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Source: *The Journal of Finance*, Vol. 42, No. 4 (Sep., 1987), pp. 1007-1021
Published by: Blackwell Publishing for the American Finance Association
Stable URL: http://www.jstor.org/stable/2328303
Accessed: 26/11/2009 08:12

THE JOURNAL OF FINANCE • VOL. XLII, NO. 4 • SEPTEMBER 1987

Optimal Hedging in Futures Markets with Multiple Delivery Specifications

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ABSTRACT

Nearly all futures contracts allow delivery of any of several qualities of the underlying asset. Consequently, the price of the futures contract is associated more with the price of the expected cheapest deliverable variety than with the price of the par-delivery variety. The delivery specifications introduce a delivery risk for every hedger in the market. We derive the optimal hedging strategies in these markets. Their hedging effectiveness is evaluated for wheat futures contracts in Chicago. Hedging optimally would have significantly reduced the variance of the rates of return on hedges while yielding similar mean returns.

NEARLY ALL FUTURES CONTRACTS on commodities and financial assets allow the seller to deliver any of several qualities of the underlying asset. The contracts specify discounts or premiums for delivering a nonpar quality, but the realized price differences can be significantly different. Consequently, the futures price on a delivery date converges to the spot price of the quality that is cheapest to deliver rather than the spot price of the par-delivery quality. Since it is uncertain which quality will be the cheapest to deliver, this introduces a delivery risk for every hedger in the market. Gay and Manaster [9] and Kane and Marcus [13] find that hedgers in the Treasury bond futures market have neither hedged nor exploited the delivery specifications in an optimal way.[1] This paper addresses how to hedge optimally in futures markets with multiple delivery specifications.[2] We derive the optimal hedging strategies and evaluate their hedging effectiveness.

Futures markets are hedging markets (Working, [20]). They enable traders to hedge the risk of unexpected changes in the spot price of their asset. The hedging effectiveness of futures markets is, therefore, an important issue. Earlier studies include Ederington [3], Figlewski [5], and Koppenhaver [16], among others. This study also differs from those studies in that we evaluate an ex ante hedging

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[1] Hieronymous [11] observes that long hedgers prefer to pay a premium by liquidating their futures position with an offsetting trade and buying the exact quality they need in the spot market, rather than accepting delivery in the futures market.

[2] Many futures contracts also allow the seller to choose among several delivery locations and among nearly all business days in the delivery month. The optimal hedges derived here are appropriate in minimizing the location risk as well. The paper does not address how to hedge the timing risk. Gay and Manaster [8, 9] report, however, that the timing option appears to be of little value in either commodity or financial futures markets.

effectiveness. Ederington [3], Figlewski [5], and Koppenhaver [16] evaluate an ex post hedging effectiveness, assuming that hedgers have perfect foresight regarding spot and futures prices. They examine strategies involving constant hedge ratios based on the true changes in spot and futures prices over the hedging period. The hedge ratios in our study are re-estimated on each hedging date and are based only on information publicly available on that day.

The relation between futures and spot prices prior to delivery largely stems from what it is expected to be at delivery. Delivery specifications are, therefore, an important determinant of the futures-spot price relation well before the delivery date. As a result, the delivery specifications are important to any participant in futures markets even though he or she may never make or take delivery (Chicago Board of Trade [2]). Kilcollin [15], Garbade and Silber [7], Gay and Manaster [8, 9], Kane and Marcus [13], and Kamara [12] provide evidence on the importance of the delivery specifications in the pricing of futures contracts.

The hedging strategy currently used by academic researchers and market participants when the hedge is held to delivery is to use the conversion factor of the hedged quality as the hedge ratio. (See, for example, Garbade [6] and Kane and Marcus [14].) (Conversion factors are fixed exchange rates, specified in the futures contract, that adjust for the fact that one futures price determines the delivery values of several different qualities.) Such a strategy is optimal in the absence of a delivery risk. It is not optimal in the presence of a valuable delivery option. By choosing a hedge ratio that is equal to his or her spot asset's conversion factor, a short hedger can lock in the sale value of his or her spot asset even when the futures contract contains multiple delivery provisions. Yet this strategy is not riskless for either short or long hedgers. It is riskless only for a short hedger who commits himself or herself, when he or she puts the hedge, to deliver his or her spot asset in fulfillment of his or her futures position. The price (loss) of this commitment is the value of the delivery option.

The following example illustrates the situation facing the short hedger. Consider the March 1974 wheat contract on February 15, 1974.

	February 15, 1974	March 1, 1974
Spot prices ($ per bushel)		
Soft wheat:	6.92	6.31
Hard wheat:	6.29	5.87
Futures price:	6.32	

Since either soft or hard wheat can be delivered at par, the conversion-factor hedging strategy is a full hedge for either quality. Short hedgers of either quality "lock in" a minimum price of $6.32 per bushel. However, on the first delivery date, short hedgers of soft wheat could have earned an instantaneous return of 7.5 percent (ignoring transactions costs). They could have sold their soft wheat in the spot market for $6.31 per bushel and liquidated their futures position at $5.87 per bushel. (The futures price on that day ranged between $5.71 and $6.00.)

