1 + r_{mt}). \tag{3}$$

This equation follows immediately from the identity:

$$c_t \equiv (1 - a_t) W_t = (1 - a_t) a_{t-1} W_{t-1}(1 + r_{mt}),$$

where a_t is the fraction of wealth invested at t, and $a = \rho^{-1}$ (independent of t) for log utility.

C. Valuation

The valuation of assets in this economy follows from the work of Rubinstein [18]. Let y_t be a real cash flow at time t. The (real) present value of this cash flow is given by

$$v_0(y_t) = \rho^{-t} \frac{\mathrm{E}_0 U'(c_t) y_t}{U'(c_0)}, \tag{4}$$

which, by the log-utility assumption, is

$$v_0(y_t) = \rho^{-t} c_0 \mathrm{E}_0(c_t^{-1} y_t) \tag{4'}$$

and

$$c_t = \rho^{-t} c_0 \prod_{s=1}^{t} (1 + r_{ms}). \tag{5}$$

In a rational-expectations equilibrium, the distribution of asset prices and, hence, rates of return is endogenously determined from the characteristics of the security cash flow and the optimal consumption schedule (derived from the investors' portfolio problem). This endogenous valuation is accomplished in equation (4').

As is evident in (4'), to calculate the value of a security, it is necessary to know the probability density associated with future (real) consumption levels. This density can be constructed from equation (5) and the specification of the market return (equation (1)). To economize on notation, define $\ln x_t = \ln(1 + r_{mt}) - \mu_\varepsilon$. Then

$$\ln c_t \sim N(\ln c_0 + t[\mu_\varepsilon - \ln \rho + m_{11}(t) \ln x_0],\ t m_{21}(t) \sigma_\varepsilon^2). \tag{6}$$

Equation (6) informs us that the log of future (real) consumption is normally distributed, conditional upon the current level of consumption and its associated growth (c_0 and x_0, respectively). Furthermore, the parameters of the distribution are determined by those of the r_{mt} process.

[3] For log utility, a_s is nonstochastic. For an infinite-lived consumer,

$$\left[\frac{(1 - a)}{a}\right]^{-1} = \rho^{-1}(1 - a)^{-1}$$

or $a = \rho^{-1}$.

The exact relationship between the parameters of real consumption and the real market process are captured in the $m_{..}$ functions. These functions are defined as

$$m_{11}(t) = \frac{\lambda_1(1 - \lambda_1^t)}{t(1 - \lambda_1)}, \tag{7}$$

$$m_{21}(t) = \frac{1}{(1 - \lambda_1)^2}\left[1 + \frac{(\lambda_1^t - 1)(2\lambda_1 - \lambda_1^2(\lambda_1^t - 1))}{t(1 - \lambda_1^2)}\right]. \tag{8}$$

In equation (6), the parameter m_{11} measures the contribution of the current deviation from the long-run mean of the consumption process, x_0, to the future mean of $\ln c_t$. For $1 > \lambda_1 > 0$, m_{11} is positive and decreasing in t (with a limit of zero). Hence, $x_0 > 1$ implies abnormal expected consumption in all future states, but decaying with time. For $-1 < \lambda_1 < 0$, m_{11} is negative and increasing in maturity, with a limit of zero. The function $m_{21}(t)$ is also positive, increasing in λ_1, and greater than one if $\lambda_1 > 0$. Thus, the effect of positive autocorrelation in the real market is to increase the variance of future (ln) consumption per period. Negative autocorrelation implies that m_{21} is positive but decreasing in t. Hence, consumption risk, per unit of time, is decreasing in maturity. The function $m_{21}(t)$ has an asymptotic value of $[1 - \lambda_1]^{-2}$, which implies an asymptote for the various term structures to be investigated.

D. Inflation

To study the pricing of nominal assets subject to purchasing-power risk, an inflation process will be incorporated into the analysis. One can argue that inflation is generally a monetary phenomenon, but the current model has no room for money. A nontrivial aspect of inflation is simply that it changes the real purchasing power of certain (nominally denominated) claims. This facet can be modeled without explicitly considering money. In particular, let I_t represent a nominal index at time t that simply describes the relationship between nominal and real consumption units at that time. By definition, real consumption, c_t, equals nominal consumption deflated by the index.

The dynamics of the index are taken as exogenous.[4] The form of this process is given in terms of the rate of change in the price index, $1 + i_t = I_t/I_{t-1}$.

$$\ln(1 + i_t) = \lambda_2[\ln(1 + i_{t-1}) - \mu_\zeta] + \zeta_t, \tag{9}$$

where

$$\zeta_t \sim N(\mu_\zeta, \sigma_\zeta^2).$$

This implies that the distribution of the future inflation index is lognormal and given by

$$\ln I_t \sim N(\ln I_0 + t[\mu_\zeta + m_{12}(t)\ln s_0], tm_{22}(t)\sigma_\zeta^2), \tag{10}$$

[4] This does not violate the equilibrium properties of the model because this process has no real implications. Calling it a monetary process is not, however, warranted because the dynamics of a monetary process, even in the absence of a real balance effect, would not be exogenous. In particular, the process would be subject to the constraint that the nominal interest rate must be non-negative.

where the $m_{..}$ are the functions previously reported, evaluated for λ_2, and $\ln s_0$ is defined as $\ln(1 + i_0) - \mu_\zeta$ (the current deviation from the long-run (in logs) mean of the inflation process). Contemporaneous correlation between market returns and inflation is represented through the relationship $\text{cov}(\varepsilon_t, \zeta_t) = \sigma_{\varepsilon\zeta}$, but noncontemporaneous correlation is assumed to be zero.[5] Finally, define

$$m_3(t) = \frac{1}{t}\sum_{j=1}^{t} \frac{(1 - \lambda_1^j)(1 - \lambda_2^j)}{(1 - \lambda_1)(1 - \lambda_2)}.$$

Then the time-zero covariance between future (ln) consumption and the future (ln) price index is given by $m_3(t)\sigma_{\varepsilon\zeta}t$.

