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ABSTRACT

Duality theory is employed to provide necessary and sufficient conditions for portfolios on the minimum-variance frontier to have positive investment proportions in all assets. These conditions involve the feasibility of portfolios that have non-negative correlation with all assets and positive correlation with at least one. Using these results, several "qualitative" results concerning the signs of investment proportions in efficient portfolios are proved. It is argued that the conditions that ensure all-positive weights in efficient portfolios are intuitively compelling and are not unique to the CAPM. With large numbers of assets, however, the signs of weights in minimum-variance portfolios can be very sensitive to slight departures from these conditions due to, for example, sampling error.

THE BEHAVIOR OF THE minimum-variance frontier is of general concern to portfolio theory and plays a central role in the study of asset pricing. An unresolved question in the theory of the minimum-variance frontier involves when it will contain portfolios with positive weights on all assets.

This paper addresses that question. We show that the conditions that ensure positive investment proportions in efficient portfolios have, by Farkas' Lemma, a simple dual. This dual, or alternative, system of inequalities puts conditions directly on the variances, covariances, and means of the returns, and these conditions can be interpreted intuitively. This contrasts with simply considering the first-order conditions for variance minimization, since these involve the inverted covariance matrix. With a large number of assets, it can be difficult to translate one's intuitive sense about how assets covary into statements about the inverse of the covariance matrix. The duality conditions we derive involve the existence of portfolios that are positively (or negatively) correlated with all assets. Indeed, we show that a necessary and sufficient condition for no portfolio on the frontier to have all-positive weights is the feasibility of an "arbitrage" or "hedge" portfolio, with weights that sum to zero, that has an expected payoff of zero and has non-negative correlation with all assets. The duality conditions also make it easy to construct examples of the types of covariability that permit or preclude positive investment proportions on all assets in frontier portfolios.

There are at least three apparent reasons for interest in the signs of weights

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in efficient portfolios. First, the static mean-variance Capital Asset Pricing Model (CAPM) predicts that a particular positively weighted portfolio, the value-weighted market portfolio, is mean-variance efficient. Roll [10] and Roll and Ross [11] have conjectured that conditions on the first two moments of asset returns that preclude positively weighted efficient portfolios might suggest ways to infer, from subsets of the universe of assets, whether the market portfolio could be efficient.

Second, one may wish to determine whether the prediction that efficient portfolios have positive investment proportions is shared by other asset-pricing models. It is clear, for example, from Chamberlain and Rothschild [1] that infinite-asset versions of the single-factor Arbitrage Pricing Theory (APT) imply that equally weighted portfolios approach the frontier as the number of assets increases. In fact, under a one-factor APT, any subset of assets will yield positively weighted portfolios that are minimum-variance portfolios. Most asset-pricing models predict that assets with large variances and large positive covariances with other assets have high expected returns. Generally speaking, this is the type of behavior required to ensure the presence of positively weighted portfolios on the ex ante minimum-variance frontier. The signs of investment proportions, however, are necessarily sensitive to small changes in parameter values in well-diversified portfolios with a large number of assets. Thus, the composition of the ex post frontier may be sensitive to sampling error.

Third, results on the signs of investment proportions can be useful to portfolio theory, independently of asset-pricing concerns. From a practical standpoint, it is valuable to know when short-sale constraints, which complicate computations, will be nonbinding. Also, most mutual funds and index funds have positive investment proportions. Since these funds exist largely to provide diversification services, one may be interested in whether such portfolios are likely to be minimum-variance portfolios.

The question of when minimum-variance portfolios have positive weights has previously been posed by Roll [10] in his critique of the CAPM and its tests. He provides a partial answer in the form of restrictions on the covariance matrix, which must hold when the global minimum-variance portfolio has all-positive weights and all assets are non-negatively correlated. These restrictions were corrected in Rudd [13], and some sufficient conditions for positive weights in the global minimum-variance portfolio are provided in Roll and Ross [11]. Roll and Ross [11] also anticipate the methods employed here by applying duality theory to the problem. However, all of these papers, citing the difficulty of obtaining more general results, focus on the global minimum-variance portfolio, which has weights that do not depend on the expected returns. Also, these papers do not provide conditions that are necessary *and* sufficient for the minimum-variance portfolio to have positive investment proportions. The results presented here, then, are more general and complete characterizations. They also highlight the role of the expected-return vector in determining portfolio weights along the efficient frontier. As a corollary to the paper's main result, we do, in fact, provide a necessary and sufficient condition for the global minimum-variance portfolio to have all-positive weights. This condition turns out to be quite simple: there must be no nontrivial "arbitrage" portfolios, with weights summing to zero, that have a non-negative correlation with all assets.

Section I introduces notation and reviews some results on the minimum-variance frontier. In Section II, the duality conditions are described, and the main result of the paper is proved. Section III provides necessary and sufficient conditions for all minimum-variance portfolios to involve short positions. A number of special cases, illustrative examples, and subsidiary issues are discussed in Section IV, and Section V concludes the paper.

I. Notation and Variance Minimization

This section introduces notation and reviews some classical results concerning the minimum-variance frontier. For the most part, we follow the development in Merton [8].

We will study portfolios that can be constructed with n risky assets. These assets have returns per dollar of $\tilde{R}_i$, $i = 1, \cdots, n$. Their expected returns are given by the n-vector E, with typical element E_i. To avoid well-known degeneracies, it is assumed that the assets do not all have the same mean return and that they are linearly independent. Thus, their variance-covariance matrix, V, which consists of entries σ_{ij}, will be strictly positive definite, as will be its inverse, V^{-1}. Let q denote an n-vector of ones. A "portfolio" with return $\tilde{R}_p$ is determined by a vector of weights x_p. These weights must sum to one, $x_p' q = 1$, where a prime denotes transposition. We will also have to deal with payoffs determined by asset purchases and short sales that sum to zero, i.e., payoffs determined by portfolios with weights $y \in R^n$ where $y'q = 0$. These portfolios will be referred to as "hedge" or "arbitrage" portfolios since they are self financing. The mean and variance of a portfolio return $\tilde{R}_p$ are given as $E_p = x_p' E$ and $\sigma_p^2 = x_p' V x_p$.