In the next section, we derive the optimal (minimum-variance) hedging strategy in futures markets with multiple delivery specifications. In Section III, we evaluate its effectiveness for futures contracts allowing delivery at par of either soft or hard wheat in Chicago, during 1970 to 1981. A review of the studies cited above on delivery uncertainty suggests that delivery uncertainty in the wheat market is small relative to delivery uncertainty in, for example, the Treasury bond market. Price switches from higher to lower occur less frequently in the spot wheat market than in the spot Treasury market. Still, we find that delivery uncertainty is important for hedging in the wheat futures market as well. Since either quality can be delivered at par value, the conversion-factor hedging strategy for either one (for hedges held to delivery) is a full hedge. Examining hedges held until the first delivery date, we find that the optimal hedges are significantly more effective than the full hedge. The variances of the rates of returns on the optimal hedges are significantly lower, while the means are of similar magnitude. We also find that the no-hedge strategy is mean-variance inferior to any one of the hedging strategies. This demonstrates the economic benefits of the wheat futures market even in the presence of a delivery risk.

I. Optimal Hedging Strategies

In this section, we study the optimal hedging strategy in a futures contract allowing a delivery of either one of two underlying assets.

Let X and Y be two spot assets, and let P_t^X and P_t^Y represent their prices at time t, where $t = 0, 1$. There exists a futures contract at time 0 with a price of f_0, calling for a delivery of one unit of either X or Y at time 1. For simplicity, we assume that either asset can be delivered at par value (that is, P^X and P^Y are the delivery-adjusted spot prices). Following Stein [19] and Ederington [3], we restrict our attention to the case of a hedger who holds only one spot asset.[3] Let R represent the return on a portfolio that includes one unit of, for example, asset X and h in futures contracts, where h is the hedge ratio. The expected return is, then

$$\mathrm{E}(R) = \mathrm{E}(P_1^X) - P_0^X + h\{f_0 - \mathrm{E}[\min(P_1^X, P_1^Y)]\}, \tag{1}$$

where the futures price at time 1 equals the minimum of the delivery-adjusted spot prices. The variance of the return is

$$V(R) = V(P_1^X) + h^2 V[\min(P_1^X, P_1^Y)] - 2h \operatorname{cov}[P_1^X, \min(P_1^X, P_1^Y)]. \tag{2}$$

To find the minimum-variance hedge ratio, we set the derivative of (2) with respect to h equal to zero and obtain

$$h^* = \frac{\operatorname{cov}[P_1^X, \min(P_1^X, P_1^Y)]}{V[\min(P_1^X, P_1^Y)]}. \tag{3}$$

[3] Like earlier studies on hedging effectiveness, we also assume the position in the spot market to be exogenous. Consequently, any interest payments and transactions costs in the spot market can be viewed as predetermined and irrelevant to the hedging decision. For simplicity, these storage and transactions costs in the futures market are assumed to be zero. Hence, we also ignore the effect of marking to the market in the futures market. The empirical evidence suggests, however, that the economic significance of this effect is very small. (See, for example, Elton, Gruber, and Rentzler [4].)

In Ederington [3], Figlewski [5], and Koppenhaver [16], the hedge ratio is easily estimated by the slope coefficient of the regression (in levels or in first differences) of spot prices on futures prices using historical data. Thus, one might think that, in our case, h^* can be efficiently computed by regressing past prices of X on past prices of $\min(P_t^X, P_t^Y)$. This view is wrong as illustrated by Figure 1. Such a procedure will yield an optimal hedge ratio equal to one (for X), even though asset Y might be the cheapest to deliver, indicating that the optimal hedge ratio should be different from one. Intuitively, information about Y should be incorporated into the estimation since Y could be the cheapest to deliver. Unfortunately, the regression approach ignores this information since, in this case, X was cheapest to deliver during the sample period. In fact, the regression approach does not utilize all available information even when prices do switch during the sample period. A different procedure is required.

Let us assume that P_1^X and P_1^Y follow a bivariate Gaussian distribution:

$$\begin{pmatrix} P_1^X \\ P_1^Y \end{pmatrix} \sim N_2\left(\begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}, \begin{pmatrix} \sigma_X^2 & \sigma_{XY} \\ \sigma_{XY} & \sigma_Y^2 \end{pmatrix}\right).$$

The optimal hedge ratio can then be expressed in terms of the parameters of the distribution of prices at time 1 using the results of the Theorem in the appendix:

$$h^* = \frac{\sigma_X^2 - (\sigma_X^2 - \sigma_{XY})N(-\zeta)}{\sigma_X^2 + (\sigma_Y^2 - \sigma_X^2)N(-\zeta) + \dfrac{\mu^2}{4} - \lambda^2}, \tag{4}$$

