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The Duration of an Adjustable-Rate Mortgage and the Impact of the Index

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ABSTRACT

With the increasing use of adjustable-rate mortgages for asset/liability management, there exists the need to properly evaluate their price sensitivity to interest rate changes. This paper provides a foundation by deriving the duration of an adjustable-rate mortgage. The properties of this duration are unique and have some important differences from those of fixed-rate securities. One important characteristic of an adjustable-rate mortgage concerns the index used to adjust the mortgage rate. It was found that the index tended to be more important than the adjustment frequency in determining the duration of an adjustable-rate mortgage.

THRIFT INSTITUTIONS AND OTHER mortgage investors need to incorporate the large quantities of adjustable-rate mortgages (ARMs) being originated into an overall portfolio strategy. An ARM is typically classified by the frequency of its rate adjustment, and thus, is often viewed as though the mortgage rate adjusts to that of the market interest rate on the adjustment date. Indeed, the Federal Home Loan Bank Board (FHLBB) treats all ARMs in this fashion when providing interest rate sensitivity reports to each member institution. While the frequency of the rate adjustment determines the speed at which the mortgage rate responds to a change in the market interest rate, other characteristics also affect the price sensitivity of this mortgage type. Among these characteristics is the index used to adjust the mortgage rate. Not only is the index an essential characteristic of all ARMs, it is also the volatility of the index that determines the possible changes in the mortgage rate.

This paper provides a foundation for evaluating the price sensitivity of an ARM by deriving the duration of this mortgage type. Duration is a statistic that measures the price sensitivity of a cash flow stream to a change in the market interest rate. This concept has been generally applied to fixed-rate securities with given cash flows. Evaluating the duration of an ARM involves the complication that the cash flow stream varies with changes in the market interest rate. This complication not only affects the duration of this mortgage type, but also results in duration characteristics which differ from securities with fixed cash flow streams.

This paper is organized into four sections. The first section presents a derivation of the duration of an ARM in which the main characteristics considered are

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the adjustment frequency and the index type. The derivation reveals that the duration can be calculated by viewing the ARM partially as a mortgage which adjusts completely to the market on the adjustment date and partially as a fixed-rate level payment mortgage, where the proportion depends on the responsiveness of the index to a change in the market interest rate. The second section examines some of the properties of the duration of an ARM. The duration is shown to vary with the term to maturity, the current market interest rate, the initial mortgage contract rate, and the adjustment period. The third section uses a simple stock adjustment model to estimate the movement of different indexes in relation to a market interest rate and illustrates the impact on the duration of an ARM. The last section summarizes the conclusions of the study.

I. Duration of an ARM

Duration can be viewed as that point in time where the future value of the cash flows is invariant to changes in interest rates.[1] Mathematically, the future value of the cash flows at the mth period can be expressed as:

$$FV(m) = (1 + r)^m \sum_{t=1}^{n} C_t(1 + r)^{-t}, \tag{1}$$

where C_t is the cash flow in period t and r the market interest rate. Setting the partial derivative of Equation (1) with respect to the market interest rate equal to zero and solving for m, yields the following duration measure:

$$m = \sum_{t=1}^{n} tC_t(1 + r)^{-t} / \sum_{t=1}^{n} C_t(1 + r)^{-t}. \tag{2}$$

This mathematical expression is referred to as the Macaulay duration measure when the market interest rate is the yield to maturity.[2] This measure also holds under the assumption of a flat yield curve in which interest rates move by a constant proportion. The weighting of the periods by the ratio of the present value of each cash flow to the present value of the cash flow stream would be unaffected by the more general assumption that the interest rates along the yield curve of zero-coupon bonds move by a constant proportion. While the remainder of this paper will use the Macaulay measure of Equation (2) because of its simplicity in exposition, the analysis can be extended to different yield curves with different stochastic shocks.

Many short-cut equations exist for calculating the duration of Equation (2). Macaulay [5] originally formulated one for a coupon bond.[3] Applying his derivation to a fixed-rate level payment mortgage where C_t is a constant, the Macaulay duration of Equation (2) can be expressed more simply as:

$$m' = (1 + r)/r - n/((1 + r)^n - 1). \tag{2'}$$

In this equation, the duration of a fixed-rate mortgage is a function of only the market interest rate and is independent of the mortgage rate. Later, it will be shown this is not the case for an ARM.