We will subscript portfolios that are on the minimum-variance frontier with Greek letters representing their mean returns. Thus, the weights, x_μ, minimize $x'Vx$ subject to $x'E = \mu$ and $x'q = 1$. The first-order conditions, which are necessary and sufficient for a solution to this problem, are

$$x_\mu = \gamma_1(\mu)V^{-1}E + \gamma_2(\mu)V^{-1}q, \tag{1}$$

where $\gamma_1(\mu)$ and $\gamma_2(\mu)$ are scalar-valued functions of the required expected return. They are the Lagrange multipliers for the two constraints in the minimization problem. Merton [8] shows that $\gamma_1(\mu)$ and $\gamma_2(\mu)$ are the following linear functions of μ:

$$\gamma_1(\mu) = \frac{C\mu - A}{D} \tag{2}$$

and

$$\gamma_2(\mu) = \frac{B - A\mu}{D}, \tag{3}$$

where $A = q'V^{-1}E$, $B = E'V^{-1}E$, $C = q'V^{-1}q$, and $D = BC - A^2$. A, B, and C are the elements of the "efficient-set information matrix," and D is its determinant. Note that B and C are quadratic forms and that V^{-1} is positive definite. Thus, $B > 0$ and $C > 0$. Merton [8, p. 1853] also shows that $D > 0$. We will make use of these facts repeatedly in the analysis below.

Let v denote the mean return on the global minimum-variance portfolio. The function $\gamma_1(\mu)$ is the Lagrange multiplier for the expected-return constraint in the variance-minimization problem. Since this constraint is nonbinding at the global minimum-variance point, $\gamma_1(v) = 0$. This implies that $v = A/C$. Further, since

$$x_v' V x_v = \sigma_v^2 = \gamma_2(v) = \frac{B - \dfrac{A^2}{C}}{BC - A^2} = \frac{1}{C}, \tag{4}$$

we also have that $1/C$ is the minimum feasible variance.

II. Characterization of the Dual

This section demonstrates and discusses the duality between positively weighted minimum-variance portfolios and the feasibility of particular types of portfolios and hedge positions.

Our interest is in conditions that ensure that the solution to (1), x_μ, is strictly positive. Thus, we are concerned with the following system of inequalities.

P1:
$$Vx_\mu = \gamma_1(\mu)E + \gamma_2(\mu)q,$$
$$x_\mu \gg 0.$$

Theorem 1 below shows that this system has a solution if and only if there exists no $y \in R^n$ solving the following "dual" or "alternative" system.

D1:
$$Vy \geq 0, \quad Vy \neq 0,$$
$$\gamma_1(\mu)y'E + \gamma_2(\mu)y'q \leq 0.$$

Solutions to the dual all have non-negative correlation with all assets, $Vy \geq 0$, and strictly positive correlation with at least one, $Vy \neq 0$. Since V is of full rank, the second condition just rules out the trivial case where y is zero in every component. Intuitively, it is not surprising that the existence of portfolios with this characteristic precludes strictly positive weights for a minimum-variance portfolio. Short positions in such portfolios will be negatively correlated with all assets and, hence, with any positively weighted portfolio. As we will see below, this negative correlation can be exploited to reduce the variance of the positively weighted portfolio with no cost in expected return. First, we show that the final constraint in the dual, which we will refer to as "the expected-return constraint," can be taken to hold with equality without loss of generality.

LEMMA: *A solution to the dual will exist if and only if there is a nontrivial $y \in R^n$ that has non-negative correlation with all assets and that satisfies*

$$\gamma_1(\mu)y'E + \gamma_2(\mu)y'q = 0. \tag{5}$$

Proof: Suppose $\hat{y}$ solves D1 and the expected-return constraint holds with strict inequality. The set of nontrivial vectors satisfying $Vy \geq 0$, $Vy \neq 0$ is a convex cone, closed everywhere but at the origin. The global minimum-variance portfolio, x_v, is in this cone since it satisfies

$$Vx_v = \gamma_2(v)q = \frac{1}{C}q \gg 0. \tag{6}$$

Further, $x_v'E = A/C$, and $x_v'q = 1$, so

$$\gamma_1(\mu)x_v'E + \gamma_2(\mu)x_v'q = \frac{B}{D} - \frac{A^2}{DC} = \frac{1}{DC}(BC - A^2) = \frac{1}{C} > 0, \tag{7}$$

and, at any mean return, x_v violates the expected-return constraint strictly. Since the constraint is linear, a convex combination of $\hat{y}$ and x_v will satisfy the constraint with equality and will also lie in the cone of vectors that are non-negatively correlated with all assets. The proof in the other direction is trivial since the last inequality in D1 is weak. Q.E.D.

The above Lemma turns out to imply that solutions to the dual are of two sorts. Either they are "arbitrage" positions, with mean returns of zero and weights that sum to zero, or they are portfolios that have expected returns equal to the zero-beta rate.

Suppose the portfolio with weights x_z is uncorrelated with the minimum-variance portfolio with mean return μ. We will denote as $r_z(\mu) \equiv x_z'E$ the mean return on this portfolio. It is the "zero-beta" rate associated with expected return μ. From the first-order condition (1),

$$0 = x_z'Vx_\mu = \gamma_1(\mu)r_z(\mu) + \gamma_2(\mu) \tag{8}$$

or

$$r_z(\mu) = -\frac{\gamma_2(\mu)}{\gamma_1(\mu)}. \tag{9}$$

Thus, we can rewrite the expected-return constraint, (5), as

$$y'E = -\frac{\gamma_2(\mu)}{\gamma_1(\mu)}y'q = r_z(\mu)y'q. \tag{10}$$

If $y'q = 0$, then solving the dual simply involves finding a hedge position that is positively correlated with every asset and that has an expected payoff of zero. A short position in such a hedge will be negatively correlated with any portfolio that has all-positive weights. Adding a sufficiently small amount of such a short position will, without changing the expected return, reduce the variance of any portfolio with positive weights. To see this, let y be the arbitrage portfolio under consideration. The variance of $x_\mu - \alpha y$, where α is scalar, will be

$$x_\mu'Vx_\mu - 2\alpha x_\mu'Vy + \alpha^2 y'Vy. \tag{11}$$

As α approaches zero, the third term will be driven to zero before the second term. If $x_\mu \gg 0$, $x_v'Vy$ will be positive. For small enough α, then, (11) will be smaller than $x_\mu'Vx_\mu$. Further, the portfolio $x_\mu - \alpha y$ is always feasible because y has weights that sum to zero. Accordingly, the existence of a zero-mean-return arbitrage portfolio that solves the dual will rule out positively weighted portfolios anywhere on the minimum-variance frontier. In fact, we show in Section III that the existence of such a hedge portfolio is not only sufficient, but it is also necessary for no portfolio on the frontier to have all-positive weights.

