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Mimicking Portfolios and Exact Arbitrage Pricing

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ABSTRACT

We characterize the sets of mimicking positions with returns that can serve in place of factors in an exact K-factor arbitrage-pricing relation for a set of N assets. All of the sets are K-dimensional nonsingular linear transformations of each other. We interpret three examples of such transformations and discuss empirical considerations. We provide conditions under which the mimicking positions can be expressed as portfolios, and we characterize the relation between mimicking portfolios and the minimum-variance frontier.

THE ARBITRAGE PRICING THEORY (APT) of Ross [16, 17] is a one-period model in which investors believe that the $N \times 1$ vector $\underset{\sim}{r}$ of returns on capital assets satisfies the generating model

$$\underset{\sim}{r} = \underline{E} + B\underset{-}{f} + \underset{\sim}{e}, \tag{1}$$

where $\underset{-}{f}$ is a $K \times 1$ vector of random factors, B is an $N \times K$ matrix of factor loadings, and $\underset{\sim}{e}$ is an $N \times 1$ vector of residuals. Empirical investigations of the APT often involve the formation of portfolios with returns that are intended to mimic the realizations of the K factors (e.g., Lehmann and Modest [14]). Estimates of factors obtained from factor analysis can be interpreted as returns on mimicking portfolios (see Grinblatt and Titman [9]). When factors are prespecified (e.g., Chen, Roll, and Ross [6]), there may also be potential benefits from constructing mimicking portfolios. As explained below, the use of mimicking portfolios allows the researcher to test additional restrictions implied by the pricing theory. This paper characterizes the sets of mimicking positions, provides conditions under which such positions can be expressed as positive-cost portfolios, and discusses several empirical considerations. Relations between mimicking portfolios and the minimum-variance frontier are also investigated.

With no loss of generality, normalize (1) to obtain $\mathrm{E}\{\underset{-}{f}\} = \mathrm{E}\{\underset{\sim}{e}\} = 0$ and $\mathrm{E}\{\underset{-}{f}\underset{-}{f}'\} = I$, where $\mathrm{E}\{\cdot\}$ denotes expectation and I is the identity matrix, so that $\underline{E}$ is the vector of mean returns. Assume further that the matrix B is of rank K. Restrictions on the diagonality of the covariance matrix $\mathrm{E}\{\underset{\sim}{e}\underset{\sim}{e}'\}$ (which is denoted by Z) and on the relation between the eigenvalues of that covariance matrix and those of BB' (as N becomes large) are required for proofs of the APT. (Huberman [10] provides a review of the APT literature.) The covariance matrix of $\underset{\sim}{r}$, denoted

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by V, is assumed to be nonsingular. An additional customary assumption, made throughout this paper, is that $\mathrm{E}\{\underline{e} \mid \underline{f}\} = 0$, which implies that $V = BB' + Z$. Finally, it is assumed that $\underline{E}$ is not proportional to $\underline{i}$, an $N \times 1$ vector of 1's.

Exact arbitrage pricing obtains if

$$\underline{E} = \underline{i}r_0 + B\underline{u}, \tag{2}$$

where r_0 is the return on a riskless asset if it exists, and $\underline{u}$ is a $K \times 1$ vector of risk premiums. Chamberlain [3], Chen and Ingersoll [5], and Connor [7] provide conditions under which (2) holds. Exact arbitrage pricing is the tested form of the APT, e.g., by Roll and Ross [15], Chen [4], Lehmann and Modest [14], and Huberman and Kandel [11].

Grinblatt and Titman [9] discuss the testability of exact arbitrage pricing on a subset of all assets available for investment. Assuming a K-factor structure, they show that maximum-likelihood factor analysis asymptotically identifies K portfolios of riskless and risky assets with returns that can replace the factors in pricing the subset's assets if and only if exact arbitrage pricing holds.

This paper examines further the characteristics of portfolios with returns that can replace the factors in the exact arbitrage-pricing relation. We make a distinction between a portfolio with a cost of 1 and an investment position with a cost that may differ from 1 and even be zero. For a given subset of assets, we characterize the collections of K factor-mimicking positions of risky assets with returns that can be used in place of the factors for pricing the subset's assets. We provide conditions under which a set of mimicking portfolios exists, and we discuss the relations between mimicking portfolios and the minimum-variance frontier. The potential advantages of forming mimicking portfolios for empirical testing are discussed briefly. We then provide three examples of mimicking positions that have an intuitive appeal:

(i) the positions with payoffs that are maximally correlated with each of the factors;
(ii) the positions with payoffs that have minimal residual variance, subject to each position having a given level of one of the factor loadings; and
(iii) the positions with payoffs that are the maximum-likelihood estimates of the factor scores (given B and Z).

