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Author(s): Edward Omberg
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A Note on the Convergence of Binomial-Pricing and Compound-Option Models

EDWARD OMBERG*

THE BINOMIAL-PRICING MODEL of Cox, Ross, and Rubinstein [2] and the compound-option model of Geske [3] are both elegant and self-contained frameworks for the valuation of American options and related contingent claims, each providing us with a distinctive valuation procedure flexible enough to be adapted to a wide variety of problems.

For American-type options on assets with (a) prices following a lognormal diffusion process and (b) a continuous-time exercise policy, such as put options on individual stocks, calls and puts on foreign exchange, and calls and puts on futures contracts, the binomial-pricing model approximates both the stochastic process and the exercise-opportunity set, the latter by restricting exercise to a discrete set of dates. The compound-option model, on the other hand, approximates only the exercise-opportunity set. However, it pays for this with a high price in computational requirements; specifically, with N exercise dates allowed, multivariate normal probabilities of every order up to and including N must be evaluated.

In their article on the valuation of put options, Geske and Johnson [4] provide an ingenious solution to this problem by applying acceleration techniques common in numerical analysis but new to the finance literature. Letting T be the time to expiration of the option and P_n the value of a compound option with exercise permitted at the n uniformly spaced dates $[T/n, 2T/n, \cdots, T]$, they generate the sequence of compound-option values $[P_1, P_2, \cdots, P_N]$. Polynomial extrapolation is then used to estimate $P = \lim n \rightarrow \infty P_n$. They found this procedure to be very accurate over a limited sample of put options with low values of N.

Though not discussed by Geske and Johnson, the reasons for the effectiveness of acceleration techniques in a compound-option model are apparent from a cursory examination. An acceleration technique will perform best when the approximating sequence converges *uniformly* on the true value of the contingent claim and worst when convergence includes complex and unpredictable oscillations. Since a compound-option model does not approximate the stochastic process for the price of the underlying asset but approximates the exercise-opportunity set by restricting it to a suboptimal set of discrete times, the values generated are lower bounds on the true value of the option; convergence is therefore always from below. In addition, it will be *uniformly* from below if each opportunity set in the sequence is at least as good as the previous one. This is not necessarily true of the Geske-Johnson sequence. For example, the three-

* Leavey School of Business, Santa Clara University.

point opportunity set $[T/3, 2T/3, T]$ used to compute P_3 does not include the previous two-point opportunity set $[T/2, T]$ and can therefore be inferior. The result may be a sequence of values $P_1 \leq P_2 > P_3$, i.e., nonuniform convergence.[1]

The difficulty above, however, is not fundamental to the compound-option-with-acceleration procedure. We can always construct the approximating sequence so that each opportunity set includes the previous one, and therefore is at least as good, by using the geometric sequence $[1, 2, 4, 8, \cdots]$ generated by successively doubling the number of uniformly spaced exercise dates, rather than the arithmetic sequence $[1, 2, 3, 4, \cdots]$ employed by Geske and Johnson. Uniform convergence from below is therefore assured, providing us with a priori confidence that acceleration methods will perform as intended.

With binomial-pricing models, on the other hand, convergence is often highly oscillatory.[2] Moreover, the oscillations are a by-product of approximating the stochastic process rather than the exercise policy; this is evident from the fact that they occur even with European options. If binomial-pricing models are necessarily subject to complex and unpredictable oscillations, the use of acceleration methods in conjunction with them seems doomed to failure. However, the binomial-pricing method has three independent parameters (for example, the up probability and the up and down jumps) subject to only two constraints: that the mean and variance converge in the limit to those of the appropriate lognormal diffusion process. Consequently, the selection of parameters has one degree of freedom in the limit and more for a finite number of steps. So far, only several of infinitely many acceptable parameter sets have appeared in the literature; specifically, those of Cox, Ross, and Rubinstein [2], Rendleman and Bartter [7], and Jarrow and Rudd [5]. The oscillations observed so far may therefore result from the employment of particular parameter choices, rather than being an inherent property of the binomial-pricing method.