When the solution to the dual, if one exists, has a positive cost ($y'q > 0$), we can normalize by the cost and consider the portfolio $y^* = y/y'q$, which has weights that sum to one. In this case, the Lemma says that y^* must be non-negatively correlated with all assets and have an expected return equal to the zero-beta rate. Suppose the frontier portfolio with mean return μ has all-positive weights. The portfolio y^* cannot be a zero-beta portfolio because $Vy^* \geq 0$, $Vy^* \neq 0$, and $x_\mu \gg 0$ preclude $x_\mu' Vy^* = 0$. Consider the arbitrage position that puts a weight of +1 on the zero-beta portfolio and a weight of −1 on the solution to the dual. This portfolio, $x_z - y^*$, is negatively correlated with the frontier portfolio because $x_\mu' Vx_z = 0$ and $-x_\mu' Vy^* < 0$. It also has weights that sum to zero and a mean return of zero. It will thus serve as a hedge position in (11), reducing the variance without changing the mean. Clearly, the existence of such a y^* will contradict all-positive weights for the frontier portfolio with mean return μ. Similarly, if $y'q < 0$, then normalizing by the cost yields a portfolio that is nonpositively correlated with all assets and that has an expected return equal to the zero-beta rate. Financing a long position in this portfolio by shorting the zero-beta portfolio again yields an arbitrage portfolio with zero mean that is negatively correlated with the frontier portfolio if it has all-positive weights. Thus, we again rule out the possibility that the minimum-variance portfolio with mean return μ has positive weights.

It is intuitively quite apparent that the existence of portfolios, or hedge positions, such as those described above would be sufficient to rule out positively weighted portfolios on the minimum-variance frontier. What is perhaps less obvious is that the existence of such opportunities is also necessary to rule out all-positive weights in frontier portfolios. Alternatively, stated in terms of the primal, to ensure that frontier portfolios have positive weights it is sufficient, as well as necessary, that there be no nontrivial positions that have non-negative correlation with all assets and satisfy equation (10). To prove this duality, one could appeal to some version of Farkas' Lemma. Since we have assumed that V has full rank, however, it is a simple matter to demonstrate the result directly. We state the theorem for expected returns other than that of the global minimum-variance portfolio, i.e., for $\mu \neq A/C$. The case $\mu = A/C$ is special because no zero-beta rate exists. The analogous result for the global minimum-variance portfolio is provided as a corollary.

THEOREM 1: *A necessary and sufficient condition for the minimum-variance portfolio with mean $\mu \neq (A/C)$ to have strictly positive weights on all assets is that there must exist no nontrivial*

(i) *hedge positions with expected payoffs equal to zero and non-negative correlation with all assets*

or

(ii) *portfolios that are either non-negatively or nonpositively correlated with all assets and have expected returns equal to the zero-beta rate, $r_z(\mu)$.*

Proof: Conditions (i) and (ii) simply restate the inequalities that constitute the dual, problem D1, as amended by the Lemma. If $y'q = 0$ and (5) holds, $y'E = 0$ since $\gamma_1(\mu) = 0$ only for $\mu = A/C$ (hence, condition (i)). Condition (ii) exploits the fact that any vector of asset purchases, y, with $q'y \neq 0$ can be normalized by its cost so that it is a portfolio with weights that sum to one. When $q'y < 0$, this normalization will yield nonpositive correlation with all assets. It remains, then, to demonstrate that the system D1 has a solution if and only if P1 does not.

Suppose both have a solution. Let $x_\mu \gg 0$ solve P1, and let $\hat{y}$ solve D1. Then, since $V\hat{y} \geq 0$ and $V\hat{y} \neq 0$, we have $x_\mu' V\hat{y} > 0$, or, by the symmetry of V,

$$\begin{aligned} \hat{y}' V x_\mu &= \hat{y}' V V^{-1}[\gamma_1(\mu)E + \gamma_2(\mu)q] \\ &= \hat{y}'[\gamma_1(\mu)E + \gamma_2(\mu)q] > 0, \end{aligned} \tag{12}$$

which contradicts $\hat{y}$ solving the dual.

Suppose neither system has a solution. Then $x_\mu = \gamma_1(\mu)V^{-1}E + \gamma_2(\mu)V^{-1}q$ has at least one negative or zero component. Suppose this is component k, and let ϵ_k be the unit vector with a one for the kth component and zeroes elsewhere. Let $\hat{y} = V^{-1}\epsilon_k$. Then, $V\hat{y} = \epsilon_k$, and

$$\hat{y}'[\gamma_1(\mu)E + \gamma_2(\mu)q] = \epsilon_k'[\gamma_1(\mu)V^{-1}E + \gamma_2(\mu)V^{-1}q] = \epsilon_k' x_\mu \leq 0. \tag{13}$$

Thus, $\hat{y}$ solves the dual, which is a contradiction. The only remaining possibility is that one system can be solved and the other cannot. Q.E.D.

The second part of this proof is instructive and illustrates the role of the duality conditions. The dual is solved by purchasing an asset that is sold short in the frontier portfolio and then linearly transforming this vector with V^{-1}. As the examples in Section IV further illustrate, this is typical of the portfolios that solve the dual system. Note also that the existence of portfolios satisfying either condition (i) or (ii) is necessary and sufficient to rule out all-positive weights in the frontier portfolio with mean return μ. As Theorem 2 will show, condition (i) is necessary and sufficient to rule out positive weights in all frontier portfolios. Finally, the global minimum-variance portfolio is a case of particular interest since it has been the focus of previous work on this problem by Roll [10], Rudd [13], and Roll and Ross [11]. The following result is at least simpler and more intuitive than the complex restrictions on the off-diagonal elements of V that those papers presented.