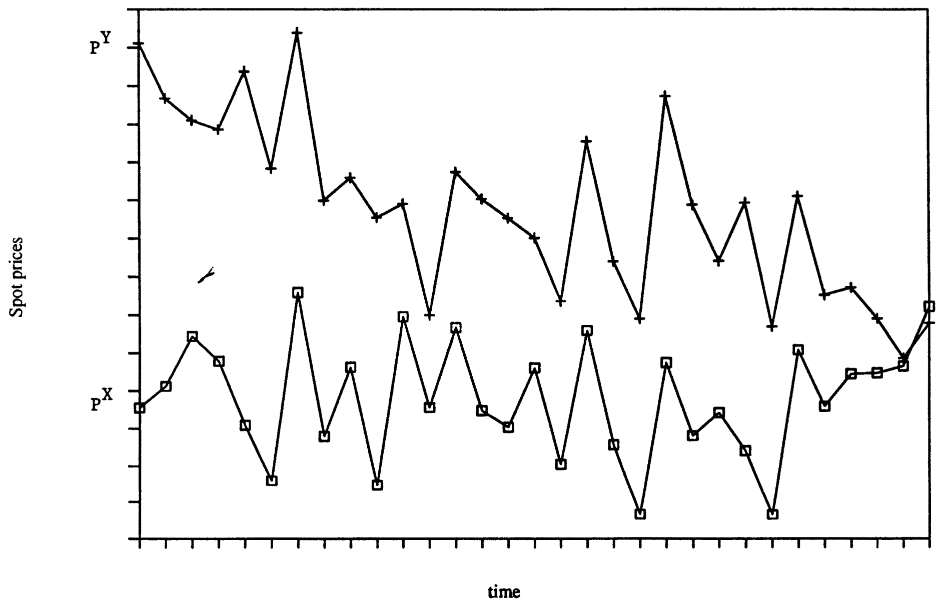

Figure 1. An Example of the Need for an Efficient Estimation Method

where

$\lambda = \frac{\mu}{2} - \mu N(-\zeta) + \mu\phi(\zeta)/\zeta,$

$\zeta = \mu/\sigma_{(Y-X)} = \mu/(\sigma_X^2 + \sigma_Y^2 - 2\sigma_{XY})^{1/2},$

$\mu = \mu_Y - \mu_X,$

$N(t) = \int_{-\infty}^{t} \phi(\tau)\, d\tau$, the cumulative distribution function of a function of a standard Gaussian, and

$\phi(t) = \sqrt{2\pi}e^{-t^2/2}$, the density function of a standard Gaussian.

Note that ζ is the standardized expected spot price spread at time 1. $N(-\zeta)$ is the probability that X, the hedged asset, will not be the cheapest to deliver:

$$\begin{aligned}\Pr(P_1^X > P_1^Y) &= \Pr\left\{\frac{(P_1^Y - \mu_Y) - (P_1^X - \mu_X)}{\sigma_{(Y-X)}} < -\frac{\mu_Y - \mu_X}{\sigma_{(Y-X)}}\right\}\\ &= N\left(-\frac{\mu_Y - \mu_X}{\sigma_{(Y-X)}}\right)\\ &= N(-\zeta).\end{aligned}$$

The minimum variance associated with the optimal hedge is given by

$$\begin{aligned}V(h^*) &= \sigma_X^2 - \frac{\{\mathrm{cov}[P_1^X, \min(P_1^X, P_1^Y)]\}^2}{V[\min(P_1^X, P_1^Y)]}\\ &= \sigma_X^2 - \frac{\{\sigma_X^2 + (\sigma_{XY} - \sigma_X^2)N(-\zeta)\}^2}{\sigma_X^2 + (\sigma_Y^2 - \sigma_X^2)N(-\zeta) + \frac{\mu^2}{4} - \lambda^2}\\ &= \sigma_X^2 - h^*\{\sigma_X^2 + (\sigma_X^2 - \sigma_{XY})N(-\zeta)\}. \end{aligned} \tag{5}$$

The hedging effectiveness of a futures market is defined as the percent reduction in the variance achieved by hedging optimally in the futures market rather than not hedging at all. The hedging strategy currently used by academic researchers and market participants for hedges held to delivery is a conversion-factor strategy. Since either X or Y can be delivered at par, these hedge ratios are unity for either asset. The increase in hedging effectiveness achieved by optimally hedging the delivery risk is, therefore,

$$\frac{V(1) - V(h^*)}{V(1)} = \frac{(1 - h^*)^2}{1 - 2h^* + \dfrac{h^*\sigma_X^2}{\{\sigma_X^2 - (\sigma_X^2 - \sigma_{XY})N(-\zeta)\}}}, \tag{6}$$

where $V(h^*)$ is the variance on the optimal hedged portfolio and $V(1)$ is the variance on a fully hedged portfolio.

II. The Equal-Variance Case

The special case in which the variances of the spot prices of the two commodities are equal represents a reasonable approximation for a situation in which X and Y represent different qualities of a given asset. Not only does this assumption significantly simplify the computational formulas and facilitate interpretation, but it can also improve statistical stability by allowing a single pooled-variance estimate to substitute for two separate (and less precise) variance estimates.

