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A Note on the Local Expectations Hypothesis: A Discrete-Time Exposition

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COX, INGERSOLL, AND ROSS [1] distinguished various forms of the expectations hypothesis of the term structure of interest rates. They proved that, with one exception, these are consistent with general equilibrium only in the trivial case in which interest rates are nonrandom. The exception is the Local Expectations Hypothesis. Because of this nonexistence result, those who regard the expectations hypothesis as a natural starting point in investigating the term structure of interest rates are motivated to understand under what restrictions the local expectations hypothesis is valid. In our experience, many readers of Cox, Ingersoll, and Ross's paper—particularly those not conversant with stochastic calculus—have difficulty following the discussion. Such readers may find it easier to work through an analysis of the local expectations hypothesis in a more familiar discrete-time framework. This paper provides such an exposition.

In continuous time, the local expectations hypothesis states that the conditional expected rates of return on bonds of all maturities over the next instant are equal to each other and to the instantaneous spot interest rate. Cox, Ingersoll, and Ross proved (p. 783 ff.) that the local expectations hypothesis is valid in an exchange economy if the dynamics of consumption are locally certain. In a stochastic constant-returns-to-scale production economy, they showed that the local expectations hypothesis will be valid under locally certain production dynamics if, in addition, agents have logarithmic utility.

The analogous definition of the local expectations hypothesis in discrete time is that the conditionally expected one-period rates of return on bonds of all maturities are equal to each other and to the one-period interest rate. Two examples below illustrate that the local expectations hypothesis is valid in discrete-time economies under the same conditions as in continuous-time economies. These examples work because a sufficient condition for the validity of the local expectations hypothesis is that the marginal utility of consumption be locally certain. This might happen either because consumption itself is locally certain (as in both of the above-mentioned examples) or because the utility function exhibits risk neutrality *and* there is strictly positive consumption in every period. A third example shows that risk neutrality alone does not imply the local expectations hypothesis if zero consumption can occur. The reason is that the implication of risk neutrality that rates of interest are nonrandom requires the exclusion of corner solutions, a restriction which is violated if zero consumption can occur.

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I. The Local Expectations Hypothesis in an Exchange Economy with Risk Aversion

Consider a one-good exchange economy in which a representative agent maximizes the expected discounted sum of some concave utility function, u:

$$\text{Max } E_0[\textstyle\sum_{t=0}^{\infty} \beta^t u(c_t)]. \tag{1}$$

In equilibrium, of course, consumption c_t is just equal to the endowment, the growth rate (s_t) of which is assumed to follow a stationary stochastic process. Since Mehra and Prescott [3] and Mehra [2] have studied this model rigorously in the Markov case, we proceed informally here. We impose a local certainty assumption, which means that at t (prior to trading), agents learn the endowment growth rate, s_t, from t to $t + 1$:

$$c_{t+1} = c_t s_t \tag{2}$$

rather than from $t - 1$ to t, as would be the appropriate specification in the absence of the local certainty restriction.

Define $p(t, \tau)$ to be the price at date t of a discount bond that matures (i.e., pays one unit of the consumption good) at $t + \tau$. This price is given by

$$p(t, \tau)u'(c_t) = \beta^\tau E_t u'(c_{t+\tau}), \tag{3}$$

since in equilibrium, the gain in current utility from a small increase in current consumption financed by a sale of bonds must just equal the expected utility of consumption of the principal at maturity.

Define the one-period rate of interest, r_t, by

$$1 + r_t \equiv \frac{1}{p(t, 1)} = \frac{u'(c_t)}{\beta E_t u'(c_{t+1})}. \tag{4}$$

By the local certainty restriction, $u'(c_{t+1})$ is known at t, therefore

$$1 + r_t = \frac{u'(c_t)}{\beta u'(c_{t+1})}. \tag{5}$$

Note for reference below that in the case of logarithmic utility, this specializes to

$$1 + r_t = \frac{c_{t+1}}{\beta c_t} \equiv \beta^{-1} s_t. \tag{6}$$

Using (5) and (3), it is easy to verify that for all $\tau > 1$

$$\frac{E_t p(t + 1, \tau - 1)}{p(t, \tau)} = \beta^{\tau-1} E_t\left[\frac{u'(c_{t+\tau})}{u'(c_{t+1})}\right] \Big/ \beta^\tau E_t\left[\frac{u'(c_{t+\tau})}{u'(c_t)}\right] = \frac{u'(c_t)}{\beta u'(c_{t+1})} = 1 + r_t.$$

The economic intuition underlying this proof is completely trivial. It is known that, in fairly general settings, the risk premium on any asset has the sign of the negative of the covariance of its one-period yield with the marginal utility of consumption. Here, due to the local certainty assumption, agents know their next-period consumption, and hence its marginal utility. The required covariance must be zero. Thus, even though interest rates do fluctuate, implying that a bond portfolio is subject to random capital gains and losses, under local certainty the one-period risk arising from this source is fully diversifiable, and hence is priced actuarially fairly.

II. The Local Expectations Hypothesis in a Production Economy with Logarithmic Utility

Next, consider a stochastic constant-returns-to-scale one-good production economy in which s_t now represents the productivity of capital. Successive values of wealth—i.e., of capital—are given by

$$w_{t+1} = (w_t - c_t)s_t. \tag{7}$$

Preferences are given by (1) restricted to the case $u(\cdot) = \log(\cdot)$.

The easiest way to solve for equilibrium prices is to derive a Pareto-optimal allocation, and then exploit the fact that this allocation must be supported by an equilibrium price vector. In fact, this equilibrium price vector will be the same as that in an exchange economy in which the exogenous consumption process is specified to coincide with the process induced on consumption in the production economy by the production process and the utility-maximizing consumption rule.