COROLLARY: *A necessary and sufficient condition for the global minimum-variance portfolio to have all-positive weights is that there exist no nontrivial arbitrage portfolios (i.e., portfolios with weights that sum to zero) that have non-negative correlation with all assets.*

Proof: The proof of Theorem 1 demonstrates the duality between P1 and D1 for any mean return. By the Lemma, the expected-return constraint can be taken to be an equality. At $\mu = A/C$, $\gamma_1(\mu) = 0$ and $\gamma_2(\mu) = 1/C$. Equality for the expected-return constraint will then imply that $y'q = 0$, regardless of the value of $y'E$. Q.E.D.

III. Frontiers with No Positively Weighted Portfolios

In this section, we describe conditions under which the dual problem will have a solution for every expected return, μ. If this is the case, there will be no minimum-variance portfolios with positive weights on all assets. The main result of this section gives necessary and sufficient conditions for this.

It is apparent that there will always be some mean returns at which the dual is solvable. Any expected return that is greater than the maximum for n assets, which we denote $\bar{\mu} \equiv \max_i\{E_i\}$, can be achieved only through short positions. Similarly, frontier portfolios with $\mu \leq \min_i\{E_i\}$ must involve negative positions in some assets. Hence, the dual must have solutions at these points. To see that this is indeed the case for $\mu \geq \bar{\mu}$, simply choose

$$y = V^{-1}(q\mu - E), \tag{14}$$

which is nontrivial since all assets do not have the same mean return. Then,

$$Vy = q\mu - E \geq 0 \tag{15}$$

since $\mu \geq \bar{\mu}$. Furthermore,

$$y'q = q'V^{-1}q\mu - q'V^{-1}E = C\mu - A \tag{16}$$

and

$$y'E = E'V^{-1}q\mu - E'V^{-1}E = A\mu - B. \tag{17}$$

Substituting these expressions into the expected-return constraint in the dual system, D1, gives

$$\begin{aligned}
\gamma_1(\mu)(A\mu - B) &+ \gamma_2(\mu)(C\mu - A) \\
&= \frac{1}{D}[(C\mu - A)(A\mu - B) + (B - A\mu)(C\mu - A)] \\
&= \frac{1}{D}(\mu - \mu)(CB - A^2) = 0.
\end{aligned} \tag{18}$$

Thus, $y = V^{-1}(q\mu - E)$ solves the dual (as amended by the Lemma) for all expected returns greater than $\bar{\mu}$. The same arguments work in reverse for extremely low expected returns. That is, if $\underline{\mu} \equiv \min_i\{E_i\}$, then $y = V^{-1}(E - q\mu)$ will solve the dual for $\mu \leq \underline{\mu}$.

If there are no portfolios on the minimum-variance frontier with positive weights on all assets, then it must be possible to solve the dual for all expected returns $\mu \in (+\infty, -\infty)$. One situation in which this will occur is when there is a nontrivial vector of asset purchases, y, which costs nothing ($y'q = 0$), has an expected payoff of zero ($y'E = 0$), and also has non-negative covariances with all assets ($Vy \geq 0$). By adding a sufficiently small short position in such an arbitrage portfolio to any positively weighted portfolio, one reduces the latter portfolio's return variance without altering its mean return. Thus, if we can find such a y, it follows that no positively weighted portfolio will be minimum-variance. Surprisingly, the feasibility of such hedge portfolios is not only a sufficient, but also a necessary condition for all frontier portfolios to involve short sales. The proof is reasonably intuitive, so technical details are left to lemmas proved in the Appendix.

THEOREM 2: *A necessary and sufficient condition for no minimum-variance portfolio to have all-positive weights is the existence of a nontrivial, zero-mean-return hedge portfolio that has a non-negative correlation with all assets.*

Proof: Sufficiency is obvious from the arguments prior to the theorem or from the fact that such a portfolio obviously satisfies the dual system, D1. For necessity, we must show that, whenever the dual can be solved for all μ, such a hedge portfolio exists. Let K denote the set of nontrivial portfolio positions that have non-negative correlation with all assets $K \equiv \{y: y = V^{-1}\xi,\ \xi \geq 0,\ \xi \neq 0\}$. K is a cone, closed everywhere but at the origin. We can rewrite the expected-return constraint in the dual as $y'(a + b\mu) \leq 0$, where

$$a \equiv \frac{B}{D} q - \frac{A}{D} E \in R^n \tag{19}$$

and

$$b \equiv \frac{C}{D} E - \frac{A}{D} q \in R^n. \tag{20}$$

As μ varies, the vectors $a + b\mu$ trace out a line in R^n. If the dual system has a solution for each μ, then $K \cap \{y: y'(a + b\mu) \leq 0\} \neq \varnothing$. We must show that this implies that $K \cap \{y: y'q = 0, y'E = 0\} \neq \varnothing$. Lemma 1A in the Appendix shows that a vector is orthogonal to both q and E, or is a zero-mean-return hedge portfolio, if and only if it solves $y'(a + b\mu) = 0$ for all μ. Let S be the set of such y. It is an $(n - 2)$-dimensional subspace of R^n. Now consider the $(n - 1)$-dimensional half space, H^-, of vectors that satisfy the expected-return constraint for all possible μ:

$$H^- \equiv \cap_{\mu \in R} \{y: y'(a + b\mu) \leq 0\}. \tag{21}$$

Lemma 2A shows that this is a half space of the $(n - 1)$-dimensional subspace orthogonal to b. It is apparent that $S \subset H^-$. Lemma 3A shows that, if $K \cap H^- \neq \varnothing$, then $K \cap S \neq \varnothing$. This is because a convex combination of any vector in $K \cap H^-$ and the global minimum-variance portfolio, which is also in K and orthogonal to b, can be constructed to produce a zero-mean-return hedge portfolio. This will also be in K since K is a cone and closed under convex combination. All that remains is to show that $K \cap H^-$ is not empty. Lemma 4A shows that this is the case by supposing the contrary, separating H^- and K with a hyperplane, H^*, and then showing that one can find a μ^* such that $a + b\mu^*$ is normal to H^*. Then, the half space $\{y: y'(a + b\mu^*) \leq 0\}$ and K are disjoint, and the dual has no solution at μ^*, a contradiction. Q.E.D.