E. Market Autocorrelation

A relevant question for the model is the implication of autocorrelations in the basic processes for the market-return process. In particular, should high autocorrelation of the inflation process imply positive autocorrelation in the market-return series given the absence of a Mundell Effect? The market return for this economy follows the process:

$$\begin{aligned}\ln(1 + R_{mt}) &= \lambda_1[\ln(1 + R_{mt-1}) - \mu_\varepsilon - \mu_\zeta] + (\lambda_2 - \lambda_1)\ln s_{t-1} + \varepsilon_t + \zeta_t \\ &= \lambda_2[\ln(1 + R_{mt-1}) - \mu_\varepsilon - \mu_\zeta] + (\lambda_1 - \lambda_2)\ln x_{t-1} + \varepsilon_t + \zeta_t. \qquad (11)\end{aligned}$$

This process is not (strictly) first-order autocorrelated unless the two autocorrelations are identical. A (structurally inaccurate) simple autoregression would produce the autocorrelation estimate:

$$\begin{aligned}\hat{\lambda} &= \lambda_1 + (\lambda_2 - \lambda_1)\left[\frac{\sigma_\zeta^2}{(1 - \lambda_2^2)} + \frac{\sigma_{\varepsilon\zeta}}{(1 - \lambda_1\lambda_2)}\right] \Big/ \text{var}(\ln(1 + R_{mt})) \\ &= \lambda_2 + (\lambda_1 - \lambda_2)\left[\frac{\sigma_\varepsilon^2}{(1 - \lambda_1^2)} + \frac{\sigma_{\varepsilon\zeta}}{(1 - \lambda_1\lambda_2)}\right] \Big/ \text{var}(\ln(1 + R_{mt})).\end{aligned}$$

Generally, the estimate will lie somewhere between λ_1 and λ_2, but negative real-inflation covariance introduces the possibility of an estimate less than λ_1 (if it is sufficient to make the quantity in brackets negative). The specifications of the model capable of mimicking Fama and French's findings of negative market autocorrelation seem to be associated with negative real autocorrelation and/or negative covariance.

II. Yields, Prices, and Equilibrium

This section investigates the intertemporal pricing mechanism inherent in the model. The first results to be presented characterize prices and yields (intertem-

[5] $\sigma_{\varepsilon\zeta} \neq 0$ is sufficient to induce a statistical relationship between expected real returns and expected inflation of the same sign. This specification is not intended to indicate a Mundell Effect, but it is in principle consistent with a weak relation of this type. In particular, Fama and Gibbons argue that it may merely reflect the particular monetary policy followed by the Federal Reserve.

poral discount rates) for "fundamental" securities. Maturity is a factor in asset price and yield because of the nonstationarity. The model exhibits a structural equation that relates the yields of securities to the observable riskless term structure. This structural equation can be interpreted as determining the appropriate yield from the yield on an equivalent maturity bond with adjustments for intertemporal consumption and inflation risk. The implications of the model for single-period pricing are then presented. In the embedded single-period equilibria, maturity, along with the type of claim, is sufficient to determine nonstochastic risk measures for each security.

A. Nominal Prices and Yields

The yield curves and prices in this economy are derived for assets with nominal cash flows that are scalar multiples of the form $\pi_t = c_t^{\alpha_1} I_t^{\alpha_2} Z_t$, where Z_t represents a noise term with mean one.[6] This form was chosen because of its relationship to several securities of interest. Specifically, nominally riskless, zero-coupon bonds would be described by $\alpha_1 = 0$, $\alpha_2 = 0$, and $Z_t = 1$ (with probability one); a (zero-coupon) bond that is riskless in real terms would be described by $\alpha_1 = 0$, $\alpha_2 = 1$, and $Z_t = 1$; and a real consumption claim (or, by virtue of the log-utility assumption, a real market claim) would be described by $\alpha_1 = 1 = \alpha_2$ and $Z_t = 1$. The set of securities evaluated in Theorem 1 is, in fact, diverse enough that any continuous function of c_t and I_t could be approximately priced.[7]

The valuation process is as follows: Given the nominal cash flow specification $\pi_t = c_t^{\alpha_1} I_t^{\alpha_2} Z_t$, the real cash flow associated with the security at time t is $c_t^{\alpha_1} I_t^{\alpha_2 - 1} Z_t$. This latter cash flow is substituted into equation (4′), in combination with the market and inflation processes ((1) and (9), respectively), to obtain the real value of the claim at $t = 0$. The resulting value is then transformed into a nominal value through multiplication by the current price level, I_0. Yields are determined by calculating the expected cash flow (either nominal or real), dividing through by the current price, and extracting the t-period accumulation factor.

The following theorem characterizes the economy's valuation process for the previously mentioned securities.

THEOREM 1: *Suppose that the nominal cash flow to a security at time t can be written in the form:*

$$\pi_t = c_t^{\alpha_1} I_t^{\alpha_2} Z_t,$$

where Z_t *is independent of* $(\varepsilon_t, \varepsilon_{t-1}, \cdots, x_0, \zeta_t, \zeta_{t-1}, \cdots, s_0)$.
(i) *Then the nominal present value of the security can be written as*

$$V_0(\pi_t) = \rho^{-t} c_0^{\alpha_1} I_0^{\alpha_2} x_0^{t(\alpha_1 - 1)m_{11}(t)} s^{t(\alpha_2 - 1)m_{22}(t)} e^{h(\alpha_1 - 1, \alpha_2 - 1, t)},$$

[6] In particular, Z_t must be independent of all prior ε and ζ. The expectation of Z_t could be stochastic and evolve over time according to information that is independent of the inflation and consumption process (hence, diversifiable). The effect of this latter specification would be to incorporate a state variable representing "firm-specific" information into the prices given in Theorem 1.

[7] Essentially any continuous function of c_t and I_t can be approximated by finite linear sums of the elements c_t^{α} and I_t^{β}. See Dunford and Schwartz [8].

where

$$h(\beta_1, \beta_2, t) = t\left[\beta_1\left(\mu_\varepsilon - \ln\rho + \frac{\beta_1 m_{21}(t)\sigma_\varepsilon^2}{2} + \beta_2 m_3(t)\sigma_{\varepsilon\zeta}\right) + \beta_2\left(\mu_\zeta + \frac{\beta_2 m_{22}(t)\sigma_\zeta^2}{2}\right)\right]$$

and the m's are the deterministic functions discussed in Section I.