We have, then, the attractive possibility that the binomial parameters can be chosen so as to guarantee uniform convergence. Acceleration methods could then be used with both binomial-pricing and compound-option models. With acceleration speeding the convergence of both, binomial-pricing models might prove to be considerably more efficient than compound-option models. In light of the discussion above, the type of uniform convergence we look for is uniform convergence from below when the number of equally spaced exercise dates is doubled; i.e., $C_{2n}(S) \geq C_n(S)$, where $C_n(S)$ is the option-valuation function with n time steps. The property with which we seek to assure uniform convergence from below is convexity of the option-valuation function; in addition to being the only property that holds for both calls and puts, it leads naturally to inequality relations.

We investigate this possibility and arrive at a negative result. In particular, we seek to find a choice of binomial-pricing parameters such that (a) the parameters

[1] A plausible instance of the above would be a deep-in-the-money put written on a low-volatility, high-dividend stock going ex-dividend once during the term of the option at $t = T/2$. There is then a high probability that the option owner will want to exercise at $T/2$ immediately after the stock goes ex-dividend. The two-date opportunity set $[T/2, T]$ may therefore capture the resulting value better than the three-date set $[T/3, 2T/3, T]$.

[2] An example is shown in Copeland and Weston [1, p. 259].

have a Taylor series expansion in $\sqrt{\Delta t}$, (b) the binomial process converges to the appropriate lognormal diffusion process, and (c) uniform convergence from below is assured for arbitrary contingent claims by the convexity of the valuation function. We show by contradiction that no such choice exists. The rest of the paper is devoted to the proof.

Consider the valuation of an American option for which the stock price follows the lognormal diffusion process

$$d(\ln S) = \mu dt + \sigma dz(t), \tag{1}$$

where μ is the drift rate of the process and $z(t)$ is a standard Gauss-Wiener process with a drift rate of zero and a variance rate of one. For a risk-neutral valuation of the contingent claim, a drift rate of $\mu' = r - \sigma^2/2$ is substituted for the actual drift rate μ of the asset price, where r is the instantaneous risk-free rate of return.

Approximate the diffusion process above by a binomial process with n steps. Letting T be the time to maturity of the option and letting $\Delta t = T/n$, the three independent binomial parameters are $p_n = p(\Delta t)$, $u_n = u(\Delta t)$, and $d_n = d(\Delta t)$, where p_n is the probability of a movement up and u_n and d_n are the magnitudes of the proportional up and down movements. Current choices for these parameters are shown in Table I.

The CRR parameters differ from the JR and RB parameters in that, with the latter two, the binomial approximation has the same mean and variance as the underlying lognormal diffusion process for any step size; the CRR parameters, on the other hand, result in the same mean for any step size but the same variance only in the limit. The RB parameters are more general than those of JR but reduce to the JR parameters with $k = 1$; RB argue in favor of this choice of k. The parameter choices above can be regarded as coming from a broader family of the form