This result can be useful in a number of contexts. By actually constructing zero-mean-return hedge portfolios that have non-negative correlation with all assets, one can show that particular structures for returns will admit no positively weighted portfolios on the frontier. This may provide an intuitive idea of what such regimes look like. Alternatively, by demonstrating that such portfolios never exist, one sees that some models, for example a single-factor APT, imply that, on any subset of assets, there are minimum-variance portfolios with all-positive weights. These questions are the concern of the next section.

IV. Special Cases

In this section, we describe some special cases that illustrate the role of the dual system in determining when minimum-variance portfolios have positive weights. We also construct some illustrative examples of covariance matrices and expected returns that admit no positively weighted portfolios on the minimum-variance frontier, and provide some other qualitative results concerning investment proportions in frontier portfolios. The results in Sections IV.A and IV.B can be obtained through other, more traditional, means. They are introduced here simply as illustrative examples in the hope that they clarify the structure of the duality conditions.

A. Uncorrelated Assets

The simplest structure one can imagine for the variance-covariance matrix, V, is that it is diagonal. In this situation, the global minimum-variance portfolio is positively weighted since $x_v = V^{-1}q(1/C) \gg 0$. Thus, from Theorem 2, it must be impossible to construct an arbitrage portfolio that has non-negative correlation with every asset. But it is obvious that this is the case. Any y such that $y'q = 0$ involves a negative position in some asset, and, as V is positive and diagonal, Vy cannot be non-negative.

For sufficiently high or low μ, however, some assets must be sold short. As noted in the discussion following Theorem 1, assets sold short in the minimum-variance portfolio correspond to assets purchased in the solution to the dual. When V is diagonal, the first-order conditions, (1), imply that

$$\sigma_j^2 x_{\mu j} = \gamma_1(\mu)E_j + \gamma_2(\mu). \tag{22}$$

If the right-hand side is negative, asset j is sold short in the frontier portfolio. The dual can be solved simply by purchasing this asset, choosing $y_j = 1$ and $y_i = 0$ for $i \neq j$. Clearly, $Vy \geq 0$, and $\gamma_1(\mu)y'E + \gamma_2(\mu)y'q$ will equal the right side of (22), which is, by assumption, negative. Thus, the expected-return constraint in the dual is satisfied.

It is also interesting to note that, when V is diagonal, only a subset of the assets is ever sold short on the upward-sloping portion of the frontier, those assets with means less than that of the global minimum-variance portfolio, A/C. The right-hand side of (22) can be rewritten as

$$(C\mu - A)E_j + B - A\mu = B - AE_j + (CE_j - A)\mu. \tag{23}$$

If $E_j \geq A/C$, the coefficient on μ on the right side of (23) is non-negative. For these assets, if $\mu \geq A/C$, (23) is greater than or equal to $B - (A^2/C) = D/C$, which is positive. Thus, assets with $E_j \geq A/C$ are never sold short on the upward-sloping portion of the frontier. When $E_j < A/C$, the coefficient on μ is negative, and thus these assets are, for sufficiently high μ, all sold short. One might intuitively expect that, as μ increases without bound, efficient portfolios place positive weight on an ever smaller subset of assets. In fact, this is not the case. One continues to diversify across all assets with expected returns greater than that of the global minimum-variance portfolio.

B. Single-Factor APT

When V is diagonal, the intuition concerning where on the frontier positively weighted portfolios occur is available directly from the first-order conditions, and a consideration of the dual does not add much. The case of a factor structure, however, is less straightforward, and the duality conditions can be more useful.

Suppose the covariance matrix of returns has a single-factor representation: $V = \beta\beta' + \Omega$, where β is an $n \times 1$ vector and Ω is a nonsingular diagonal matrix of residual variances. Note that one can always choose a vector of expected returns that ensures or precludes positive weights for the minimum-variance portfolio with a given mean return. Ross's [12] Arbitrage Pricing Theory (APT), however, restricts the expected returns to be related in a particular way to the covariance matrix: $E = \lambda_0 q + \lambda_1 \beta$.

This added restriction implies that, if returns are consistent with a one-factor model and APT holds, then, for any subset of assets, there will be some portfolios on the minimum-variance frontier that have positive weights on all assets. By Theorem 2, there will be no positively weighted portfolios on the frontier if and only if there exists a nontrivial y such that $y'q = 0$, $y'E = 0$, and $Vy \geq 0$. Suppose such a y exists. Since $y'q = 0$ and $E = \lambda_0 q + \lambda_1 \beta$, we must have $y'\beta = 0$. But then $Vy = \beta\beta'y + \Omega y = \Omega y$. If y is to have non-negative covariance with all assets, since Ω is positive and diagonal, y must be non-negative. That would imply that $y'q > 0$, which is a contradiction, so no such y exists.

It is often argued that the mean-variance CAPM and single-factor APT are empirically indistinguishable. In the present context, both will imply that there are positively weighted portfolios on the minimum-variance frontier. As emphasized in Grinblatt and Titman [3], however, APT also restricts the composition of minimum-variance portfolios for subsets of assets. Here we have another example of this behavior. CAPM will not imply analogous restrictions on subsets.

C. Multi-Factor APT

With more than one factor, there may or may not be positively weighted minimum-variance portfolios. We can construct a hedge position that solves the dual for all mean returns as long as Ωy is not required to be non-negative, as in the zero-factor or one-factor case. For example, with two factors, "exact" APT pricing implies that $E = \lambda_0 q + \lambda_1 \beta_1 + \lambda_2 \beta_2$, where β_1 and β_2 are n-vectors of factor loadings that are columns in the $n \times 2$ matrix β. To construct a zero-mean-return hedge portfolio, we must find $y \in R^n$ such that $y'q = 0$ and $y'E = y'(\lambda_1 \beta_1 + \lambda_2 \beta_2) = 0$. Note that this need not imply that y is orthogonal to β_1 and

β_2 separately. It requires only that $y'\beta_1 = -(\lambda_2/\lambda_1)y'\beta_2$. Therefore,

$$Vy = \beta\beta' y + \Omega y = \beta[\beta_1' y, \beta_2' y]' + \Omega y = \beta \begin{bmatrix} -(\lambda_2/\lambda_1)\beta_2' y \\ \beta_2' y \end{bmatrix} + \Omega y$$

$$= \beta \begin{bmatrix} -\lambda_2/\lambda_1 \\ 1 \end{bmatrix} \beta_2' y + \Omega y. \tag{24}$$

The first term in the final expression need not be zero in every component. Large positive values in the vector $\beta\beta' y$ can offset negative values in Ωy, as the following example illustrates.