(ii) *The nominal expected yield to maturity on this security is*

$$\ln(1 + Y_\pi) = \mu_\varepsilon - \frac{m_{21}(t)\sigma_\varepsilon^2}{2} - m_3(t)\sigma_{\varepsilon\zeta} + m_{11}(t)\ln x_0 + \mu_\zeta - \frac{m_{22}(t)\sigma_\zeta^2}{2} + m_{12}(t)\ln s_0 + \alpha_1[m_{21}(t)\sigma_\varepsilon^2 + m_3(t)\sigma_{\varepsilon\zeta}] + \alpha_2[m_{22}(t)\sigma_\zeta^2 + m_3(t)\sigma_{\varepsilon\zeta}].$$

(iii) *The real expected yield to maturity on this security is*

$$\ln(1 + y_\pi) = \mu_\varepsilon - \frac{m_{21}(t)\sigma_\varepsilon^2}{2} - m_3(t)\sigma_{\varepsilon\zeta} + m_{11}(t)\ln x_0 + \alpha_1 m_{21}(t)\sigma_\varepsilon^2 + \alpha_2 m_3(t)\sigma_{\varepsilon\zeta}.$$

The primary characteristics of asset prices evidenced in Theorem 1 can be summarized as follows: Nominal prices are functions of the four state variables (c_0, I_0, x_0, s_0). The (level) state variables c_0 and I_0 appear according to their form in the time-t nominal cash flow (α_1, α_2). The (rate) state variables, indicating the current status of expected consumption growth and inflation, appear according to the (certainty-equivalent) characteristics $(\alpha_1 - 1, \alpha_2 - 1)$ and the maturity factors associated with the cash flow $(m_{11}(t), m_{12}(t))$. Result (i) implies that future prices are lognormally distributed and that relative prices of securities are functions of the four state variables and maturity.

The characteristics of yields are similar. Because of the dependence on the (rate) state variables, (one plus) nominal yields are lognormally distributed. Note, however, that the (level) state variables c_0 and I_0 do not enter into the determination of yields. Also, real yields are independent of the inflation state, simply reflecting that inflation has no real consequences. The impact of the current state variable is smoothed through time with an asymptotic effect of zero.[8] Finally, in the absence of nonstationarity (autocorrelation), real and nominal yields are nonstochastic and, hence, known for all future periods.

The term structure for nominal bonds will be analyzed in the next section. However, note from Theorem 1 that the yields for consumption claims (market claims), $Y_C(t)$, may be written as

$$\ln(1 + Y_C(t)) = \mu_\varepsilon + \mu_\zeta + m_{11}(t)\ln x_0 + m_{12}(t)\ln s_0 + \sigma^2(t)/2,$$

where

$$\sigma^2(t) = m_{21}(t)\sigma_\varepsilon^2 + 2m_3(t)\sigma_{\varepsilon\zeta} + m_{22}(t)\sigma_\zeta^2,$$

while the yield on nominally riskless bonds, $Y_N(t)$, may be written as

$$\ln(1 + Y_N(t)) = \mu_\varepsilon + \mu_\zeta + m_{11}(t)\ln x_0 + m_{12}(t)\ln s_0 - \sigma^2(t)/2.$$

[8] There is, of course, a stability issue. If $\lambda_1 = 1$ and $\lambda_2 = 0$ so that the production rate of return follows a random walk, then so does the interest rate. Then $m_{11}(t) = 1$ and $m_{21}(t) = (t - 1)/2$, and long-term yields on market claims diverge to $+\infty$, while zero-coupon bond yields converge to -100%.

Ignoring the effect of the state variables, which average zero, the term structure for market claims is the mirror image of bonds. If the average term structure for nominal bonds is upward sloping, then the average term structure for market claims must be downward sloping. More in line with the rational-expectations interpretation, if nominal risk per unit of time (associated with consumption claims) is decreasing, then bond yields are rising. That the two curves exhibit exact symmetry is surprising and is probably a feature of the specific assumptions of the model. However, it is nontrivial to note that this observation does not coincide with the intuition that a rising term structure implies a rising structure for intertemporal market rates. This result indicates one use of this type of model—they are necessary for making inferences concerning market discount rates from an observable term structure.

The general structure of nominal yields is obtained by noting that the securities priced in Theorem 1 all conform to

$$\ln(1 + Y_\pi(t)) = \ln(1 + Y_N(t)) + \alpha_1(m_{21}(t)\sigma_\varepsilon^2 + m_3(t)\sigma_{\varepsilon\zeta}) + \alpha_2(m_3(t)\sigma_{\varepsilon\zeta} + m_{22}(t)\sigma_\zeta^2). \quad (12)$$

The bracketed quantities of equation (12) can be interpreted as risk premia and represent, respectively, the average real risk (per period) in nominal return $t^{-1}\mathrm{cov}(\prod_{s=1}^{t} \ln(1 + r_{ms}), \prod_{s=1}^{t} \ln(1 + R_{ms}))$ and the average (per period) inflation risk $t^{-1}\mathrm{cov}(\prod_{s=1}^{t} \ln(1 + i_s), \prod_{s=1}^{t} \ln(1 + R_{ms}))$. The risk measures are α_1 and α_2. In this respect, equation (12) serves as a formal model for the risk structure of yields that can be used, after observing nominal yields on riskless securities, to infer various other intertemporal discount factors. The calculation requires only the properties of the cash flow (α_1, α_2) and the risk-adjustment parameters explicit in equation (12), in direct contrast to the standard textbook methodology, in which all state-variable information, in particular the term structure, is neglected in the process of assigning an intertemporal discount rate computed through beta, the current risk premium, and the current risk-free rate.

Also of interest is the yield curve for indexed bonds, described by $\alpha_1 = 0$ and $\alpha_2 = 1$. According to equation (12), the yield curve for indexed bonds differs from the yield curve for nominally riskless bonds by the quantity $m_3(t)\sigma_{\varepsilon\zeta} + m_{22}(t)\sigma_\zeta^2$. As mentioned previously, Benninga and Protopapadakis [2] examine the term structure in an economy sans inflation. The assets priced are therefore indexed bonds, and the term structure for those claims can be drastically different from that for nominally riskless securities.

Finally, Theorem 1 can be used to construct a general format for the study of inflation premia in this model. First, note that the natural log of the expected inflation rate over the period $(0, t)$ can be written:

$$t^{-1}[\ln \mathrm{E}_0 I_t - \ln I_0] = \mu_\zeta + [1 - m_{22}(t)]\sigma_\zeta^2/2 + m_{12}(t)\ln s_0.$$

By properties (ii) and (iii) of the theorem,

$$\ln(1 + Y_\pi) - \ln(1 - y_\pi) = t^{-1}[\ln \mathrm{E}_0 I_t - \ln I_0] + \alpha_1 m_3(t)\sigma_{\varepsilon\zeta} + (\alpha_2 - 1)\sigma_\zeta^2. \quad (13)$$

The general form of the inflation adjustment is consistent with the Fisher Effect under limited circumstances. The Fisher Effect is the appropriate adjustment if inflation is nonstochastic ($\lambda_2 = \sigma_\zeta^2 = 0$) or if $\alpha_1 = 0$ and $\alpha_2 = 1$, implying an inflation-hedged asset with no consumption risk. Of the securities investigated in this paper, this applies only to the inflation-indexed, default-free bond. It is important to note that, for the Fisher Effect to totally represent the inflation adjustment, it is not sufficient that the security be inflation hedged (i.e., $\alpha_2 = 1$). The existence of a correlation between inflation rates and consumption growth implies an extra term in the adjustment related to market risk. This also applies to market claims in this model where the adjustment is the Fisher Effect plus $m_3(t)\sigma_{\varepsilon\zeta}$.