Under logarithmic utility the planner maximizes expected utility of the representative agent by allocating consumption according to:

$$c_t = (1 - \beta)w_t, \tag{8}$$

as can be verified by elementary methods. Note that (8) says that s_t, the productivity of invested capital from t to $t + 1$, does not influence the current saving decision. This invariance reflects the exact cancellation of income and substitution effects that is characteristic of logarithmic utility. Iterating on (7) and (8), we obtain

$$c_{t+\tau} = (1 - \beta)\beta^{\tau} w_t s_t \cdots s_{t+\tau-1}. \tag{9}$$

The equilibrium prices we seek for our production economy correspond exactly to those of an exchange economy in which consumption is described by (9). Therefore, equilibrium bond prices are derived by substituting (8) and (9) into (3). Recalling that u is logarithmic, we obtain

$$p(t, \tau) = E_t(s_t \cdots s_{t+\tau-1})^{-1}. \tag{10}$$

The one-period interest rate, in turn, is given by

$$1 + r_t \equiv 1/p(t, 1) = s_t. \tag{11}$$

Note the similarity between (6) characterizing bond prices in an exchange economy with logarithmic utility and the corresponding (11) for a production economy: the only difference is that β^{-1} appears in (6), but not in (11). The interpretation is that, in an exchange economy, high rates of time-preference (β low) show up in high rates of return on bonds, since agents must be induced to consume no more than the endowment. In the production economy, however, agents can adjust aggregate consumption in favor of the present by adopting a low savings rate [from (8), β is just the savings rate out of wealth], so that the time-preference parameter need not affect equilibrium interest rates.

Just as in the exchange economy studied above, it is easily verified from (10) and (11) that $p(t, \tau)$ satisfies

$$\frac{E_t p(t+1, \tau-1)}{p(t, \tau)} = (1 + r_t),$$

so that under local certainty, the local expectations hypothesis holds in a production economy if utility is logarithmic. The necessity for the restriction to logarithmic utility in the production setting arises from the fact that, in general, local certainty in the productivity of capital does not assure local certainty in consumption: c_{t+1} will depend on s_{t+1}. With random marginal utility of consumption, the covariance of one-period rates of return on bonds with the marginal utility of consumption will be nonzero. Hence, the same will be true of term premia, and the local expectations hypothesis will fail. But with logarithmic utility, consumption at $t + 1$ is a constant fraction of wealth at $t + 1$, and wealth at $t + 1$ is known at t. Hence, the dependence of the marginal utility of consumption at $t + 1$ on the random s_{t+1} disappears.

III. Risk Neutrality without the Local Expectations Hypothesis

Next we show that risk neutrality does not guarantee the validity of the local expectations hypothesis if zero consumption can occur. As in the second example, consider a production economy with a one-good, stochastic constant-returns-to-scale technology. But assume that consumption occurs only in one final period, with utility proportional to final wealth. Hence, the representative agent maximizes $E_0(w_T)$, with wealth following the stochastic difference equation

$$w_{t+1} = w_t s_{t+1}, \quad w_0 \text{ given}. \tag{12}$$

A crucial characteristic of (12) is that, in contrast to (7), the relevant productivity of capital from t to $t + 1$ is not determined until $t + 1$ (wealth is not locally certain). In such an economy, the prices of bonds obey the following relation:

$$p(t, \tau) = [E_t(s_{t+1}, s_{t+2}, \cdots, s_{t+\tau})]^{-1}. \tag{13}$$

This is so because one unit of wealth can be successively used in production, leading to $s_{t+1}, \cdots, s_{t+\tau}$ units of consumption at $t + \tau$. Because risk-neutral agents will be indifferent at t between this and $E_t(s_{t+1}, \cdots, s_{t+\tau})$ units of consumption at τ with certainty, and also between investing one unit of wealth in this way and buying $1/p(t, \tau)$ bonds, (13) must result. Observe, incidentally, the contrast between (13) and (10).

It may be proved from (13) that

$$E_t\left[\frac{p(t+1, \tau-1)}{p(t, \tau)}\right] = E_t(s_{t+1}, s_{t+2}, \cdots, s_{t+\tau}) \cdot E_t[E_{t+1}(s_{t+2}, \cdots, s_{t+\tau})]^{-1}.$$

But now, since s_{t+1} is random at t, and as long as the shocks are serially dependent, we will have in general:

$$E_t\left[\frac{p(t+1, \tau-1)}{p(t, \tau)}\right] \neq E_t[s_{t+1}] = 1 + r_t$$

so that the local expectations hypothesis does not hold. On the other hand, if the shocks are serially independent, interest rates are non-random, so we are back to the trivial version of the expectations hypothesis.

IV. Summary

The first example above exhibits an exchange economy with a representative risk-averse agent who knows his or her next-period's endowment. Consumption is then locally certain, and the local expectations hypothesis holds. The second example exhibits a production economy. The representative agent knows the next-period rate of growth of wealth and his or her logarithmic preferences make consumption again locally certain; thus, the local expectations hypothesis holds also. In both cases, one-period rates of return on bonds of different maturities are random, so the validity of the local expectations hypothesis is not trivial. In a third example, a risk-neutral agent has zero consumption (a corner solution) until a final date. In this case, the local expectations hypothesis does not hold. We see that the local expectations hypothesis holds if the marginal utility of consumption is locally certain. For this condition to be satisfied, risk neutrality is neither necessary nor sufficient.

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