Example 1: Choose Ω to be the identity matrix, $\lambda_0 = .1$, $\lambda_1 = \lambda_2 = 1$, and suppose there are three assets with factor loadings $\beta_1' = [6, 7, 11]$ and $\beta_2' = [4, 2, 0]$. If we short the first asset and buy equal proportions of the second and third with the proceeds, we have $y'q = -1 + ½ + ½ = 0$. This choice of y is also orthogonal to $\beta_1 + \beta_2 = [10, 9, 11]$ since the first term is an average of the second two. The only remaining requirement is the following inequality:

$$0 \le Vy = \beta[-(\lambda_2/\lambda_1), 1]'\beta_2' y + \Omega y$$

$$= [-(\lambda_2/\lambda_1)\beta_1 + \beta_2]\beta_2' y + \Omega y. \tag{25}$$

Since $\lambda_2/\lambda_1 = 1$, this will only be satisfied if

$$\Omega y = y \ge (\beta_1 - \beta_2)(\beta_2' y) = (-3) \begin{bmatrix} 2 \\ 5 \\ 11 \end{bmatrix} = \begin{bmatrix} -6 \\ -15 \\ -33 \end{bmatrix} \tag{26}$$

and y satisfies this inequality. Thus, y has weights summing to zero and a zero-mean return, and it has positive correlation with all three assets. There will be no positively weighted portfolios on the frontier.

To see that there may be two-factor APT regimes with positively weighted frontier portfolios, simply consider the case where β_1 or β_2 is constant. Then, any portfolio that is orthogonal to q and to the means is orthogonal to both vectors of factor loadings and $\beta\beta' y = 0$. This case is the same as the single-factor case in that we end up requiring $\Omega y \ge 0$, which is inconsistent with $y'q = 0$.

D. Examples with No Positively Weighted Frontier Portfolios

Example 1 above has an interesting property. The mean-return vector and covariance matrix are

$$E = \begin{bmatrix} 10.1 \\ 9.1 \\ 11.1 \end{bmatrix}, \quad V = \begin{bmatrix} 53 & 50 & 66 \\ 50 & 54 & 77 \\ 66 & 77 & 122 \end{bmatrix}.$$

Asset 1 has an average mean return but has a variance and covariances with the other assets that are relatively small.

This is typical of the situations in which one finds no positively weighted portfolios on the minimum-variance frontier. It is intuitively apparent from Theorem 2 why this should be the case. There are no positively weighted frontier portfolios if and only if we can construct a portfolio, non-negatively correlated with all assets, where long and short positions offset each other in cost ($y'q = 0$) and mean return ($y'E = 0$). If a portfolio with weights summing to zero is to have non-negative correlation with all assets, it must, loosely speaking, involve long positions in assets that have large variances and large positive covariances with other assets. The assets held short must have smaller variances and covary less with other assets. If the mean returns are generally higher for the types of assets that must be held long and lower for those held short, the mean returns on the long and short positions will not be offset. Thus, we will be unable to solve the dual for all μ, and there will be positively weighted portfolios on the frontier. The single-factor APT and CAPM both imply that assets with large variances and covariances with other assets have high mean returns and, hence, that the frontier includes portfolios with positive weights. Multiple-factor APT's do not necessarily have this property, as Example 1 shows, because "no-arbitrage" simply delivers linearity in the factor loadings. It does not restrict in any way the "prices" of the factors. The example exploits the freedom to price them in a pathological way. The assets that are most responsive to the first factor are the least responsive to the second factor, and, yet, both factors have the same "price" (i.e., $\lambda_1 = \lambda_2 = 1$). One would suspect that general-equilibrium versions of the APT, where factor prices depend on marginal rates of substitution, would rule out this type of behavior over reasonably sized subsets. These issues are explored in detail in Tiemann [14].

The next two examples also involve situations in which assets with larger variances and covariances do not have larger mean returns. The numerical example given in Roll [10, p. 173] of an efficient frontier with no positively weighted portfolios exhibits the behaviors summarized by Example 3. In Roll's example, asset 2 has an average mean return, but its variances and covariances are all greater than or equal to the average. Given the above discussion, however, we would qualify Roll's conclusion: "There seems no necessary economic reason to consider such examples pathological," [10, pp. 173–74]. Example 3 is also used in Section IV.E in discussing the behavior of sample estimates of the efficient frontier.

Example 2: Suppose there are two of the $n > 2$ assets, subscripted j and k, related as follows. For every i, $\sigma_{ji} > \sigma_{ki}$. Note that this implies that $\sigma_j^2 > \sigma_{jk} > \sigma_k^2 > 0$ so that the two returns must be positively correlated. Now consider spending one dollar on asset j and shorting an equal amount of asset k. We choose $\hat{y}_j = 1$, $\hat{y}_k = -1$, and $\hat{y}_i = 0$ for $i \neq j, k$. This portfolio has zero cost ($\hat{y}'q = 0$) and positive covariance with all assets ($V\hat{y} > 0$). From the Corollary to Theorem 1, the global minimum-variance portfolio must involve short positions. If $E_j > E_k$, $\hat{y}$ will solve the dual for all $\mu > A/C$ where $\gamma_1(\mu) > 0$, and there will be no positively weighted portfolios on the upward-sloping portion of the frontier. If $E_j < E_k$, there will, analogously, be no positively weighted portfolios along the minimum-variance inefficient portion of the frontier where $\mu < A/C$. Finally, if $E_j = E_k$, there will be no positively weighted portfolios anywhere on the frontier since $\hat{y}$ will solve the dual for all mean returns.