B. Single-Period Risk Return

The relationship between expected return and risk that exists in this model is derived by considering the expected (nominal) price at the end of the period and current price for the claims of Theorem 1. This relationship is characterized in the following corollary:

COROLLARY 1: *If the nominal value of a security at the end of this period can be written as*

$$V_1 = Kc_1^{\alpha_{11}}x_1^{\alpha_{12}}I_1^{\alpha_{21}}s_1^{\alpha_{22}},$$

where K is a constant, then the nominal expected rate of return on this security over the period, ER, *is given by*

$$\ln(1 + \mathrm{E}R) = \ln(1 + Y_N(1)) + (\alpha_{11} + \alpha_{12})(\sigma_\varepsilon^2 + \sigma_{\varepsilon\zeta}) + (\alpha_{21} + \alpha_{22})(\sigma_\zeta^2 + \sigma_{\varepsilon\zeta}). \tag{14}$$

The corollary indicates that the log of expected return is obtained by adding two risk adjustments to the log of the nominal single-period bond yield. The risk premia in each case are unrelated to maturity and are covariance risks associated with real activity and inflation. The risk measure $\alpha_{11} + \alpha_{12}$ is the elasticity of the security (for end of period, nominal value) with respect to the real state variable, x, while $\alpha_{21} + \alpha_{22}$ is the corresponding value elasticity with respect to inflation. Finally, state variables affect the expected return only through the level of the single-period risk-free rate; hence, the risk premia for fundamental securities are nonstochastic.[9]

From Theorem 1, the risk measures associated with fundamental securities are easily identified as

$$\alpha_{11} + \alpha_{12} = \alpha_1 + (\alpha_1 - 1)(t - 1)m_{11}(t - 1),$$
$$\alpha_{21} + \alpha_{22} = \alpha_2 + (\alpha_2 - 1)(t - 1)m_{12}(t - 1).$$

For positive autocorrelation, the elasticities are increasing functions of the cash flow elasticities (α_1 and α_2) and contain a maturity component associated with autocorrelation of the respective process. With negative autocorrelation in the

[9] This is not to be interpreted as a two-factor model. It is approximately so because the two state variables (factors) enter through the risk-free rate, but they enter in a log-linear fashion.

real process, the elasticities will decrease (increase) with maturity if the respective α is greater than one (< 1).

The discussion of the expected-returns premium indicated in (14) is simplified by noting that the risk measure reduces to a single dimension. The importance of this observation is that there is a simple, yet meaningful, way to combine the separate elasticities into an index that still conforms to the risk-return relationship and is thus indicative of total risk. To see this, define

$$\Phi_n \equiv \sigma_\varepsilon^2 + 2\sigma_{\varepsilon\zeta} + \sigma_\zeta^2,$$

$$\Delta \equiv \frac{\sigma_\varepsilon^2 + \sigma_{\varepsilon\zeta}}{\Phi_n}.$$

Φ_n is the nominal risk premium, while Δ is the fraction of nominal risk related to real returns. Then

$$\beta = \Delta(\alpha_{11} + \alpha_{12}) + (1 - \Delta)(\alpha_{21} + \alpha_{22}). \tag{15}$$

Equation (14) now becomes

$$\ln(1 + \mathrm{E}R) = \ln(1 + Y_N(t)) + \beta\Phi_n, \tag{14'}$$

indicating that the risk premium of fundamental securities can be regarded as arising from an SML-like equation with risk endogenously specified. To determine the risk parameter, β, for fundamental securities, one needs the cash flow parameters α_1 and α_2 and an understanding of the basic processes of the economy. In this sense, β is not simply a regression parameter; it accurately incorporates inflation and real elasticities.

From equation (15), the betas for the nominally riskless bond, the indexed bond, and the consumption claim are

$$\beta_N = -\Delta(t-1)m_{11}(t-1) - (1-\Delta)(t-1)m_{12}(t-1),$$

$$\beta_F = -\Delta(t-1)m_{11}(t) + (1-\Delta),$$

$$\beta_C = 1.$$

Consumption claims exhibit the simplest structure; their beta is one and independent of maturity. The beta of an indexed bond is constant if there is no real autocorrelation. If real autocorrelation is negative, then the single-period risk measure for this security will increase with maturity. The behavior of betas for nominal bonds will be discused in Section III.

Cox, Ingersoll, and Ross [5] have examined the ability of the unbiased-expectations theory to describe yield data for bonds. They employ the "modern" version of the theory, which states that unbiased expectations are characterized by an equivalent expected return over a single period for all maturities of the security in question. Examining the betas of the above securities, we see that this holds, strictly, only for consumption claims. It might also hold for the indexed bonds, but we must have $\lambda_1 = 0$. The explanation for this result is straightforward. Betas indicate whether the risks are the same over the next period for securities of different maturities. If they are not (i.e., if risks in a one-period context are a function of maturity), then the "modern" version cannot hold because it contradicts the equilibrium risk-return mechanism. It is interesting to observe that it may apply to one class of securities while not to others in a particular equilibrium. That the model's market claims exhibit this property seems to be due to the peculiarities of the log-utility function that lead to consumption claims for all maturities having constant relative prices.

The fact that the beta equivalents of fundamental claims are nonstochastic in this economy does not imply that this holds for claims to a linear combination of the basic cash flows; nor are the betas of such claims simply weighted averages of the component betas. For example, suppose we construct a security with a cash flow that can be represented as

$$(\pi_t, \pi_{t'}) = (c_t^{\alpha_1} I_t^{\alpha_2}, c_t^{\alpha_1'} I_t^{\alpha_2'}). \tag{16}$$

Value additivity, in combination with Theorem 1, can be used to value such a claim for any $\tau < \min(t', t)$. The expected return for each component over a single period follows from the corollary, and the expected return for the security is simply the weighted average. To formalize this, let

$$1 + \mathrm{E}R_c = w_1(1 + \mathrm{E}R_1) + (1 - w_1)(1 + \mathrm{E}R_2),$$

where $\mathrm{E}R_c$ denotes the expected return on the composite claim, $\mathrm{E}R_1$ and $\mathrm{E}R_2$ are the expected returns indicated by the corollary for the two components of $(\pi_t, \pi_{t'})$, and w_1 is the proportional (by value) time-τ value of the first cash flow component. From equation (15), we see that

$$1 + \mathrm{E}R_c = (1 + Y_N(1))[w_1 e^{\beta_1 \Phi_n} + (1 - w_1)e^{\beta_2 \Phi_n}].$$