I. Mimicking Portfolios and Mimicking Positions

Consider a set of N risky assets, and denote an investment position in these assets by $\underline{v}$, an $N \times 1$ vector. The cost of the position $\underline{v}$ is $\underline{v}'\underline{i}$, its payoff is $\underline{v}'\underline{r}$, and its return is $\underline{v}'\underline{r}/\underline{v}'\underline{i}$ (if its cost is not zero). A portfolio is a position that costs 1. The systematic and unsystematic risks of the position $\underline{v}$ are $\underline{v}'BB'\underline{v}$ and $\underline{v}'Z\underline{v}$. The type-$k$ systematic risk of the position $\underline{v}$ is $\underline{v}'\underline{b}_k\underline{b}_k'\underline{v}$, where $\underline{b}_k$ is the kth column of the matrix B (i.e., the type-k systematic risk is the square of the kth factor loading).

A set of K positions of the N risky assets is represented by an $N \times K$ matrix A, where the columns of A are linearly independent. The kth column of A, $\underline{a}_k$, is the position with a cost and payoff equal to $\underline{a}_k'\underline{i}$ and $\underline{a}_k'\underline{r}$, respectively. Let $\underline{R} =$

$A'\underline{r}$ be the $K \times 1$ vector of the payoffs on the K positions. Let C denote the $N \times K$ covariance matrix, $\operatorname{cov}(\underline{r}, \underline{R})$, and note that $C = VA$.

The positions represented by the matrix A are defined as factor-mimicking positions (of the original N risky assets) if their payoffs can be used in place of the factors for pricing the N assets if exact arbitrage pricing holds. This mimicking property should hold for all values of the vector of risk premiums $\underline{u}$ because a researcher who constructs the mimicking positions typically does not know the risk premiums. Formally, the positions described by the matrix A are factor mimicking if and only if, for any $\underline{u}$, whenever (2) holds, there exists a vector $\underline{w}$ such that

$$\underline{E} = \underline{i}r_0 + C\underline{w}. \tag{3}$$

PROPOSITION 1: *The positions represented by A are factor mimicking if and only if $A = V^{-1}BL$, where L is a nonsingular $K \times K$ matrix.*

Proof: By the above definition, the positions in A are factor mimicking if and only if the column spaces of B and C $(=VA)$ coincide, and this condition is equivalent to that stated in the proposition.

Although mimicking positions always exist, a set of mimicking positions can be transformed to a set of mimicking portfolios only under the characterization given in Proposition 2. That characterization entails the systematic risk of the global minimum-variance portfolio, the portfolio of risky assets with a return variance that is minimal.

PROPOSITION 2: *A set of mimicking portfolios exists if and only if the global minimum-variance portfolio has positive systematic risk.*

Proof: The global minimum-variance portfolio of the N risky assets is $(\underline{i}'V^{-1}\underline{i})^{-1}V^{-1}\underline{i}$. Its systematic risk is $(\underline{i}'V^{-1}\underline{i})^{-2}\underline{i}'V^{-1}BB'V^{-1}\underline{i}$, which is positive if and only if $\underline{i}'V^{-1}B \neq \underline{0}'$. ($\underline{0}$ is the K-dimensional zero vector.) This condition holds if and only if there is a nonsingular $K \times K$ matrix L such that $\underline{i}'V^{-1}BL = \underline{i}_K'$, where $\underline{i}_K$ is a $K \times 1$ vector of 1's. Given Proposition 1, this last condition is both necessary and sufficient for the existence of a set of mimicking portfolios.

II. Mimicking Portfolios and the Minimum-Variance Frontier

In this section, we study the relations between exact arbitrage pricing and the number of portfolios on the minimum-variance frontier of the N risky assets that are portfolios of mimicking portfolios (as defined in Section I). The relations are summarized in Table I.