Example 3: This example generalizes the intuition underlying the previous one. Suppose an asset's return has the property that its variance is larger than its average covariance with other assets' returns, and its covariance with any other asset's return is larger than the average covariance of that return with each of the others. With no loss of generality, suppose this is the return on asset 1. Construct $\hat{y}$ by buying asset 1 with the proceeds from shorting equal fractions of the other assets. Thus, $\hat{y}_1 = 1$, $\hat{y}_j = -1/(n-1)$ for $j > 1$, and $\hat{y}'q = 0$. The components of $V\hat{y}$ will be

$$\sigma_1^2 - \sum_{j=2}^{n} \frac{\sigma_{1j}}{n-1} > 0, \tag{27}$$

$$\sigma_{1i} - \sum_{j=2}^{n} \frac{\sigma_{ij}}{n-1} > 0, \qquad i = 1, \cdots, n. \tag{28}$$

Again, by the Corollary, the global minimum-variance portfolio must involve negative weights. If $E_1 < \sum_{j=2}^{n} E_j/(n-1)$ and asset 1 has a lower-than-average mean return, there will be no positively weighted portfolios on the upward-sloping part of the frontier; if it has a relatively high mean return, there will be none on the inefficient portion of the frontier. If $\tilde{R}_1$ has an average mean, there will be no positively weighted portfolios anywhere on the frontier.

E. Efficient Frontiers Computed from Sample Moments[1]

The previous subsection argued that the conditions that ensure all-positive weights in some efficient portfolios are intuitively reasonable and might be shared by many asset-pricing models. When the number of assets is large, however, very slight deviations from these conditions can lead to short sales in all efficient portfolios. The signs of the portfolio weights become sensitive to small changes in the parameters for a very simple reason. If the number of assets is very large and all assets have positive weights, some of these weights must be very close to zero. Continuity of the weights in the parameters, then, necessarily implies that small perturbations may change their signs. These perturbations could be due to sampling error. Thus, the examples presented below may shed some light on empirical experience, which suggests that efficient portfolios computed from sample moments for a reasonably large cross-section almost always involve short sales or binding short-sale constraints.[2]

One example of this behavior is evident from the analysis of the constant-correlation case in Elton, Gruber, and Padberg [2]. Minor manipulation of their equation (11) shows that the weight on asset i in the minimum-variance portfolio with mean return μ is proportional to

$$\frac{E_i - r_z(\mu)}{\sigma_i} - \frac{n\rho}{1-\rho+n\rho}\left[\frac{1}{n}\sum_j \frac{E_j - r_z(\mu)}{\sigma_j}\right], \tag{29}$$

where ρ denotes the constant correlation coefficient between assets. As n increases, $n\rho/(1-\rho+n\rho)$ approaches unity. For the weights to be uniform in sign,

[1] I thank R. Grauer for bringing these issues to my attention.

[2] For example, Kroll, Levy, and Markowitz [6], using sample moments for twenty securities from 1949–68, find that only nine ever receive positive weight and, on much of the frontier, only four or five securities do. Pulley [9] and Kallberg and Ziemba [4] report similar results.

therefore, the reward-to-variability ratios must lie in an ever smaller interval around their average. In the limit they must be equal.

The assumption of constant correlation puts a great deal of structure on the first two moments of asset returns. Assets are distinguished from each other solely through their means and own variances, and all assets with the same variance have the same market beta. The single-factor (or single-index) model for the return-generating process offers more flexibility since assets with the same beta can have different variances. The duality conditions in Theorem 2, however, can be used to show that similar behavior arises with the single-index model. We assume that $\beta_i > 0$, for $i = 1, \cdots, n$. This is not an unreasonable assumption when n is comparable to the number of assets used in computing sample efficient sets.[3] Let $\sigma^2(\varepsilon_i)$ denote the residual variance of the return on asset i, and assume that the variance of the "factor" is normalized to unity so that $\sigma_i^2 = \beta_i^2 + \sigma^2(\varepsilon_i)$ and $\sigma_{ij} = \beta_i\beta_j$. Suppose, without loss of generality, that asset 1 has a higher-than-average beta. We will attempt to solve the dual as in Example 3 of Section IV.D by purchasing asset 1 and shorting equal proportions of the remaining assets. If this arbitrage portfolio is to satisfy $V\hat{y} \geq 0$, it must satisfy equation (27) for this case. Letting $\bar{\beta}$ denote the average beta, this requires that

$$\sigma_1^2 = \beta_1^2 + \sigma^2(\varepsilon_1) \geq \textstyle\sum_{i=2}^{n} \dfrac{\beta_1\beta_i}{n-1} \tag{30}$$

or, since $\beta_1 > 0$, that

$$\beta_1 + \frac{n-1}{n}\frac{\sigma^2(\varepsilon_1)}{\beta_1} \geq \bar{\beta}. \tag{31}$$

This will always hold if asset 1 has a higher-than-average beta. Equation (28) requires, in this case, that, for $i = 2, \cdots, n$,

$$\beta_1\beta_i \geq \textstyle\sum_{j=2}^{n} \dfrac{\beta_j\beta_i}{n-1} + \dfrac{\sigma^2(\varepsilon_i)}{n-1} \tag{32}$$

or

$$\beta_1 \geq \bar{\beta} + \frac{\sigma^2(\varepsilon_i)}{n}. \tag{33}$$

This, too, can always be satisfied for large enough n, as long as asset 1 has an above-average beta. The expected payoff on this position is $E_1 - \sum_{j=2}^{n} E_j/lj(n-1)$. If this were zero or negative, there would be no positively weighted portfolios anywhere on the efficient frontier since we would have solved the dual for all μ greater than or equal to the expected return on the global minimum-variance portfolio. This will not be the case if $E_i = \lambda_0 + \lambda_1\beta_i$, as in the single-factor APT or CAPM, since an above-average beta must be associated with an above-average mean return. Note, however, that the construction above can, if the number of assets is large enough, be applied to any asset with an above-average beta, including assets with betas extremely close to the average. In

[3] For example, the Merrill Lynch "Beta Book" [7] for 1986 lists only 122 of 3732 stocks with betas that are less than or equal to zero. In 1982, it reported only 128 such stocks out of 4393.

equilibrium, expected returns for these assets will be very close to the cross-sectional average, and very slight measurement error in the mean returns could lead to a nonpositive expected return on the arbitrage position. Thus, even though statistical tests would not reject the equilibrium restrictions on the mean returns, the sample moments would imply short positions in all frontier portfolios.

Note that these examples could be reinterpreted in terms of "equilibrium" expected returns. A model that specified expected returns very close to those implied by CAPM or APT might not preserve the implication that there are efficient portfolios that have all-positive weights, when the number of assets is sufficiently large.