Theorem 1 implies that relative prices of securities are not constant in this model (unless $\alpha_1 = \alpha_1'$, $\alpha_2 = \alpha_2'$, and maturity is identical) but are random functions of the state variables. This, in turn, implies that the weights are functions of the state variables and therefore stochastic. The beta of this security can be inferred from the relationship:

$$e^{\beta \Phi_n} = w_1 e^{\beta_1 \Phi_n} + (1 - w_1)e^{\beta_2 \Phi_n}, \tag{17}$$

but equation (17) implies that $\beta > w_1\beta_1 + (1 - w_1)\beta_2$ because of the convexity of the function e. As an example, the beta of a coupon bond is in fact stochastic and has an average higher than indicated by the linear combination of betas associated with pure discount bonds. This means a certain amount of stochasticity in the holding-period term premia of nominal coupon bonds. Similar comments would apply to any combination security, provided that its components were not strictly consumption claims.

III. Nominally Riskless Bonds

This section applies the general model to nominally riskless bonds. The implications for the shape of the term structure, the relationship of forward rates to expected spot rates, and the single-period risk of bonds are addressed. It is demonstrated that either negative real autocorrelation and/or negative real-inflation covariance can lead to an upward-sloping term structure and that the term structure for holding-period premia is not related to the term structure for yields.

A. *The Term Structure*

Theorem 1 identifies the term structure for nominal bonds as

$$\ln(1 + Y_N(t)) = \mu_\varepsilon - m_{21}(t)\sigma_\varepsilon^2/2 - m_3(t)\sigma_{\varepsilon\zeta} + m_{11}(t)\ln x_0 + \mu_\zeta - m_{22}(t)\sigma_\zeta^2/2 + m_{12}(t)\ln s_0. \tag{18}$$

In the absence of autocorrelations, this specializes to

$$\ln(1 + Y_N(t)) = \mu_\varepsilon + \mu_\zeta - \sigma_\varepsilon^2/2 - \sigma_{\varepsilon\zeta} - \sigma_\zeta^2/2,$$

and the term structure is nonstochastic and flat. It follows that the autocorrelations of Section I influence the term structure in two ways: first, through impact of the current state variables, x_0 and s_0, and, second, through the determination of average risks per unit of time (embodied in m_{2*} and m_3). Because the expected value for the state variables is zero, the average shape of the term structure is determined through the latter factors.

The state variables' effects on the term structure are simply the average real productivity and inflation over future periods. The presumed positive autocorrelation in inflation rates means that a high current state for inflation will lead to increased yields, decaying with maturity. Negative autocorrelation in the real process would mean that high productivity is associated with an upward-sloping component to the term structure. The combined effect of the two processes is complicated by the inflation-real covariance, but, noting that the conditional variance of next period's spot rate is

$$\lambda_1^2\sigma_\varepsilon^2 + 2\lambda_1\lambda_2\sigma_{\varepsilon\zeta} + \lambda_2^2\sigma_\zeta^2,$$

negative covariance and negative autocorrelation in real returns would increase the volatility of short-term rates.

The fact that both m_{11} and m_{12} go to zero implies that the term structure approaches an asymptote independent of the current state variables. This asymptote is given by

$$\ln(1 + Y_N(\infty)) = \mu_\varepsilon + \mu_\zeta - \frac{\sigma_\varepsilon^2}{2(1-\lambda_1)^2} - \frac{\sigma_{\varepsilon\zeta}}{(1-\lambda_1)(1-\lambda_2)} - \frac{\sigma_\zeta^2}{2(1-\lambda_2)^2}. \tag{19}$$

Note that positive autocorrelation tends to cause the asymptote to be lower, while negative autocorrelation tends to raise its value (the exact opposite of market yields).

The average term structure of interest rates, termed the primitive term structure in the remainder of this paper, follows from the properties of the m_{2*}, m_3, and the structure of variances and covariance for the economy's two processes. Positive autocorrelations, in combination with positive inflation-real covariance, can lead only to downward-sloping (primitive) term structures. The rationale is that, per unit of time, consumption risk is increasing in such an economy, and

Table I

$\lambda_1 = 0;\ \sigma_{\varepsilon\zeta} = -\ 0.0014$

Maturity	Yield	Forward Rate	Expected Interest Rate	β
1	0.066200	*	*	0.000000
2	0.066501	0.066802	0.066298	0.092105
3	0.066703	0.067107	0.066346	0.156579
4	0.066843	0.067263	0.066370	0.201711
5	0.066943	0.067344	0.066381	0.233303
6	0.067017	0.067388	0.066387	0.255417
7	0.067074	0.067411	0.066389	0.270897
8	0.067117	0.067424	0.066391	0.281733
9	0.067152	0.067432	0.066392	0.289319
∞	0.067444	0.067444	0.066392	0.307018

Table II

$\lambda_1 = -0.3;\ \sigma_{\varepsilon\zeta} = 0.0$

Maturity	Yield	Forward Rate	Expected Interest Rate	β
1	0.064800	*	*	0.000000
2	0.065886	0.066972	0.065348	0.261538
3	0.065897	0.065920	0.065437	0.156154
4	0.065875	0.065806	0.065464	0.168923
5	0.065797	0.065489	0.065476	0.151900
6	0.065717	0.065316	0.065481	0.147772
7	0.065639	0.065169	0.065484	0.142546
8	0.065568	0.065068	0.065485	0.139589
9	0.065504	0.064995	0.065486	0.137309
∞	0.064819	0.064819	0.065487	0.132150

riskless assets are desirable. To establish equilibrium, the yields on such assets must be relatively low. Alternatively (but equivalently), nominal bonds are good hedges against unfortunate shifts in real productivity. When real productivity is low, on average so is inflation and, therefore, interest rates. Hence, the bond price is high.

Maintaining the assumption of positive autocorrelation for inflation, alternative shapes must be associated with negative real autocorrelation and/or negative real-inflation covariance. Tables I to III examine the primitive term structure for economies all described by $\mu_\varepsilon = .02$, $\mu_\zeta = .05$, $\sigma_\epsilon^2 = .01$, $\sigma_\zeta^2 = .0004$, and $\lambda_2 = .7$, but with differing real autocorrelation and inflation-real covariance. Table I demonstrates that a negative covariance, by itself, can lead to an upward-sloping yield structure.[10] The negative covariance undermines the ability of the bond to hedge against declines in future real rates and results in a risk premium. Tables II and III describe term structures for economies with negative real autocorrelation. Both such economies exhibit a "hump-backed" structure similar (in spirit) to that observed by Fama [10]. The interaction of the downward pull of inflation and the upward force of the negative real autocorrelation, possibly combined

[10] The tables are intended to illustrate shapes rather than magnitudes.