Relations between exact arbitrage pricing and mean-variance efficiency are obtained by Chamberlain [3] for the infinite economy and by Grinblatt and Titman [9] and Jobson and Korkie [12] for a finite set of assets. Jobson and Korkie consider the special case where the K factors are traded assets (or portfolios) and there exists a risk-free asset with a return that differs from the expected return on the minimum-variance portfolio of risky assets. They show that exact arbitrage pricing implies that the Sharpe-Lintner tangent portfolio is

Table I

Relations between Portfolios of Mimicking Portfolios on the Minimum-Variance Frontier and Exact Arbitrage Pricing

Portfolios of Mimicking Portfolios on the Frontier	Exact Arbitrage Pricing Holds	Exact Arbitrage Pricing Does Not Hold
none	$r_0 = \mathrm{E}\{r_{\mathrm{GMV}}\}$ $\underset{\sim}{i}$ is not in col(B)	possible
one, not GMV	$r_0 \neq \mathrm{E}\{r_{\mathrm{GMV}}\}$ $\underset{\sim}{i}$ is not in col(B)	impossible
one, GMV	impossible	possible
entire frontier	r_0 unrestricted, $\underset{\sim}{i}$ is in col(B)	impossible

Note: Exact arbitrage pricing is given by equation (2). GMV is the global minimum variance portfolio; $\mathrm{E}\{r_{\mathrm{GMV}}\}$ is its expected return; and col(B) is the column space of the matrix B.

a portfolio of the K factors. Grinblatt and Titman define a set of K reference portfolios to be a collection of K portfolios such that no combination of these portfolios is the global minimum-variance portfolio. Their Proposition I shows that exact linear pricing with respect to the K reference portfolios holds if and only if a portfolio of these portfolios is on the mean-variance frontier.

Proposition 3 provides necessary and sufficient conditions for the existence of only one portfolio of mimicking portfolios on the mean-variance boundary, where this portfolio is not the global minimum-variance portfolio. This proposition is an implication of the definition of mimicking portfolios and Proposition I of Grinblatt and Titman. As Proposition I of Grinblatt and Titman and its proof include many parts that are not directly related to our Proposition 3, we provide a short and direct proof for Proposition 3.

PROPOSITION 3: *There exists only one portfolio of mimicking portfolios on the minimum-variance frontier of the risky assets, where this portfolio is not the global minimum-variance portfolio, if and only if* (*a*) *exact arbitrage pricing holds with* r_0 *not equal to the expected return on the global minimum-variance portfolio and* (*b*) $\underset{\sim}{i}$ *is not in the column space of* B.

Proof: (Necessity) Let $\underset{\sim}{x}$ denote the unique frontier portfolio of the mimicking portfolios, and note, using Proposition 1, that

$$\underset{\sim}{x} = V^{-1}B\underset{\sim}{z}, \tag{4}$$

where $\underset{\sim}{z}$ is unique. Recall that $\underset{\sim}{x}$ is on the minimum-variance frontier (and is not the global minimum-variance portfolio) if and only if there exist unique scalars p and q such that

$$\underset{\sim}{E} = p\underset{\sim}{i} + qV\underset{\sim}{x}, \tag{5}$$

where q is nonzero and p does not equal the expected return on the global minimum-variance portfolio. Substituting (4) into (5) gives $\underset{\sim}{E} = p\underset{\sim}{i} + B(q\underset{\sim}{z})$, which is the exact arbitrage-pricing relation. Since $q\underset{\sim}{z}$ is unique, $\underset{\sim}{i}$ is not in the column space of B.

(Sufficiency) Note that the exact arbitrage-pricing relation (2) can be written as $\underline{E} = r_0\underset{\sim}{i} + qV\underset{\sim}{x}$, where $\underset{\sim}{x} = q^{-1}V^{-1}B\underset{\sim}{u}$ and $q = \underset{\sim}{i}'V^{-1}B\underset{\sim}{u}$. To see that $q \neq 0$, premultiply both sides of (2) by $(\underset{\sim}{i}'V^{-1})/(\underset{\sim}{i}'V^{-1}\underset{\sim}{i})$, the weights in the global minimum-variance portfolio. Since the left-hand side does not equal r_0 by assumption, $\underset{\sim}{i}'V^{-1}B\underset{\sim}{u} \neq 0$. Thus, $\underset{\sim}{x}$ lies on the minimum-variance frontier (and is not the global minimum-variance portfolio). By Proposition 1, $\underset{\sim}{x}$ is a combination of mimicking positions, and, since $\underset{\sim}{i}'V^{-1}B \neq \underline{0}'$, Proposition 2 implies that there exists an L such that $\underset{\sim}{x} = q^{-1}V^{-1}BL(L^{-1}\underset{\sim}{u})$, where $V^{-1}BL$ is a set of mimicking portfolios. Since $\underset{\sim}{i}$ is not in the column space of B, $B\underset{\sim}{u}$ is unique, and thus $\underset{\sim}{x}$ is unique. (To see how Proposition I of Grinblatt and Titman can be used to prove Proposition 3, note that mimicking portfolios are reference portfolios if and only if $\underset{\sim}{i}$ is not in the column space of B.)