Let

$$\sigma_X^2 = \sigma_Y^2 = \sigma^2$$

and

$$\rho = \frac{\sigma_{XY}}{\sigma^2}.$$

Using the fact that $\mu = \sigma\zeta\sqrt{2(1-\rho)}$, we may write the optimal hedge as a function of ρ and ζ as

$$h^* = h^*(\rho, \zeta) = \frac{1-(1-\rho)N(-\zeta)}{1+\frac{1-\rho}{2}\zeta^2 - \frac{1-\rho}{2}\{\zeta - 2\zeta N(-\zeta) + 2\phi(\zeta)\}^2}. \tag{7}$$

Note that $N(-\zeta)$, the probability that X (the hedged asset) will be more expensive at time 1, is a monotonically decreasing function of ζ. Consequently, we may think of the optimal hedge as a function of the correlation ρ and the probability that the hedged quality will be more expensive. Figure 2 shows the

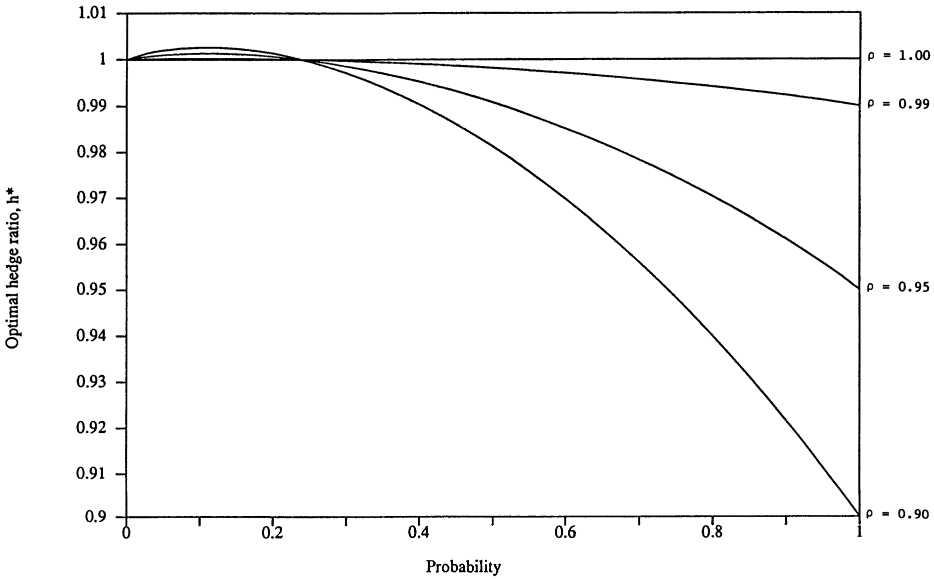

Figure 2. Optimal Hedge Ratios as a Function of the Probability That the Hedged Quality Will Be More Expensive

optimal hedge ratio for X as a function of the probability $N(-\zeta)$ for correlations of $\rho = 1$, 0.99, 0.95, and 0.90. The optimal hedge ratio is close to one when the correlation is very close to one (regardless of the probability that X will be more expensive) and when the probability that X will be most expensive is small (regardless of the correlation). The optimal hedge differs most from the full hedge when the hedged commodity is likely to be more expensive and the correlation is low.

Define $P_1^X - \min(P_1^X, P_1^Y)$ as the basis at time 1. In the case of perfect correlation, i.e., $\rho = 1$, $P_1^X - P_1^Y$ is constant since $\sigma_X = \sigma_Y$; consequently, the basis is constant and the full hedge is optimal and riskless. In the case when $N(-\zeta) = 0$, X is certain to be the cheapest to deliver; again, a full hedge of X is riskless and optimal. The optimal hedge ratio also equals one (for any ρ) when the probability that the hedged quality will not be cheapest to deliver equals 0.239. The optimal hedge ratio is greater (smaller) than one when this probability is smaller (greater) than 0.239. Finally, in the case when $N(-\zeta) = 1$, X is certain not to be the cheapest asset to deliver; the optimal hedge ratio in this case is $\mathrm{cov}(P_1^X, P_1^Y)/V(P_1^Y)$ or ρ when the variances are equal.

From the point of view of hedging theory, note that the variance of the hedger's return can also be written as

$$V\{\text{basis} - (h - 1)\min(P_1^X, P_1^Y)\}.$$

That is, the hedge's variance contains a basis variance and a minimum-spot-price variance. A risk-averse hedger will, therefore, choose an optimal hedge ratio greater (smaller) than unity when the basis and the minimum spot price are positively (negatively) correlated.

The increase in the hedging effectiveness achieved by hedging optimally rather than fully is displayed in Figure 3. This shows the increase in hedging effectiveness:

$$\frac{V(1) - V(h^*)}{V(1)} = \frac{(1 - h^*)^2}{1 - 2h^* + \dfrac{h^*}{1 - (1 - \rho)N(-\zeta)}}$$

A comparison of the value of the optimal hedge h^* in Figure 2 with the increase in its effectiveness in Figure 3 shows that the increase is greater, the greater the difference between the optimal hedge and the full hedge. For example, if the correlation is 0.9 and the hedged asset has a twenty-five percent probability of being the cheapest to deliver, then the variance would be reduced by a factor of 2.1 percent by hedging optimally rather than fully.