$$p = f(\Delta t) \tag{2a}$$

$$\ln u = g(\Delta t) \quad \text{or} \quad u = e^{g(\Delta t)} \tag{2b}$$

$$\ln d = h(\Delta t) \quad \text{or} \quad d = e^{h(\Delta t)}, \tag{2c}$$

with each function having a Taylor series expansion in $\sqrt{\Delta t}$

$$p = p_0 + f_1\sqrt{\Delta t} + f_2\Delta t/2 + \cdots \tag{3a}$$

$$\ln u = g_1\sqrt{\Delta t} + g_2\Delta t/2 + \cdots \tag{3b}$$

$$\ln d = -h_1\sqrt{\Delta t} + h_2\, \Delta t/2 + \cdots \tag{3c}$$

Table I

Current Choices of Binomial-Pricing Parameters

	$\ln u$	$\ln d$	p
Cox, Ross, and Rubinstein (CRR)	$\sigma\sqrt{\Delta t}$	$-\sigma\sqrt{\Delta t}$	$1/2 + (\mu'/2\sigma)\sqrt{\Delta t}$
Jarrow and Rudd (JR)	$\mu'\Delta t + \sigma\sqrt{\Delta t}$	$\mu'\Delta t - \sigma\sqrt{\Delta t}$	$1/2$
Rendleman and Bartter (RB)	$\mu'\Delta t + k\sigma\sqrt{\Delta t}$	$\mu'\Delta t - k^{-1}\sigma\sqrt{\Delta t}$	$1/(1 + k^2)$

In order for the binomial process to converge to the lognormal diffusion process, the Taylor series coefficients above must be selected so that the binomial process in the limit has the same first and second moments as the diffusion process; i.e.,

$$\lim_{\Delta t\to 0}\mathrm{E}(\ln x) = \lim_{\Delta t\to 0}[p\ln u + (1-p)\ln d] = \mu'\Delta t \tag{4a}$$

$$\lim_{\Delta t\to 0}\mathrm{E}(\ln^2 x) = \lim_{\Delta t\to 0}[p\ln^2 u + (1-p)\ln^2 d] = \sigma^2\Delta t + (\mu'\Delta t)^2. \tag{4b}$$

The above imply that

$$p_0 g_1 + (1-p_0)(-h_1) = 0 \tag{5a}$$

$$p_0 g_2/2 + f_1 g_1 + (1-p_0)h_2/2 + f_1 h_1 = r - \sigma^2/2 \tag{5b}$$

$$p_0 g_1^2 + (1-p_0)h_1^2 = \sigma^2. \tag{5c}$$

The first of the three conditions above results from noting that the coefficient of $\sqrt{\Delta t}$ in (4a) must vanish; otherwise, $\mathrm{E}(\ln x)$ would be proportional to $\sqrt{\Delta t}$ rather than Δt in the limit, and the drift rate of the process $(1/\Delta t)\mathrm{E}(\ln x)$ would approach infinity as Δt approached zero. Together, (5a) and (5c) are equivalent to

$$p_0 = h_1/(g_1 + h_1) \tag{6a}$$

$$\sigma^2 = g_1 h_1. \tag{6b}$$

For both the CRR and JR parameter sets, $g_1 = h_1 = \sigma$ and $p_0 = ½$; for the RB parameter set, $g_1 = k\sigma$, $h_1 = k^{-1}\sigma$, and $p_0 = 1/(1 + k^2)$. All three parameter sets satisfy the above conditions.

We now commence with the proof. Consider two valuations of the option, one with n time steps of length $\Delta t = T/n$ and a second one with $2n$ steps of length $\Delta t/2$. For the first valuation, the binomial-process parameters are $p_n = f(\Delta t)$, $\ln u_n = g(\Delta t)$, and $\ln d_n = h(\Delta t)$; for the second valuation, $p_{2n} = f(\Delta t/2)$, $\ln u_{2n} = g(\Delta t/2)$, and $\ln d_{2n} = h(\Delta t/2)$. The hypothesis under examination is that the functions $p = f(\Delta t)$, $\ln u = g(\Delta t)$, and $\ln d = h(\Delta t)$ can be chosen so that, when the number of steps is doubled, the option's value is at least as great as it was previously.

Let $C(S)$ be the valuation function for the option at time t and $G(S)$ its valuation function at time $t + \Delta t$. The desired property is $C_{2n}(S) \geq C_n(S)$, where the subscript denotes the number of binomial steps over the term $[0, T]$ of the option. An improved value $C_{2n}(S) > C_n(S)$ may result from the additional exercise opportunity at $t + \Delta t/2$; however, we cannot count on this and therefore ignore it. Additionally, we simplify notation by ignoring discounting from $t + \Delta t$ back to t; there is no loss of generality since the discounting is the same for both valuations.