Table III

$\lambda_1 = -0.2;\ \sigma_{\varepsilon\zeta} = -0.001$

Maturity	Yield	Forward Rate	Expected Interest Rate	β
1	0.065800	*	*	0.000000
2	0.066691	0.067582	0.066238	0.264286
3	0.066911	0.067352	0.066274	0.256429
4	0.067024	0.067363	0.066301	0.289500
5	0.067079	0.067299	0.066312	0.304936
6	0.067108	0.067249	0.066318	0.317284
7	0.067122	0.067205	0.066321	0.325618
8	0.067128	0.067172	0.066322	0.331515
9	0.067130	0.067147	0.066323	0.335630
∞	0.067083	0.067083	0.066323	0.345238

with negative covariance, makes alternative shapes possible. Table III, which combines the effects, indicates a higher overall level of rates, with the hump occurring further out (period nine versus period three).

B. Forward Rates and Expected Spot Rates

Early work on the term structure focused on the relationship between expected spot rates and forward rates.[11] These efforts were misguided in the sense that the forward rate is the marginal certainty equivalent for going one period further into the term structure. Lacking a portfolio theory, one cannot evaluate this certainty equivalent and its relationship to the expected future rate. However, the question is not without merit and is taken up within the scope of the model.

To formalize the forward rate-expected rate relationship, let f_t denote the forward rate for $(t, t+1)$ and U_t' the marginal utility of real consumption at time t. It is straightforward to derive

$$f_t = \frac{\mathrm{E}_0 R_{Ft} + \mathrm{cov}_0\left(R_{Ft}, \dfrac{U_{t+1}'}{1 + i_{t+1}}\right)}{1 + B}, \tag{20}$$

where

$$B = \left(\mathrm{E}_0 \frac{U_{t+1}'}{1 + i_{t+1}} \mathrm{E}_0 I_t^{-1}\right)^{-1} \mathrm{cov}_0\left(\frac{U_{t+1}'}{1 + i_{t+1}}, I_t^{-1}\right).$$

Concentrating on the two terms in the numerator, the forward rate is approximately the expected interest rate plus the covariance of the real interest rate (on nominal bonds) for $(t, t+1)$ and marginal utility at $t+1$ (scaled by $1 + B$). It is evident, however, that it will be difficult to establish the sign of the bias in terms of the general expression (20). Fortunately, the model allows for numerical evaluation of this bias.

From Theorem 1, the riskless rate in future periods is lognormally distributed.

[11] For a short survey of this literature and an extensive reference list, see Van Horne [20] (Chapter 5).

The interest rate expected to prevail at time t, $\mathrm{E}_0 R_{Ft}$, is

$$\ln(1 + \mathrm{E}_0 R_{Ft}) = \mu_\varepsilon + \mu_\zeta + \lambda_1^{t+1}\ln x_0 + \lambda_2^{t+1}\ln s_0 + (d_1(t) - 1)\sigma_\varepsilon^2/2 + (d_2(t) - 1)\sigma_{\varepsilon\zeta} + (d_3(t) - 1)\sigma_\zeta^2/2, \quad (21)$$

where

$$d_1(t) = \frac{\lambda_1^2(1 - \lambda_1^{2t})}{(1 - \lambda_1^2)},$$

$$d_2(t) = \frac{\lambda_2^2(1 - \lambda_2^{2t})}{(1 - \lambda_2^2)},$$

$$d_3(t) = \frac{\lambda_1\lambda_2(1 - \lambda_1^t\lambda_2^t)}{(1 - \lambda_1\lambda_2)}.$$

The difference between the (ln) forward rate ($t\ln(1 + Y_N(t)) - (t-1)\ln(1 + Y_N(t-1)))$) and the expected interest rate is easily computed but is difficult to evaluate analytically because it contains polynomial terms in the various λ's.[12] Tables I to III tabulate evaluations of forward rates and expected spot rates for the three economies. In Table I, forward rates are upward-biased estimates of future spot rates, and the bias seems to be increasing in maturity. In Table II, the bias in forward rates reverses at time six, first overestimating spot rates and then underestimating spot rates. Table III, combining negative real autocorrelation and negative covariance, reflects forward rates that are upward biased, but the bias is decreasing with maturity. In these examples, negative autocorrelation appears to be connected with an initially positive but decreasing bias, while negative covariance seems to lead to a more stable (positive) bias. Negative autocorrelation would cause real interest rates and marginal utility to covary in a positive manner, contributing to a risk premium in equation (20), but the effect tends to decay very quickly, allowing the inflation effect ultimately to dominate. The negative-covariance effect is not so easily explained. As mentioned previously, this undermines the ability of the long-term bond to act as an opportunity-set hedge, seemingly implying a risk premium, but that premium is connected with the expected holding-period return and may not carry over into the forward rate. The theoretical expression for the forward rate does not incorporate covariance in a self-evident way. It seems that negative covariance, along with positive autocorrelation in the real and inflation processes, would lead to a negative covariance between marginal utility and real rates, which would tend to reduce or cause a negative sign for the bias. The examples of this section make it clear that this is not the case. Evidently, the most profound effect of negative covariance is a reduction in B in equation (20).

The tables may be interpreted as indicating that an (on average) upward-sloping yield curve is associated with upward-biased forward rates, while a hump-backed yield curve seems to imply a decreasing (and, possibly, reversing) bias. At this time, however, such implications are only suppositions.

[12] However, it does not depend upon the current state variables. This property will be discussed in Section IV.

C. Bond Betas

Section II derived the beta of a nominal bond as

$$\beta_N = -\Delta(t-1)m_{11}(t-1) - (1-\Delta)(t-1)m_{12}(t-1).$$

If λ_1, λ_2, and $\sigma_{\varepsilon\zeta}$ are all positive, then bond betas are negative and decreasing with maturity. Again, two components of the model can contribute to positive bond betas: negative autocorrelation in real returns and negative covariance. Table I illustrates the betas associated with a negative real-inflation covariance. These betas are positive and increasing with maturity. Tables II and III reflect the impact of negative autocorrelation on bond betas. The striking feature of these (and other tables not presented) is a similarity to liquidity preference. If $\lambda_1 < 0$, all other bonds tend to differ dramatically from a one-period bond. This is particularly evident in Table III, as contrasted with Table I.