V. Concluding Remarks

What, then, can we conclude about when there will be positively weighted portfolios on the minimum-variance frontier and, particularly, on the mean-variance efficient frontier? First, the restrictions that ensure that there are positively weighted portfolios on the frontier do not, absent more structure, lead directly to restrictions that are testable on subsets of a larger universe of assets. If, for some set of $n - 1$ assets, there are positively weighted frontier portfolios, then we can construct an nth asset, as in Example 2, that rules out all-positive weights for any minimum-variance portfolio. Alternatively, if there are no positively weighted portfolios on the frontier for the selected $n - 1$ assets, then we can choose some vector of positive portfolio weights and use Theorem 1 from Kandel [5] to choose an nth asset that makes this portfolio efficient. Nevertheless, the restrictions described in this paper may provide new directions in which to look for the added structure to obtain testable restrictions on subsets. For example, Tiemann [14] derives restrictions on factor prices that guarantee that positively weighted portfolios are efficient under APT.

In a more fundamental sense, however, this paper's results suggest that some of the behaviors that lead to positively weighted efficient portfolios under CAPM may be common to other asset-pricing regimes. Most asset-pricing models, the weight of empirical evidence, and common economic intuition tell us that assets with returns that have high variances and large positive covariances with other assets' returns should also have high expected returns. This will make it difficult to construct the types of portfolios called for by Theorems 1 and 2, which solve the dual for mean returns above the global minimum-variance returns. Portfolios with returns that are positively correlated with returns on all assets are likely to have large mean returns. The following qualification should be kept in mind, however. With large numbers of assets, well-diversified portfolios will necessarily involve weights that are very small, even if they are all positive. Continuity of the weights in the means and variances must therefore imply that the signs of the weights are sensitive to small changes in these moments due to, for example, sampling error.

Appendix

LEMMA 1A: *$y'(a + b\mu) = 0$ for all μ if and only if $y'q = 0$ and $y'E = 0$.*

Proof: One direction (if) is obvious since a and b are linear combinations of q

and E. To demonstrate necessity, then, suppose that

$$\begin{aligned} 0 &= y'(a + b\mu) \\ &= [(B/D)y'q - (A/D)y'E] + [(C/D)y'E - (A/D)y'q]\mu \\ &= c + d\mu, \end{aligned} \tag{A1}$$

where $c = (B/D)y'q - (A/D)y'E$ and $d = (C/D)y'E - (A/D)y'q$. It is apparent that c and d must both be zero for this to hold for all μ. Also, if $y'q = 0$ and $y'E \neq 0$, then $d = (C/D)y'E \neq 0$ since $C = q'V^{-1}q > 0$. Thus, $c = 0$ implies that $(y'E/y'q) = (B/A)$, and $d = 0$ implies that $(y'E/y'q) = (A/C)$. But $(A/C) = (B/A)$ implies that $BC - A^2 = D = 0$, which is a contradiction since (see Merton [8, p. 1853]) D is the determinant of a strictly positive definite matrix. Q.E.D.

LEMMA 2A: *If $y \in H^- = \{y: y'(a + b\mu) \leq 0$ for all $\mu\}$, then $y'b = 0$ and $y'a \leq 0$.*

Proof: Suppose $y'b >(<) 0$. Then, by choosing $\mu^* > (<)[y'a/y'b]$, we find that $y'(a + b\mu^*) > 0$. But this implies that $y \notin H^-$. Therefore, $y'b = 0$, and it must be the case that $y'a \leq 0$. Q.E.D.

LEMMA 3A: *If there exists $y \in K$ such that $y'(a + b\mu) \leq 0$ for all μ, then there exists $\hat{y} \in K$ such that $\hat{y}'E = 0$, $\hat{y}'q = 0$.*

Proof: Consider $y \in K \cap H^-$, where H^- is defined in (21). By Lemma 2A, $y \in H^-$ implies that $y'b = 0$ and $y'a \leq 0$. If $y'a = 0$, Lemma 1A ensures that it is orthogonal to E and q, and we are done. Suppose $y'a < 0$. The global minimum-variance portfolio, x_v, is in K, by (6). We have, from (19) and (20), $x_v'b = 0$, so, from (7), $x_v'a = (1/C) > 0$. For $w \in (0, 1)$, define $\hat{y} \equiv wy + (1 - w)x_v$. Since K is a cone and both y and x_μ are in K, $\hat{y} \in K$. Since y and x_v are both orthogonal to b, $\hat{y}$ is, too. For $w \equiv x_v'a/(x_v'a - y'a) \in (0, 1)$, we also have $\hat{y}'a = 0$. Thus, $\hat{y} \in S \cap H^-$. Q.E.D.

LEMMA 4A: *If, for each μ, $K \cap \{y: y'(a + b\mu) \leq 0\} \neq \varnothing$, then $K \cap H^- \neq \varnothing$.*

Proof: Suppose $K \cap H^- = \varnothing$. Both K and H^- are convex disjoint sets in R^n, and we can separate them with an $(n - 1)$-dimensional hyperplane through the origin and the $(n - 2)$-dimensional subspace S, which is disjoint from K. Call this hyperplane H^*. Because K is closed everywhere but the origin and is convex, it is not an open half space. Therefore, H^* can be chosen to strictly separate K and $(H^- - S)$ so that $H^* \cap H^- \neq H^-$. We now choose μ^* such that $a + b\mu^*$ is the normal vector for this hyperplane. Let z be a vector of unit length in H^*, which is orthogonal to S. Because H^* does not contain H^-, $z'b \neq 0$. Then we can write any vector in H^* in terms of the orthogonal decomposition $x = \alpha y + \beta z$ for some $y \in S$. Choose $\mu^* = -z'a/z'b$. Then,

$$\begin{aligned} x'(a + b\mu^*) &= \alpha[y'a + y'b\mu^*] + \beta(z'a + z'b\mu^*) \\ &= \alpha \cdot 0 + \beta \cdot 0 = 0. \end{aligned} \tag{A2}$$

So $a + b\mu^*$ is orthogonal to all $x \in H^*$. Since the half space $\{y: y'(a + b\mu^*) \leq 0\}$ contains H^-, and H^* separates K from H^-, it must be the case that

$$K \cap \{y: y'(a + b\mu^*) \leq 0\} = \varnothing. \tag{A3}$$

But this contradicts our original hypothesis that the dual can be solved for all μ. Q.E.D.

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