III. An Evaluation of the Hedging Strategies

This section compares the effectiveness of the optimal hedging strategies with a full hedge and with a no-hedge strategy. The comparison with a no-hedge strategy evaluates the hedging benefits of the wheat futures market. The strategies are evaluated in hedging either number-two soft red wheat or number-two hard red winter wheat in Chicago. The hedges are held over periods of two and four weeks,

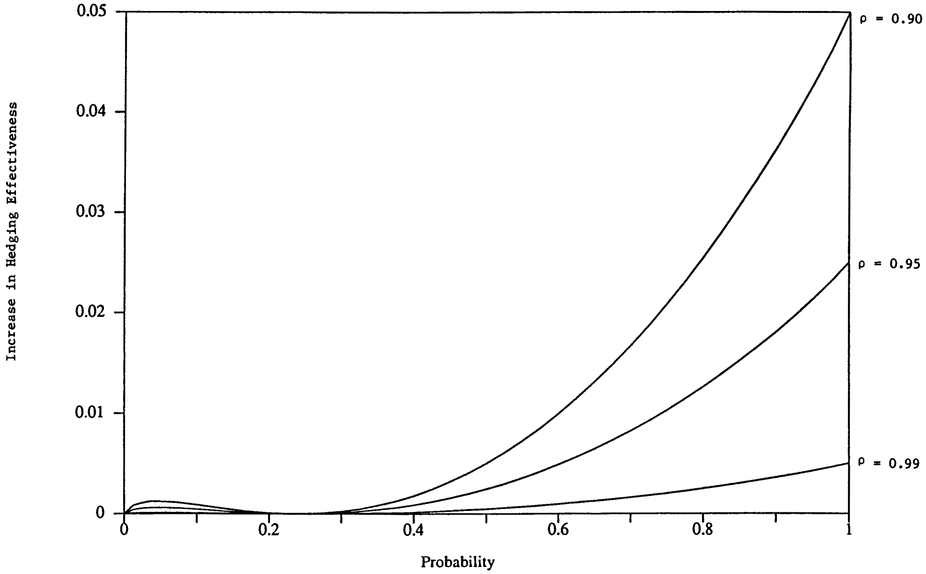

Figure 3. The Increase in Hedging Effectiveness as a Function of the Probability That the Hedged Quality Will Be More Expensive

respectively, and are closed on the first delivery day. By assuming that the hedges are held until delivery, we are able to concentrate only on the basis risk due to delivery uncertainty.

Our data sample consists of daily futures and spot prices of number-two soft red and number-two hard red winter wheat in Chicago from January 1970 through March 1981. The prices were taken from the various issues of the Chicago Board of Trade *Statistical Annual* [1].[4] The futures price used here is the settlement price. This price is usually the price of the last recorded transaction of the day. The futures market closes at 1:15 P.M. Central time. The spot prices are Chicago elevator bid prices and are quoted at 1:15 P.M. Central time. Daily transaction spot price series for that period are unavailable. The effect of the measurement error on our results is unclear.[5]

The futures contracts on wheat traded on the Chicago Board of Trade have deliveries scheduled every March, May, July, September, and December. Consequently, we have fifty-four observations for either hedging period. The contract allows eleven different qualities of wheat to be delivered at either Chicago or

[4] Our sample ends on March 1981 since we have found many mistakes in the spot prices reported thereafter.

[5] Let P^T and P^O represent true and observed spot prices, respectively. The variance of the observed basis, representing the delivery basis risk, is thus

$$\begin{aligned} V(B^O) &= V(P^O - f) = V(P^O - P^T + P^T - f) \\ &= V(e + B^T) = V(e) + V(B^T) + 2\operatorname{cov}(e, B^T), \end{aligned}$$

where B^O and B^T are the true basis and observed basis, respectively, and e is the measurement error. Thus, the effect depends on the sign of $\operatorname{cov}(e, B^T)$ and its magnitude relative to $V(e)$.

Toledo, at any time during the delivery month. However, Gay and Manaster [8] report that most wheat deliveries consist of number-two hard red winter or number-two soft red and are made at Chicago. Either of these two qualities can be delivered at par to satisfy the futures contract. Each contract is on 5000 bushels.

Swaps of soft and hard wheat on the first delivery date would have generated an average instantaneous (nonannualized) rate of return of 2.74 percent of the minimum spot price (with a standard error of 0.36 percent). The highest rate of return in our sample period was 8.66 percent on the December 1973 contract.

The empirical experiment was conducted in two steps. In the first step, the optimal hedge ratios were calculated for each time a hedge was placed. The hedge ratios were estimated from spot prices over the twenty business days preceding the hedging day. In the second step, the hedges were executed and the rates of return were calculated. Consequently, our study differs from Ederington [3], Figlewski [5], and Koppenhaver [16]. Those studies evaluate an ex post effectiveness. They assume that the hedgers have perfect foresight with respect to spot and futures prices. Our study, on the other hand, evaluates an ex ante hedging effectiveness.