It is the betas that determine the expected excess holding-period return of the various bonds in this economy. Note that positive (and increasing) bond betas do not imply a rising (primitive) term structure in this model. As can be seen in Table III, despite the fact that bond betas are increasing functions of maturity, the data imply a generally downward-sloping yield-to-maturity curve. Examples can be constructed in which bond betas are monotonically increasing but the primitive term structure is decreasing. Hence, neither the existence of positive bond betas—nondecreasing in maturity—nor that of positive term premia can be an unambiguous signal of an upward-sloping term structure. The relationship between forward rates and betas reflects a similar ambiguity. Table I exhibits forward rates and betas that are increasing in maturity. Table II also exhibits a similar relationship—both forward rates and holding-period returns peaking at period two and then declining. However, in Table III, beta is increasing while the forward rate is decreasing. In [10], p. 544, Fama argues that the time period he examines is representative, based on the congruence between forward rates and average holding-period returns. This argument is correct for the economies represented in Tables I and II, but not correct for the economy of Table III.

IV. Relationship to Empirical Work

This section examines some recent empirical work on the term structure and its relationship to the model.

A. Recent Empirical Work

Fama [9, 10] documents a number of features of the term structure for the period 1953 through 1982. These features include the following:

1. Forward, spot, and term premia display reliably positive autocorrelation. The autocorrelation of spot rates (one month) is particularly strong.
2. Forward rates display greater volatility than future spot rates.
3. The volatility of the difference between the forward rate and the current spot rate of interest is related (positively) to the next month's holding-period return on the bond.

4. The volatility of the difference between the forward rate and the current spot rate is weakly related to the future changes in the spot rate.
5. The average holding-period-premium term structure rises out to about nine months and then declines markedly. Beyond one year, no statistical inferences can be drawn (due to the size of the standard errors).

The first two conclusions had been documented by prior researchers.[13] Fama's contribution was to offer an explanation of the source of the volatility indicated in point two. That source, according to Fama, was the structural relationship between forward rates and holding-period returns as indicated in point three.

To explore the source of the volatility of forward rates, Fama makes use of the two regressions:

$$HPR(\tau)_{t+1} - r_t = \alpha_1 + \beta_1(F(\tau)_t - r_t) + \varepsilon_{t+1}, \qquad \text{(Fama's 11)}$$

$$r_{\tau-1} - r_t = \alpha_2 + \beta_2(F(\tau)_t - r_t) + \delta_{\tau-1}, \qquad \text{(Fama's 12)}$$

where $F(\tau)_t$ is the forward rate for a bond maturing at time τ (observed at t), r_t is the spot rate on a one-month T-bill, and $HPR(\tau)_{t+1}$ is the holding-period return on the bond of maturity τ over $[t, t+1]$. Fama notes that, if forward rates are caused by factors in addition to the expected spot rate, both of the above equations are tainted by omitted variables. In this case, β_1 and β_2 only provide information on the relative contribution of holding-period return and future spot rate to the volatility of forward rates.[14] In general, Fama finds that holding-period returns account for substantially more of the volatility of forward rates than do future spot rates.

A problem that troubles Fama's analysis is the possible presence of errors in the variables induced by measurement errors in the current bond price. To see this, suppose that today's measured price happens to be an underestimate of the equilibrium price. The forward rate is then overestimated (on average). If the error in next period's price is unrelated to today's error, then it will show up in the first period's holding return. Hence, β_1 would be significant if there were significant errors in the variables.[15] Furthermore, since the forward rate is determined by two prices measured with error, while the spot rate contains only one error, one expects more volatility in forward rates.[16]

Stambaugh [19] examines the measurement-error problem associated with Fama's equation (11). Using bills of one-week difference in maturity to calculate holding-period returns and forward rates, he finds that much of the effect observed by Fama in point three disappears. This seems to indicate a severe measurement-error problem in Fama's study.

Stambaugh also performs a test that is relevant to the model of this paper. He observed that, in a model where (log) prices are linear functions of state variables, and if the expected values of future state variables are linear functions of the

[13] See Fama's cited papers for references.

[14] Two points need to be emphasized: First, the betas are not then unbiased estimates of the structural relationships (i.e., we cannot infer from β_2 whether forward rates contain unbiased forecasts of future interest rates). Second, the betas are not unbiased measures of the volatility contribution unless any remaining omitted variables are orthogonal.

[15] Fama admits related possibilities in [9], p. 527.

[16] The added volatility is only on the order of an additional ten percent. See [9] (Table 3, p. 515).

current states, then all (log) forward rates and expected holding-period returns are linear functions of the current state variables. Formally, let $\mathbf{H}_t$ be the $N \times 1$ cross-sectional vector of expected holding-period returns over $(t, t + 1)$, let $\mathbf{X}_t$ be the $k \times 1$ vector of state varibles observed at time t, and let $\mathbf{F}_t$ be the $N \times 1$ vector of forward rates computable at t; then,

$$\mathbf{H}_t = \mathbf{A} + \mathbf{B}\mathbf{X}_t + \mathbf{V}_t, \tag{22}$$

$$\mathbf{F}_t = \mathbf{D} + \mathbf{E}\mathbf{X}_t + \mathbf{Q}_t, \tag{23}$$

where $\mathbf{A}$ and $\mathbf{D}$ are $N \times 1$ vectors of constants, $\mathbf{B}$ and $\mathbf{E}$ are $N \times k$ matrices, and $\mathbf{V}$ and $\mathbf{Q}$ are error vectors caused by measurement error. There exists some matrix $\mathbf{G}$, $k \times N$, such that

$$\mathbf{G}\mathbf{F}_t = \mathbf{G}\mathbf{D} + \mathbf{X}_t + \mathbf{G}\mathbf{Q}_t,$$

so that

$$\mathbf{H}_t = \mathbf{A} - \mathbf{B}\mathbf{G}\mathbf{D} + \mathbf{B}\mathbf{G}\mathbf{F}_t + \mathbf{V}_t - \mathbf{G}\mathbf{Q}_t. \tag{24}$$

Stambaugh notes that the rank of the $(N \times N)$ matrix $\mathbf{BG}$ can at most be k by the theory. He tests this rank restriction and finds that there appear to be at least two state variables. This test is of relevance because it applies not only to the Cox, Ingersoll, and Ross model, but also to the model of this paper. As noted in Section III, expected holding-period returns are nonstochastic in this model, and therefore $\mathbf{BG}$ should have rank zero. Unfortunately, the issue remains clouded because the error term in equation (24) is not uncorrelated with the independent variable(s). (Note that the measurement error, $\mathbf{Q}_t$, is a component of $\mathbf{F}_t$ from (23).) This is known to bias the coefficient estimates and presumably must also bias the rank estimate. The only apparent solution to this problem is to identify the state variables and estimate equation (22) directly.