Finally, we assume that the hypothesis under consideration is true "so far"; that is, as we perform a recursive valuation from expiration T backwards, the hypothesis has been true up to time $t + \Delta t$. We now seek to prove that it holds for time t as well. Since the hypothesis is true "so far," $G_{2n}(S) \geq G_n(S)$. To be safe, we assume the worst case with respect to continuation of the hypothesis, namely that $G_{2n}(S) = G_n(S) = G(S)$; we know this to be true in at least one case, when $t + \Delta t = T$ so that $G(S)$ is the value of the option at expiration.

Since we have ignored exercise opportunities and discounting, the option-valuation functions at t are given by

$$C_n(S) = p_n G(Su_n) + q_n G(Sd_n) \tag{7a}$$

$$C_{2n}(S) = p_{2n}^2 G(Su_{2n}^2) + 2p_{2n}q_{2n}G(Su_{2n}d_{2n}) + q_{2n}^2 G(Sd_{2n}^2), \tag{7b}$$

where $q_n = 1 - p_n$ and $q_{2n} = 1 - p_{2n}$.

Now suppose that the one-step jumps $[u_n, d_n]$ are convex combinations of the two-step jumps $[u_{2n}^2, u_{2n}d_{2n}, d_{2n}^2]$; i.e.,

$$u_n = \lambda_{uu}u_{2n}^2 + \lambda_{ud}u_{2n}d_{2n} + \lambda_{dd}d_{2n}^2 \tag{8a}$$

$$d_n = \omega_{uu}u_{2n}^2 + \omega_{ud}u_{2n}d_{2n} + \omega_{dd}d_{2n}^2, \tag{8b}$$

with

$$0 \le \lambda_{uu}, \lambda_{ud}, \lambda_{dd} \le 1 \quad 0 \le \omega_{uu}, \omega_{ud}, \omega_{dd} \le 1$$

$$\lambda_{uu} + \lambda_{ud} + \lambda_{dd} = 1 \quad \omega_{uu} + \omega_{ud} + \omega_{dd} = 1.$$

We know this to be true at least in the limit as $\Delta t \to 0$ since then, from (3b, c),

$$d_{2n}^2 = 1 - 2h_1\sqrt{(\Delta t/2)} < u_n = 1 + g_1\sqrt{\Delta t} < 1 + 2g_1\sqrt{(\Delta t/2)} = u_{2n}^2 \tag{9a}$$

$$d_{2n}^2 = 1 - 2h_1\sqrt{(\Delta t/2)} < d_n = 1 - h_1\sqrt{\Delta t} < 1 + 2g_1\sqrt{(\Delta t/2)} = u_{2n}^2. \tag{9b}$$

The options-valuation function $G(S)$ is convex provided that the distribution of returns is independent of the stock price level itself.[3] Thus, we can be assured that

$$C_n(S) \le (p_n\lambda_{uu} + q_n\omega_{uu})G(Su_{2n}^2) + (p_n\lambda_{ud} + q_n\omega_{ud})G(Su_{2n}d_{2n}) + (p_n\lambda_{dd} + q_n\omega_{dd})G(Sd_{2n}^2). \tag{10}$$

The right-hand side of the above expression can be interpreted as the valuation of the option with a compound binomial-trinomial process over the interval $[t, t + \Delta t]$, rather than the original binomial process. Specifically, a binomial process with probabilities $[p_n, q_n]$ is sampled; the result is either of two trinomial processes, with probability distributions $[\lambda_{uu}, \lambda_{ud}, \lambda_{dd}]$ and $[\omega_{uu}, \omega_{ud}, \omega_{dd}]$, respectively. The trinomial process is then sampled to determine a jump taken from the set $[u_{2n}^2, u_{2n}d_{2n}, d_{2n}^2]$. Denoting by $C_n^{BT}(S)$ the value of an option with the above binomial-trinomial process substituted for the original binomial process over the step $[t, t + \Delta t]$, we have the inequality

$$C_n(S) \le C_n^{BT}(S). \tag{11}$$

If we can construct the compound binomial-trinomial process so that

$$C_n^{BT}(S) \le C_{2n}(S), \tag{12}$$

then $C_{2n}(S) \ge C_n(S)$, and uniform convergence from below is guaranteed. We now show by contradiction that we cannot do so.