Let us assume that

$$\begin{pmatrix} \Delta P^X \\ \Delta P^Y \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}\right),$$

where ΔP^X and ΔP^Y are the daily changes in soft and hard wheat prices, respectively, and σ_1, σ_2, and σ_{12} define the variance-covariance matrix of daily spot price changes. (Labys and Granger [17] provide evidence that daily spot prices of many commodities, including wheat, follow a martingale process.) Assuming further that the daily changes are independent and identically distributed over time, the price changes from the trading day until the closing day of the hedge, T days ahead, are

$$\textstyle\sum_{t=1}^{T} \begin{pmatrix} \Delta P^X \\ \Delta P^Y \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, T\begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}\right).$$

The corresponding sample estimates s_1, s_2, and s_{12} (based on the preceding twenty business days) were used to calculate the optimal hedge ratios.

Our optimal hedge ratios, however, suffer from estimation errors. To reduce the likelihood and severity of these errors, we have also estimated the optimal hedge under the assumption that the two spot prices have equal variances. That is, after estimating s_1^2, s_2^2, and s_{12} separately, we used the simple average of s_1^2 and s_2^2 as the estimated variance of daily changes in either spot price in the hedge-ratio formulas. We tested the assumption that the variances of soft and hard wheat prices are equal and could not reject it at the coventional level.

Table I presents the summary statistics of the optimal hedge ratios in the presence of a delivery risk. The means of the optimal hedge ratios in Table I are always lower than unity. When the variances of spot price changes are allowed to be unequal, the difference from unity is statistically significant at the conventional level (0.05) only for two-week hedges of hard wheat. Yet, when the

Table I

Optimal Hedge Ratios When There is a Delivery Risk with and without the Assumption of Equal Variances

	Soft Wheat		Hard Wheat	
	Unequal Variances	Equal Variances	Unequal Variances	Equal Variances
Two-Week Hedges:				
Mean	0.99	0.96[a]	0.95[b]	0.97[a]
Standard Deviation	0.11	0.06[c]	0.13	0.06[c]
Standard Deviation of Mean	0.02	0.01	0.02	0.01
Highest Value	1.28	1.01	1.15	1.01
Lowest Value	0.43	0.73	0.35	0.72
Four-Week Hedges:				
Mean	0.98	0.96[a]	0.97[b]	0.97[a]
Standard Deviation	0.10	0.06[c]	0.12	0.06[c]
Standard Deviation of Mean	0.01	0.01	0.02	0.01
Highest Value	1.17	1.01	1.27	1.01
Lowest Value	0.57	0.69	0.34	0.68

Note: Sample period: January 1970 through March 1981; a total of fifty-four observations in either hedging period.

[a] The optimal hedge ratio is different from unity at a 0.001 significance level.

[b] The optimal hedge is different from unity at a 0.02 significance level.

[c] The variance of the hedge ratios assuming equal variances is smaller than the variance of the hedge ratios assuming unequal variances at a 0.00001 significance level (a two-tailed test).

variances are assumed to be equal, those differences from unity are statistically significant at a 0.0001 level in all the cases. The effect of the equal-spot-variances assumption is clearly to move the estimated hedge ratios closer to their mean. In fact, the difference between the means is statistically insignificant, while the reduction in the variance of the hedge-ratios series is statistically significant at a 0.0001 level for either of the four paired series.[6]

In the second stage, we calculated the rates of return for each strategy assuming a short hedge. The rate of return for the jth hedging strategy is

$$\{(P_1^i - P_0^i) + h_j[f_0 - \min(P_1^X, P_1^Y)]\}/P_0^i,$$

where $i = X, Y$.

Table II reports the summary statistics of the rates of return of the four strategies for either spot quality. It is important to note that these rates are not

[6] The equality of the means is tested using a paired t-test. The test of equal variances for two series A and B, when A and B may be correlated contemporaneously but either series is independent over time, is

$$t_{n-2} = \frac{\sigma_A^2 - \sigma_B^2}{\sqrt{(1 - r_{AB}^2)\sigma_A^2\sigma_B^2}} \frac{\sqrt{n-2}}{2},$$

where n is the number of observations in either series and r_{AB} is the coefficient of correlation. This test is equivalent to testing whether the correlation between $A + B$ and $A - B$ is zero. Further details can be found in Morgan [18].

Table II

Effectiveness of Chicago Wheat Futures in Hedging Soft and Hard Wheat at Chicago

	Unhedged	Full Hedge	Optimal Hedge	Optimal Hedge with Equal Variances
Two-Week Hedges:				
Soft Wheat:				
Mean Return	−0.31[a]	1.92	2.19	1.91
Standard Deviation	5.07	4.02	3.86	3.93
Effectiveness	—	37.00	41.82	39.75
Hard Wheat:				
Mean Return	−0.31[a]	1.93	1.81	1.88
Standard Deviation	5.08	4.00	3.95	3.96
Effectiveness	—	38.29	39.46	39.44
Four-Week Hedges:				
Soft Wheat:				
Mean Return	1.06	1.73	1.69	1.81
Standard Deviation	7.35	4.28	4.07	4.04
Effectiveness	—	66.01[b]	69.28[b]	69.83[b]
Hard Wheat:				
Mean Return	1.35	1.98	1.97	2.01
Standard Deviation	7.34	4.08	4.08	3.92
Effectiveness	—	69.09[b]	69.11[b]	71.38[b]

Note: Mean and standard deviations are in percentages over their respective periods; they are not annualized. Effectiveness is the percent reduction in variance relative to an unhedged position. Sample period: January 1970 through March 1981.