The next proposition provides necessary and sufficient conditions for the mimicking portfolios to span the mean-variance frontier of the risky assets. Propositions 5 and 6 then cover two special cases.

PROPOSITION 4: *Every portfolio on the minimum-variance frontier is a portfolio of mimicking portfolios if and only if (a) exact arbitrage pricing holds and (b) $\underset{\sim}{i}$ is in the column space of B.*

Proof: The proof is similar to that of Proposition 3 and is therefore sketched only briefly.

(Necessity) Using (4) and (5), there exist $r_{01} \neq r_{02}$, $q_1 \neq q_2$, and $\underset{\sim}{z}_1 \neq \underset{\sim}{z}_2$ such that $\underline{E} = r_{01}\underset{\sim}{i} + q_1B\underset{\sim}{z}_1 = r_{02}\underset{\sim}{i} + q_2B\underset{\sim}{z}_2$, which establishes arbitrage pricing and implies that $\underset{\sim}{i}$ is in the column space of B.

(Sufficiency) Substituting the right-hand side of (2) for $\underline{E}$ in (5) and solving for $\underset{\sim}{x}$ gives $\underset{\sim}{x} = q^{-1}V^{-1}[(r_0 - p)\underset{\sim}{i} + B\underset{\sim}{u}]$, which can be written as $\underset{\sim}{x} = V^{-1}B\underset{\sim}{z}$ since $\underset{\sim}{i}$ is in the column space of B. The latter condition also implies that $\underset{\sim}{i}'V^{-1}B \neq \underline{0}'$, so $\underset{\sim}{x}$ can be represented as a portfolio of mimicking portfolios (by Proposition 2).

PROPOSITION 5: *If the global minimum-variance portfolio is the only portfolio of mimicking portfolios on the minimum-variance frontier, then exact arbitrage pricing does not hold.*

Proof: The portfolio on the frontier can be described as $V^{-1}B\underset{\sim}{z}$ (because it is a portfolio of the mimicking portfolios) and as $V^{-1}\underset{\sim}{i}(\underset{\sim}{i}'V^{-1}\underset{\sim}{i})^{-1}$ (because it is the global minimum-variance portfolio). Therefore, $V^{-1}B\underset{\sim}{z} = V^{-1}\underset{\sim}{i}(\underset{\sim}{i}'V^{-1}\underset{\sim}{i})^{-1}$, which (after multiplying both sides by V) implies that $\underset{\sim}{i}$ is in the column space of B. Invoke Proposition 4 to conclude that exact arbitrage pricing cannot hold under this proposition's condition.

PROPOSITION 6: *Suppose exact arbitrage pricing holds. No portfolio of mimicking portfolios is on the minimum-variance frontier if and only if r_0 is the expected return on the global minimum-variance portfolio and $\underset{\sim}{i}$ is not in the column space of B.*

Proof: (Necessity) To see that $\underline{i}$ is not in the column space of B, invoke Proposition 4. To see that r_0 is the expected return on the global minimum-variance portfolio, invoke Proposition 3.

(Sufficiency) Invoke Proposition 3.

III. Mimicking Positions in Empirical Tests

If the regression function $\mathrm{E}\{\underline{r} \mid \underline{R}\}$ is linear in $\underline{R}$, i.e.,

$$\underline{r} = g + G\underline{R} + \underline{\varepsilon}, \tag{6}$$

where $\mathrm{E}\{\underline{\varepsilon} \mid \underline{R}\} = \underline{0}$, then (3) can be written as

$$\underline{E} = \underline{i}r_0 + G\underline{h}, \tag{7}$$

with $\underline{h} = A'VA\underline{w}$, since $G = \mathrm{cov}(\underline{r}, \underline{R})[\mathrm{cov}(\underline{R}, \underline{R})]^{-1} = VA(A'VA)^{-1}$. In other words, the linear pricing relation can also be expressed in terms of the multiple-regression coefficients obtained by regressing the N asset returns on the payoffs of the K mimicking positions. The latter representation is used in testing the exact arbitrage-pricing relation using mimicking portfolios in a multivariate regression (e.g., Lehmann and Modest [14]).