[1] This is also the point in time where the change in the value of the reinvestment income is offset by the change in the value of the asset. See Bierwag [2].

[2] See Macaulay [5].

[3] Macaulay [5], pp. 48–9.

Bierwag [1, 2], Khang [4], and others have explored duration measures assuming different shapes and stochastic interest rate shocks in the yield curve. In their analyses, the cash flows were always assumed unchanged for different market rates of interest. However, the cash flows of ARMs can vary after the first adjustment period, with a change in the market interest rate. To solve this problem of a potential change in the mortgage payments, the future value expression of Equation (1) must assume that each cash flow after the first adjustment period is a continuous function of the market interest rate.

$$FV(m) = (1 + r)^m(\textstyle\sum_{t=1}^{j} C_t(1 + r)^{-t} + \sum_{t=j+1}^{n} C(r)(1 + r)^{-t}), \qquad (1')$$

where j is the period of the first payment adjustment and $C(r)$ a cash flow that depends on the market interest rate. This equation views the cash flows or mortgage payments over the life of the loan as known, but different, for each specific market interest rate. Any change in the market interest rate now affects the future value indirectly by changing the mortgage payments in a nonstochastic manner.

The duration for a variable cash flow stream now can be derived by setting to zero the partial derivative of Equation (1′) with respect to the market interest rate and solving for m.

$$m = (\textstyle\sum_{t=1}^{n} tC_t(1 + r)^{-t}$$
$$-(1 + r)\textstyle\sum_{t=j+1}^{n} (dC/dr)(1 + r)^{-t})/\sum_{t=1}^{n} C_t(1 + r)^{-t}, \qquad (3)$$

where dC/dr represents the partial derivative of the cash flow with respect to the market interest rate. Note that if the cash flows are invariant to interest rate movement, then dC/dr would be zero and Equation (3) simplifies to the Macaulay duration measure of Equation (2).

To convert the duration of a variable cash flow stream into one specific for an ARM, the formula for the mortgage payment must be substituted for the cash flows in Equation (3). In particular, the mortgage payment per dollar of principal balance can be specified as a function of the mortgage rate as follows:

$$C = r^*/(1 - (1 + r^*)^{-n}), \qquad (4)$$

where r^* is the mortgage rate and n the term to maturity. To determine how the mortgage payment varies with a change in the market interest rate, the derivative of Equation (4) with respect to r is taken, giving

$$dC/dr = C(1 + r^*)^{-1}m^*(dr^*/dr), \qquad (5)$$

where dr^*/dr is the partial derivative of the mortgage rate with respect to the market interest rate and m^*, $(1 + r^*)/r^* - n/((1 + r^*)^n - 1)$, the duration of a fixed-rate level payment mortgage depicted in Equation (2′), where the interest rate is the mortgage rate, r^*, instead of the market interest rate. For an ARM indexed to the market interest rate, dr^*/dr equals one, and thus, would drop out of the equation.

Substituting Equation (5) into Equation (3) and using m' from Equation (2′) gives an expression for the duration of an ARM where the first adjustment occurs in the j[th] payment.

$$m = (1 - [a_t(1 + r)m^*/(1 + r^*)m'])m' + [a_t(1 + r)m^*/(1 + r^*)m'] \cdot m'\left(\sum_{t=1}^{j} (1 + r)^{-t}/\sum_{t=1}^{n} (1 + r)^{-t}\right), \tag{6}$$

where a_t is $\sum_{t=j+1}^{n} (dr^*/dr)(1 + r)^{-t}/\sum_{t=j+1}^{n} (1 + r)^{-t}$ and can be referred to as the adjustability coefficient of the index. Note that when the index lags in adjusting to the market interest rate, the adjustability coefficient is calculated like the Macaulay duration formulation in that it is the weighted average of the changes in the index relative to the market interest rate, where the weights are the present value of each period as a percentage of the total present values of all the periods. In Equation (6), the duration of an ARM is a weighted average of the durations of a fixed-rate level payment mortgage and of an ARM that adjusts fully to the market interest rate on its adjustment date, where the weight is $a_t(1 + r)m^*/(1 + r^*)m'$. If the mortgage rate equals the market interest rate, then the weight simplifies to a_t. Equation (6) represents the duration of a fully adjusted ARM (D_{ARM}) when the index adjusts perfectly to the market interest rate, or when a_t equals one and r^* equals r.