B. The Model in Relation to the Empirical Evidence

The model of this paper is, in principal, capable of explaining the observed autocorrelation in risk-free rates and forward rates. To verify this, note that the process for the risk-free rate may be written as

$$\begin{aligned} \ln(1 + R_{Ft}) &= K_F + \lambda_1(\ln(1 + R_{Ft-1}) - K_F) + \lambda_2(\lambda_2 - \lambda_1)\ln s_{t-1} + \text{error} \\ &= K_F + \lambda_2(\ln(1 + R_{Ft-1}) - K_F) + \lambda_1(\lambda_1 - \lambda_2)\ln x_{t-1} + \text{error}, \end{aligned} \tag{25}$$

where $K_F = \mu_\varepsilon + \mu_\zeta - (\sigma_\varepsilon^2 + 2\sigma_{\varepsilon\zeta} + \sigma_\zeta^2)/2$, so that this period's interest rate depends upon last period's and one of the two state variables (s or x). This result accords with the findings of Gultekin and Rogalski [13], indicating that a two-factor model is a better description of bond returns than the traditional CAPM, and the study by Marsh and Rosenfeld [17], which indicates that the lognormal dynamics specification fits interest rate data better than the alternatives they tested.[17] A simple (but structurally inaccurate) autoregression would produce the

[17] It must, however, be mentioned that Hansen and Singleton [14] reject the specifications of this model because there appears to be significant predictive power through information available in the capital markets with regard to the errors produced through this model's structure.

slope estimate:

$$\hat{\lambda} = \lambda_1 + \lambda_2(\lambda_2 - \lambda_1)\left[\frac{\lambda_2\sigma_\zeta^2}{(1-\lambda_2^2)} + \frac{\lambda_1\sigma_{\varepsilon\zeta}}{(1-\lambda_1\lambda_2)}\right]\Big/ \text{var}(R_F)$$

$$= \lambda_2 + \lambda_1(\lambda_1 - \lambda_2)\left[\frac{\lambda_1\sigma_\varepsilon^2}{(1-\lambda_1^2)} + \frac{\lambda_2\sigma_{\varepsilon\zeta}}{(1-\lambda_1\lambda_2)}\right]\Big/ \text{var}(R_F).$$

Fama [10] reports interest rate autocorrelation of ≈.95. If inflation autocorrelation is less than .95 and if real autocorrelation is negative, to produce this level of autocorrelation in the model would require that inflation-real covariance be significantly positive. On the other hand, if the real autocorrelation is positive, then inflation-real covariance must be significantly negative. With negative autocorrelation in real returns and a negative covariance (or with both positive), the observed autocorrelation of the interest rate will be greater than λ_1 but less than λ_2. Relating this to the discussion of previous sections, the most likely specification of the model that combines negative market autocorrelation, an upward-sloping yield curve, and high interest rate autocorrelation is the negative covariance of the inflation and real processes, but with non-negative real correlation.

As noted in Section III, the model does not provide for stochastic (log) term premia (β is a nonstochastic function for zero-coupon bonds with no default risk); nor does this model allow for autocorrelation in these premia. The term-premia autocorrelation observed by Fama is low and could, in principal, be explained by some low-level autocorrelation in measurement errors, nonconstancy in the parameters of the model, or possibly even tax effects. Similarly, Stambaugh's estimate of at least two significant factors may indicate a shortcoming of the model or, again, simply the presence of measurement errors. Whichever the case, in its current form, the model predicts that the only source of volatility in forward rates is the expectation of the future interest rate ($\beta_1 = 0$, $\beta_2 = 1.0$ in Fama's equations (11) and (12)).

Jordan [15] argues that taxes must be a part of any attempt to understand the empirical term structure, while Constantinides [4] demonstrates that taxes result in options connected with bonds. Incorporating tax options into the model will be a difficult task but may be necessary if the model is to be empirically viable. If the model is to be tested in its current form, two alternatives suggest themselves: one could either estimate the structural parameters of the model (λ_1, λ_2) and attempt to fit the model in that fashion, or nonlinear estimation techniques could be used in a cross-sectional analysis in an attempt to estimate the parameters λ_1, λ_2, and the various $\sigma_{..}$. The possible presence of measurement errors suggests that testing should probably proceed in terms of prices, focusing on the question of significant pricing errors. The difference between the two approaches is philosophical; cross-sectional testing would allow for variation in the parameters, but that variation must at most be a function of time. Otherwise, it would be necessary to price it in a rational-expectations sense.[18]

[18] Similar remarks apply to Brown and Dybvig's [3] test of the Cox, Ingersoll, and Ross [7] model, a model that does account for term premium randomness. The way they fit the model incorporates changes in structural parameters that Cox et al. regard as fixed. The model cannot (in general) be rational expectations under these conditions.

V. Summary

This paper has modeled the interest and market rate uncertainty inherent in a specific rational-expectations equilibrium. The uncertainty of capital market rates arose because of nonstationarities in the inflation and real production processes of the economy. Pricing, however, incorporates these risks, and the paper has dealt with their implications for present value.

The model is characterized by a number of interesting features. The term structure of interest rates was shown to depend upon the state variables for inflation and real returns, the way in which inflation and real risks interact to form total risk, and the maturity of the bond in question. The slope of the term structure always depends upon the state variables, but its primitive (average) form can have an upward slope only if a sufficiently strong negative correlation exists between unanticipated inflation and real returns or if negative autocorrelation exists in real returns. Furthermore, if the term structure was upward sloping, then the term structure for market claims was downward sloping. The determinants of single-period systematic risk were also investigated and were shown to depend upon the type of claim, the dynamics of the inflation and real processes, and, through those dynamics, maturity.

Further investigations using the model could follow two directions. Within the current assumptions, the model would be utilized to price interest rate-dependent options such as the standard call (in the face of interest rate risk) or a callable bond. Either would be accomplished along the lines of Rubinstein [18] although the callable bond certainly would involve numerical approximations. Empirically, the model could be tested by attempting to fit the inflation and real-return processes and then testing the specifications against the observed term structure. Hansen and Singleton [14] and Fama [9] provide evidence against the model in its current form, but their papers do not consider the question of whether we better understand the term structure because of such a model. Furthermore, Jordan [15] argues that taxes must be a part of any attempt to understand the empirical term structure, while Constantinides [4] demonstrates that taxes result in options connected with bonds. These considerations were beyond the scope of this paper and will be difficult to explicitly incorporate into the model. Finally, the issue of the relationship of other yields to the visible term structure demands further investigation.

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