Comparing (7b) and (10), the only way we can assure (12) for arbitrary

[3] See Merton [6].

contingent claims is for the inequalities

$$p_n \lambda_{uu} + q_n \omega_{uu} \le p_{2n}^2 \tag{13a}$$

$$p_n \lambda_{ud} + q_n \omega_{ud} \le 2p_{2n} q_{2n} \tag{13b}$$

$$p_n \lambda_{dd} + q_n \omega_{dd} \le q_{2n}^2 \tag{13c}$$

to hold. However, each side of the above set of inequalities represents the probabilities of a distribution with the same three outcomes—in one case (left-hand side), the compound binomial-trinomial process, in the other (right-hand side), the two-step binomial process. Both sides therefore sum to one. As a result, all of the above must be equalities. This implies that the compound binomial-trinomial process has both the same probabilities and the same outcomes as the two-step binomial process and, therefore, the same option valuation. Then, if the inequality (12) holds for arbitrary contingent claims, it must be an equality.

Thus, we can guarantee uniform convergence if and only if we can construct the compound binomial-trinomial process so that its probability distribution is identical to two steps of the binomial process. We now show that we cannot do so, at least in the limit as $\Delta t \to 0$.

Consider the limit as $\Delta t \to 0$ of (8a, b) and (13a, b, c), now noting that we know that the latter must hold as equalities. For the former set of equations, the dominant nontrivial terms are of order $\sqrt{\Delta t}$; for the latter set, these terms are of order zero. Neglecting higher order terms results in

$$g_1 = 2\lambda_{uu} g_1/\sqrt{2} + \lambda_{ud}(g_1 - h_1)/\sqrt{2} + 2\lambda_{dd}(-h_1)/\sqrt{2} \tag{14a}$$

$$-h_1 = 2\omega_{uu} g_1/\sqrt{2} + \omega_{ud}(g_1 - h_1)/\sqrt{2} + 2\omega_{dd}(-h_1)/\sqrt{2} \tag{14b}$$

$$p_0 \lambda_{uu} + (1 - p_0)\omega_{uu} = p_0^2 \tag{15a}$$

$$p_0 \lambda_{ud} + (1 - p_0)\omega_{ud} = 2p_0(1 - p_0) \tag{15b}$$

$$p_0 \lambda_{dd} + (1 - p_0)\omega_{dd} = (1 - p_0)^2. \tag{15c}$$

Substituting $\lambda_{ud} = 1 - \lambda_{uu} - \lambda_{dd}$ into (14a) above and rearranging terms give

$$\lambda_{uu} = (\sqrt{2} - 1)g_1/(g_1 + h_1) + h_1/(g_1 + h_1) + \lambda_{dd}. \tag{16}$$

From (6a), $p_0 = h_1/(g_1 + h_1)$; from (6b), g_1 and h_1 are either both strictly positive or both strictly negative so that $g_1/(g_1 + h_1)$ is strictly positive. Since λ_{dd} is non-negative, this implies the inequality

$$\lambda_{uu} > h_1/(g_1 + h_1) = p_0. \tag{17}$$

A second inequality,

$$\lambda_{uu} \le p_0, \tag{18}$$

follows from equation (15a) and the non-negativity of ω_{uu}. Combining both inequalities gives

$$p_0 < \lambda_{uu} \le p_0, \tag{19}$$

which cannot hold. The proof by contradiction is then complete.

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