Suppose that in (1) the vector of residuals $\underline{e} = \underline{r} - \mathrm{E}\{\underline{r} \mid \underline{f}\}$ has a diagonal covariance matrix. This property does not apply to the residuals from the regression of the assets' returns on the mimicking positions' payoffs. Indeed, since $\underline{R} = L'B'V^{-1}\underline{r}$, the covariance matrix of $[\underline{r} - \mathrm{E}\{\underline{r} \mid \underline{R}\}]$ is $V - B(B'V^{-1}B)^{-1}B' = Z - B[(B'V^{-1}B)^{-1} - I]B'$, which in general is not diagonal even if Z is diagonal. It is noteworthy that the covariance matrix of $[\underline{r} - \mathrm{E}\{\underline{r} \mid \underline{R}\}]$ does not depend upon the choice of the nonsingular transformation L.

The regression approach using mimicking positions provides the opportunity to test restrictions implied by the pricing relation that cannot be tested using nontraded factors. Premultiply (7) by A' and recall that $A'G = I$, the identity matrix, and that $A'\underline{E}$ is $\mathrm{E}\{\underline{R}\}$, the $K \times 1$ vector of expected payoffs on the mimicking positions. Thus,

$$\underline{h} = \mathrm{E}\{\underline{R}\} - A'\underline{i}r_0. \tag{8}$$

When mimicking portfolios are used to express the pricing relation as in (7), then $A'\underline{i} = \underline{i}_K$ and the vector of risk premiums $\underline{h}$ equals $\mathrm{E}\{\underline{R} - \underline{i}_K r_0\}$, the vector of expected excess returns on the mimicking portfolios. This restriction, which is well known in the case of $K = 1$ (e.g., Black, Jensen, and Scholes [2]), also holds for larger K, as discussed by Shanken [18] and Huberman and Kandel [11]. (See also Ross [17] and Admati and Pfleiderer [1] for related results in the infinite-economy approximate APT.) The condition that the risk premiums equal expected excess portfolio returns can be imposed as an additional restriction in tests of (4) using multivariate regression (e.g., Gibbons [8]). In that case, the restriction tested in (6) is $\underline{g} = r_0(\underline{i} - G\underline{i}_K)$. When the tests are conducted using nontraded factors, with $B\underline{f}$ replacing $G\underline{R}$ in (6), then only the linearity of the relation can be tested. The restriction tested in that case is $\underline{g} = \underline{i}r_0 + B[\underline{u} - \mathrm{E}\{\underline{f}\}]$. The last point is discussed by Stambaugh [19, 20] in the case of $K = 1$.

The multivariate-regression model in (6) can also be used to investigate whether a set of mimicking portfolios spans the minimum-variance frontier (cf. Proposition 4). Huberman and Kandel [11] show that the portfolios with returns that are given by $\underline{R}$ span the minimum-variance frontier if and only if $\underline{g} = \underline{0}$ and $G'\underline{i} = \underline{i}_K$.

IV. Examples of Mimicking Positions

The three examples of mimicking positions mentioned in the introduction are given by different specifications of L in Proposition 1.

Example 1: $L = I$; $A = V^{-1}B$.

In this example each column of A represents the position having maximal correlation with the corresponding factor. For instance, $\underline{a}_k$, the kth column, solves the problem

$$\text{maximize corr}(\underline{a}'\underline{r},\, f_k) = (\underline{a}'\underline{b}_k)/(\underline{a}'V\underline{a})^{0.5}, \tag{9}$$

where $\text{corr}(x, y)$ denotes the correlation between x and y, and the vector $\underline{b}_k$ is the kth column of the matrix B.

The same maximum-correlation principle applies to any model that identifies a specific mean-variance-efficient benchmark portfolio. Kandel and Stambaugh [13], for instance, show that the portfolio of a subset of assets that is maximally correlated with the market portfolio is a mimicking portfolio for testing the Capital Asset Pricing Model (CAPM). That is, the maximum-correlation portfolio can replace the market portfolio in pricing the subset of assets if the CAPM holds.

Problem (9) has infinitely many solutions. The vector $\underline{a}_k$ is also the position with the minimum total variance among positions with a predetermined loading on the kth factor. It is the solution of the following parametric quadratic program:

$$\text{minimize } \underline{a}'V\underline{a} \tag{10a}$$

$$\text{subject to } \underline{a}'\underline{b}_k = (\underline{b}_k'V^{-1}\underline{b}_k). \tag{10b}$$

Example 2: $L = (B'Z^{-1}B + I)$; $A = Z^{-1}B$.