[a] The mean rate of return of the unhedged strategy is significantly lower than the means of either of the other strategies at a 0.01 level.

[b] The variance of the rate of return of this strategy is significantly lower than that of the unhedged strategy at a 0.0002 significance level (a two-tailed test).

annualized rates but are expressed per their respective hedging periods. The first conclusion is that not hedging in the futures market is suboptimal for either quality. Hedging would have increased the mean of the rate of return and decreased its standard deviation. By hedging, holders of either quality could have increased the mean rate of return by at least 0.6 percent in the four-week case and at least two percent in the two-week case. The increase in the mean rate of return in each of the two-week cases is statistically significant at a one percent level (with a two-tailed test). At the same time, hedging would have reduced approximately forty percent of the variance in the two-week case and approximately seventy percent in the four-week case. The reduction in the variances of the rate of return in each of the four-week cases is statistically significant at a 0.0002 level (with a two-tailed test). This represents the hedging benefits of the wheat futures market even in the presence of a delivery risk.

Comparing the optimal hedging strategies with the full hedge, we find that the mean rates of return for the optimal hedges are sometimes higher and sometimes lower than that of the full hedge, but none of these differences is statistically significant at the ten percent level. More importantly, however, the standard

Table III

Hedging Effectiveness of the Optimal Hedges Relative to a Full Hedge

	Soft Wheat		Hard Wheat	
	Unequal Variances	Equal Variances	Unequal Variances	Equal Variances
Two-Week Hedges:	7.64	4.35	1.89	1.86
t-Statistics	0.86	2.12	0.41	1.19
Significance Level	0.39	0.04	0.68	0.24
Four-Week Hedges:	9.60	11.20	0.12	7.45
t-Statistics	1.71	3.28	0.03	3.62
Significance Level	0.09	0.00	0.98	0.00

Note: Sample period: January 1970 through March 1981.

deviations for the optimal hedging strategies are always lower than that of the full hedge.

Table III reports the percent reduction in variance achieved by optimally hedging the delivery risk rather than fully hedging. This reduction represents the increase in the hedging effectiveness achieved by minimizing the delivery risk in the wheat futures market. The percent reduction is always greater for soft wheat than for hard wheat. By hedging optimally, traders of soft wheat could have reduced the variance by 4.35 to 7.64 percent in the two-week hedges and by 9.6 to 11.2 percent in the four-week hedges. Traders in hard wheat could have reduced the variance by approximately 1.9 percent in the two-week hedges and 0.12 to 7.45 percent in the four-week hedges. The reduction in variance is statistically significant (a two-tailed test) when the spot variances are assumed to be equal in both hedging periods for soft wheat and in four-week hedges of hard wheat. The reduction is always insignificant (at the 0.05 level) when the variances of spot prices are allowed to be unequal.[7]

Finally, the increases in hedging effectiveness are greater for four-week hedges than for two-week hedges. This is consistent with theoretical predictions and with the empirical evidence in Gay and Manaster [8] since longer time to delivery is associated with greater uncertainty regarding the relative spot prices on delivery date.

The finding that "minimizing the delivery risk has a greater effect on soft wheat hedgers than on hard wheat hedgers" provides further support to the evidence reported in Kamara [12]. Kamara reports that delivery risk is more important in pricing Chicago wheat futures contracts relative to soft wheat than

[7] It is interesting to note that the assumption that the variances of the two spot prices are equal to each other has produced more effective hedges in the four-week hedges but less effective hedges in the two-week hedges. Yet the increases in the effectiveness relative to the full hedge are always more statistically significant when the variances are assumed to be equal. The assumption of equal spot-price variances has eliminated the extreme and unreasonable highs and lows of the general optimal hedge ratio. This has increased the correlation between the rates of returns of the optimal hedge and the full hedge, thereby increasing the values of the *t*-statistics. The differences in either the means or variances of the rates of return on the optimal strategies (assuming either equal or unequal spot variances) are, however, statistically significant (at a 0.05 level) in all the cases.

relative to hard wheat. The result is interesting. The Chicago wheat futures market is widely regarded as a soft wheat market and, during 1970 through 1981, has clearly been dominated by soft wheat hedgers (Gray and Peck [10]). Yet hard wheat prices were lower than or equal to soft wheat prices, on the first delivery date, in thirty-four out of the fifty-four observations (sixty-three percent) in our sample. Consequently, changes in wheat futures prices were associated more with changes in hard wheat prices than with changes in soft wheat prices.