$$D_{ARM} = m'\left(\sum_{t=1}^{j} (1 + r)^{-t}/\sum_{t=1}^{n} (1 + r)^{-t}\right). \tag{6'}$$

The ratio of the present value of the cash flows up to the first adjustment period to the present value of the mortgage loan is equivalent to the proportion of the mortgage balance amortized by the first adjustment period. Therefore, the duration of the fully adjusted ARM is the product of the duration of a fixed-rate level payment mortgage and the proportion of the mortgage balance amortized by the first adjustment period.

II. Properties of the Duration of an ARM

To better understand the duration of an ARM, some of its properties are examined, especially those that relate to the fully adjusted ARM. The first, and most obvious, property relates to the adjustment period. As the adjustment period approaches the term to maturity (j approaches n), the proportion of the mortgage amortized approaches one, and consequently, the duration would equal that of a fixed-rate level payment mortgage. This result implies that the duration of a fixed-rate mortgage can be viewed as a special case of the duration of a fully adjusted ARM, one in which the adjustment period equals the term to maturity. Contrariwise, as the adjustment period becomes smaller, or approaches zero, the proportion of the mortgage balance amortized, and consequently the duration of the ARM, will approach zero. For an instantaneous adjustment period, the mortgage becomes a variable-rate mortgage with a duration of zero. Hence, depending on its adjustment period, the duration of a fully adjusted ARM can range between zero and that of a fixed-rate level payment mortgage.

Another interesting property is that the duration of the fully adjusted ARM is affected by the term to maturity. This property arises because the amortization of the mortgage principal changes as the mortgage rate changes. For a variable-rate financial instrument that does not change the amortization of the principal, such as the variable payments of a swap transaction, the duration can be calculated by including the principal amount on the adjustment date similar to that of a balloon payment. Hence, the term to maturity would be irrelevant when calculating the duration. This is not the case for a fully adjusted ARM, which has a duration that is less than or equal to that of a mortgage with a balloon payment on the adjustment date. To illustrate, a 1-year ARM amortized over 30 years making annual payments at a current rate of 12 percent will have a duration of 0.92 years or about 11 months. This duration is less than 1 year, which is the duration of the mortgage assuming a balloon payment on the adjustment date. However, as the term to maturity of a fully adjusted ARM gets longer, or approaches infinity, the duration approaches $(1 - (1 + r)^{-j})(1 + r)/r$, which is the duration of a bond priced at par that matures in j periods.[4] For the example of a 1-year ARM making annual payments at a current rate of 12 percent, the duration would now equal 1 year and be equivalent to that of a mortgage with a balloon payment on the adjustment date.

A third property of the duration of a fully adjusted ARM deals with the initial level of the market interest rate. For fixed-rate securities, the duration is a decreasing function of the market interest rate. This is not the case for an ARM. The duration of an ARM initially increases, and then declines as the market interest rate rises. This property is exhibited since the duration is the product of the proportion of the mortgage principal amortized by the first adjustment period, which is an increasing function of the market interest rate, and of the duration of a fixed-rate level payment mortgage, which is a decreasing function of the market interest rate. Interestingly, the duration of a 1-year ARM amortized monthly over 30 years will reach a maximum value when the market interest rate is 18.5 percent. Since this rate is higher than the mortgage rate history of conventional fixed-rate mortgages, the duration of an ARM can be viewed practically as an increasing function of the market interest rate contrary to that of securities with fixed cash flows.

A fourth property of the duration of an ARM involves the mortgage rate. If the mortgage rate is less than the market interest rate, then the duration would be less than one that fully adjusts to the market interest rate. For instance, if the market interest rate on a 1-year ARM amortized yearly is 12 percent while the mortgage rate is 11.75 percent, then the duration would be 9.6 months instead of the 11 months for one paying the market interest rate. Again, this characteristic differs from that of securities with fixed cash flows. Bonds have higher duration the lower the coupon rate, while fixed-rate mortgages have a duration that is unaffected by the mortgage rate.