To see that $A = Z^{-1}B$, note that $V^{-1} = (BB' + Z)^{-1} = Z^{-1} - Z^{-1}B(B'Z^{-1}B + I)^{-1}B'Z^{-1}$. In this case, $\underline{a}_k$, the kth column of A, represents the position with minimal unsystematic risk subject to having a predetermined loading on the kth factor. In other words, $\underline{a}_k$ is the solution of the problem

$$\text{minimize } \underline{a}'Z\underline{a} \tag{11a}$$

$$\text{subject to } \underline{a}'\underline{b}_k = (\underline{b}_k'Z^{-1}\underline{b}_k). \tag{11b}$$

Note that the solution of (11a)–(11b) is not identical with the solution of (10a)–(10b).

Example 3: $L = [I + (B'Z^{-1}B)^{-1}]$; $A = Z^{-1}B(B'Z^{-1}B)^{-1} = V^{-1}B(B'V^{-1}B)^{-1}$.

The elements of A are the coefficients on $\underline{r}$ used in computing the factor scores. (In maximum-likelihood (ML) factor analysis, the ML estimates of B and Z are substituted.) Grinblatt and Titman [9] discuss this set of factor-mimicking positions.

As in the previous examples, the positions here can be described as proportional to solutions of quadratic programs. Indeed, $\underset{\sim}{a}_k$ is the solution of

$$\text{minimize } a' V \underset{\sim}{a} \tag{12a}$$

$$\text{subject to } \underset{\sim}{a}' B = \underset{\sim}{e}_k, \tag{12b}$$

where $\underset{\sim}{e}_k$ is the $K \times 1$ vector with the kth component equal to 1 and the other components equal to zero. Substituting the matrix V for the matrix Z in the objective function (12a) gives an equivalent problem with the same solution.

The mimicking positions corresponding to the factor scores possess some unique properties that may be useful in empirical applications. First, in the pricing equation (7), $G = VA(A'VA)^{-1} = B$ and $\underset{\sim}{h} = \underset{\sim}{u}$. This implies that, for these mimicking positions, all of the risk premiums are equal to the risk premiums associated with the factors. Second, $\mathrm{E}\{\underset{\sim}{R} \mid \underset{\sim}{f}\} = \mathrm{E}\{\underset{\sim}{R}\} + \underset{\sim}{f}$, which follows by substituting the second expression for A above into $\mathrm{cov}(\underset{\sim}{R}, \underset{\sim}{f}) = A'B = I$. Note, however, that, while $\mathrm{cov}(\underset{\sim}{f}, \underset{\sim}{f}) = I$, $cov(\underset{\sim}{R}, \underset{\sim}{R}) = (B' V^{-1} B)^{-1}$, which, in general, is not diagonal.

In each of the three examples above, the nonsingular matrix L can be right-multiplied by a diagonal matrix D, where the role of the kth diagonal element of D is, if possible, to normalize the kth column of the matrix A to transform the position it represents into a portfolio. This normalization is possible if the elements of the kth column of A do not add to zero. Proposition 2 gives conditions under which there will exist some L for which every column of A sums to a nonzero value. Even when those conditions are met, however, there can be other choices of L for which some of the columns of A sum to zero. To illustrate this possibility, consider a case where $N = 3$ and $K = 2$, and let

$$B = \begin{bmatrix} -1 & -7/9 \\ 2 & 2 \\ 3 & 1 \end{bmatrix} \quad \text{and} \quad Z = I. \tag{13}$$

In this case, neither column of A sums to zero for the choices of L in Examples 1 and 2, but the second column of A sums to zero for the positions corresponding to the factor scores in Example 3.

V. Conclusions

The K mimicking positions with returns that can serve in place of factors in the exact arbitrage-pricing relation are $K \times K$ nonsingular linear transformations of $V^{-1}B$. These positions can be transformed into a set of mimicking portfolios if and only if the global minimum-variance portfolio has systematic risk. We provide relations between the minimum-variance frontier and mimicking portfolios, and we interpret three examples of sets of mimicking positions.

Tests using mimicking positions can investigate an additional restriction that is inappropriate in tests using nontraded factors. When nontraded factors are used, pricing with respect to factors imposes no restrictions on the vector of risk premiums. When mimicking positions are used as factors, the risk premiums are related to the expected payoffs on the positions [equation (8)]. Hence, mimicking positions offer a potential advantage in tests of asset pricing. Given that the weights in the mimicking positions must be estimated, however, it remains for future research to determine whether this additional restriction provides increased power.

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