IV. Conclusions

We have derived the optimal hedging strategies in futures markets allowing delivery of more than one quality of the underlying asset. An ex ante evaluation of hedging effectiveness across four different strategies and two hedging periods demonstrates the importance of the delivery uncertainty in formulating hedging strategies in futures markets. The use of optimal hedging strategies would have led to a significant increase in the effectiveness of the hedges. The rates of returns on the optimal hedges have significantly lower variances but similar means.[8]

The applications of the hedging strategies developed in this paper extend well beyond the wheat futures market. Multiple delivery specifications exist in nearly all commodity and financial futures contracts. Our study demonstrates that hedging the delivery risk improves the hedge performance significantly even in a market where the delivery uncertainty is relatively small. Soft and hard wheat prices have tracked each other very closely. Their prices have switched from higher to lower very infrequently, and price spreads were relatively small. Previous studies on the importance of the delivery uncertainty suggest that it is greater in financial futures contracts than in commodity futures contracts. The price spreads are larger, and it is more difficult to predict which quality will be the cheapest to deliver. Therefore, we expect that these optimal hedging strategies would be even more effective for hedgers of financial futures contracts.

Appendix

In this appendix, we derive the formula for the optimal hedging ratio. Consider a bivariate normal distribution:

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N_2\left(\begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}, \begin{pmatrix} \sigma_X^2 & \sigma_{XY} \\ \sigma_{XY} & \sigma_Y^2 \end{pmatrix}\right).$$

Also, define $\mu = \mu_Y - \mu_X$, $\sigma_{(Y-X)} = \sqrt{\sigma_X^2 + \sigma_Y^2 - 2\sigma_{XY}}$, and $\zeta = \mu/\sigma_{(Y-X)}$.

[8] Kane and Marcus [14] report, using simulations, that a strategy of simply estimating cov(spot, futures)/V(futures) as the hedge ratio could have significantly improved the hedging performance of traders in the Treasury bond futures markets. We compared the optimal hedging strategy with such a strategy as well. We found that the optimal hedges are significantly more effective than hedge ratios estimated on cov(spot, futures)/V(futures) at less than a two percent level in each of the eight cases examined.

THEOREM: *First- and second-order moments involving min(X, Y) are given by*

(a) $E[min(X, Y)] = \mu_X + \mu N(-\zeta) - \sigma_{(Y-X)}\phi(\zeta)$,

(b) $Var[min(X, Y)] = \sigma_X^2 + (\sigma_Y^2 - \sigma_X^2)N(-\zeta)$

$$+ \frac{\mu^2}{4} - \left[\frac{\mu}{2} - \mu N(-\zeta) + \sigma_{(Y-X)}\phi(\zeta)\right]^2,$$

(c) $cov[X, min(X, Y)] = \sigma_X^2 - (\sigma_X^2 - \sigma_{XY})N(-\zeta)$.

The proof of this theorem will first require two basic lemmas. Let Z be a standard normal random variable with mean zero and variance one. Define the indicator function $I(Z < \alpha)$ to be one if $Z < \alpha$ and zero otherwise.

LEMMA 1: *Moments involving Z and $I(Z < \alpha)$ are given by*

(1) $E[ZI(Z < \alpha)] = -\phi(\alpha)$

(2) $E[Z(Z - \alpha)I(Z < \alpha)] = N(\alpha)$.

Proof: This follows from definitions and integration by parts.

LEMMA 2: *Let U and V be jointly normally distributed with $E(U) = 0$:*

$$\begin{pmatrix} U \\ V \end{pmatrix} \sim N_2\left(\begin{pmatrix} 0 \\ \mu_V \end{pmatrix}, \begin{pmatrix} \sigma_U^2 & \sigma_{UV} \\ \sigma_{UV} & \sigma_V^2 \end{pmatrix}\right).$$

Then

$$E[UVI(V < 0)] = \sigma_{UV}N(-\mu_V/\sigma_V).$$

Proof: Write U as a linear combination of $(V - \mu_V)$ and a term statistically independent of V. Substitute $Z = (V - \mu_V)/\sigma_V$, and use Lemma 1, Part (2), with $\alpha = -\mu_V/\sigma_V$.

Proof of Theorem: Define $T = Y - X$, which is normally distributed with mean μ and standard deviation $\sigma_{(Y-X)}$. Define $Z = (T - \mu)/\sigma_{(Y-X)}$, which is standard normal. We will use the following representation of the minimum:

$$\min(X, Y) = X + TI(T < 0).$$

Part (a) follows from Lemma 1. To prove (b), observe that

$$\text{var}[\min(X, Y)] = \sigma_X^2 + 2\,\text{cov}[X, TI(T < 0)] + \text{var}[TI(T < 0)].$$

Evaluate the covariance term using Lemma 2 with $U = X - \mu_X$. The variance term may be written as

$$\text{var}[TI(T < 0)] = \text{E}[(T - \mu)TI(T < 0)] + \mu\text{E}[TI(T < 0)] - \{\text{E}[TI(T < 0)]\}^2.$$

Applying Lemma 2 to the first term above with $U = T - \mu$ and $V = T$, then completing the square, we find

$$\text{var}[\min(X, Y)] = \sigma_X^2 + 2\sigma_{XT}N(-\zeta) + \sigma_{(Y-X)}^2N(-\zeta) + \mu^2/4 - \left\{\frac{\mu}{2} - \text{E}[TI(T < 0)]\right\}^2.$$

Substitute for the remaining expectation from Part (a); use the result to establish (b). Part (c) follows from Lemma 2 with $U = X - \mu_X$ and $V = T$.

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