The last property of the duration of an ARM concerns the index type used to adjust the mortgage rate. An index could adjust the mortgage rate completely to the market, making the duration equal to that of the fully adjusted ARM or the index may not adjust at all to the market, resulting in a duration equal to that of a fixed-rate level payment mortgage. In the example of a 1-year ARM amortized

[4] See footnote 3 and the proposition in the Appendix for the proof.

over 30 years making annual payments at a current rate of 12 percent, the adjustability of the index relative to the market implies a range between 11 months for a fully adjusted ARM and 8.3 years for a fixed-rate level payment mortgage. However, it is also possible to use an index that overreacts to changes in the market interest rate, that is, dr^*/dr would be greater than one. This may occur when using a market interest rate with a term less than the adjustment period. In this latter case, the duration could be substantially less than the 11 months of the fully adjusted ARM. Lastly, although not too likely, it is theoretically possible to have an index that moves in the opposite direction of the market interest rate. In this case, dr^*/dr would be negative and the duration would be longer than that of a fixed-rate level payment mortgage. Thus, it is important for thrift institutions and other investors in ARMs to know how the index type used to adjust the mortgage rate varies with the market interest rate when trying to assess the interest rate sensitivity of an ARM.

III. Adjustability of ARM Indexes

To obtain a better understanding of the actual importance of the index type chosen, the relationships between different indexes and the market interest rate were examined. A simple stock adjustment model is used to illustrate how a specific index is related to the market interest rate.

$$I_t = a + bR_t + cI_{t-1} + U_t, \tag{7}$$

where I is the specific index, R the market interest rate, and U an error term. The coefficient b depicts how much the market interest rate affects the index each period, while c is the adjustment coefficient relating how fast the index adjusts. If b equals one and c zero, the index adjusts perfectly to the market; while if b is zero and c one, the index is unresponsive to changes in the market interest rate. For incomplete adjustment to the market interest rate, the adjustment for j periods equals $\sum_{i=1}^{j} bc^i$, while the total adjustment would be $b/(1-c)$.

Four rate adjustment frequencies have accounted for nearly all the ARMs made (6 months; 1 year; 3 years; and 5 years) and consequently were chosen for estimating Equation (7). The market rates of interest for each are taken to be the 6-month Treasury bill rate and the 1-, 3-, and 5-year Treasury security yields, respectively.[5] The indexes examined are the FHLBB Monthly Median Annualized Cost of Funds for FSLIC-Insured Institutions (cost-of-funds index), the FHLBB National Average Mortgage Contract Rate for Major Lenders on the Purchase of Previously Occupied Homes (mortgage contract rate index), and three Treasury bill rates (3 months, 6 months, and 1 year). The Treasury bill rates are those quoted in the secondary market and reported by the Federal Reserve. The data used were monthly for the period January 1980 to June 1985. Using ordinary least squares, the results are presented in Table I. Since the estimation of the equations that contain the Treasury bill rates as indexes revealed an autocorrelation problem, the equations were reestimated using the Cochrane-Orcutt technique. The adjustability coefficients of the indexes and

[5] The rates are those in the Federal Reserve statistical release G.13 or H.15.

their durations are calculated using a 12 percent mortgage, amortized monthly over 30 years with the market interest rate being 12 percent.

The cost-of-funds index has the lowest adjustability coefficients of the indexes examined, ranging between 53.2 percent for the 1-year adjustment frequency to 63.6 percent for the 5-year adjustment frequency. According to the adjustability coefficient, a little under half the volume of the ARMs tied to this index type should be treated as fixed-rate level payment mortgages. Consequently, the durations of the 6-month and 1-year ARMs would be 3.4 and 4.0 years, respectively. In comparison, the fully adjusted ARMs would have durations of only 5.4 and 10.5 months, respectively. Interestingly, an adjustment frequency of either 3 or 5 years does not substantially increase the duration of an ARM tied to this index. The 3- and 5-year ARMs have durations of 4.6 and 5.0 years, respectively. Hence, this index type appears to be the more important characteristic in determining the duration of an ARM than its adjustment frequency.

The mortgage contract rate index responds much more quickly than the cost of funds index to changes in the market interest rate. The adjustability coefficients ranged between 89.4 percent for the 1-year adjustment frequency and 119.0 percent for the 5-year adjustment frequency. This index nearly doubles the durations of the ARMs with the 6-month and 1-year adjustment frequencies from those that fully adjust to the market. However, for the 3-year adjustment frequency, the adjustability coefficient is slightly greater than one, resulting in the 3-year ARM tied to this index having a price sensitivity close to that of the fully adjusted ARM with the same adjustment frequency. For the 5-year adjustment frequency, the adjustability coefficient is substantially greater than one, resulting in a duration less than the one that fully adjusts. This latter case is an example of the index adjusting more quickly than the market interest rate. Hence, when using the mortgage contract rate index, a 3-year adjustment frequency would be the most appropriate if one wished the mortgage to have an interest rate sensitivity close to the one that fully adjusts to the market.

By examining the 3-month Treasury bill rate as an index, the effect on duration of using a market rate index with a term shorter than the adjustment frequency can be assessed. For all four adjustment frequencies, the adjustability coefficient is greater than 100 percent, revealing that the index changes by a greater amount than the market interest rate. Specifically, the adjustability coefficient ranges between 108.3 percent for the 1-year adjustment frequency and 146.1 percent for the 5-year adjustment frequency. The resulting durations for the 1- and 3-year ARMs are 3.8 and 5.0 months, respectively, only a 1.2-month difference. The 6-month ARM actually has a negative duration of 3.6 months. An asset with a negative duration can simply be viewed as a short position in the market where one promises to sell a security in the future at a given price. The 3- and 5-year ARMs that use the 6- and 12-month Treasury bill rates as indexes also have adjustability coefficients greater than 100 percent. Generally, using a market rate index with a term less than the adjustment frequency can substantially decrease the duration of the ARM from one that fully adjusts to the market. One exception can be seen in Table I where the 6-month Treasury bill rate is used in a 1-year ARM. For this case, the adjustability coefficient is only 98.8 percent.

Table I

Estimates of the Adjustability of ARM Indexes and the Effect on the Duration of ARMs

Dependent Variable	Constant	Independent Variables TB6	TY1	TY3	TY5	COF_{-1}	MR_{-1}	$TB3_{-1}$	$TB6_{-1}$	$TB12_{-1}$	Regression Statistics R^2	DW	Adjustability Coefficient* (%)	Duration* (months)
COF	0.25	0.05				0.93					0.95	1.64	58.8	40.56
	(0.85)	(4.39)				(33.30)								
COF	0.31		0.05			0.91					0.95	1.64	53.2	48.00
	(1.06)		(4.34)			(32.44)								
COF	0.36			0.07		0.89					0.95	1.59	57.6	55.00
	(1.24)			(4.28)		(29.38)								
COF	0.29				0.08	0.88					0.95	1.58	63.6	59.76
	(1.00)				(4.39)	(28.44)								
MR	0.23	0.10					0.90				0.98	1.96	94.4	10.14
	(0.92)	(8.44)					(41.97)							
MR	0.25		0.10				0.89				0.98	1.99	89.4	19.04
	(1.01)		(8.53)				(40.64)							
MR	0.05			0.13			0.88				0.97	1.67	105.1	24.93
	(0.18)			(7.23)			(35.07)							
MR	−0.17				0.14		0.88				0.97	1.48	119.0	32.69
	(0.57)				(6.63)		(33.98)					RHO		
TB3	−1.16	1.05						0.05			0.99	0.77	110.5	−3.58
	(3.45)	(35.53)						(1.67)				(9.54)		
TB3	−1.73		0.95					0.12			0.98	0.79	108.3	3.85
	(2.69)		(19.55)					(2.56)				(9.66)		
TB3	−4.48			1.08				0.21			0.95	0.86	136.8	5.05
	(3.33)			(11.44)				(2.84)				(12.03)		
TB3	−5.51				1.15			0.21			0.93	0.86	146.1	19.45
	(3.17)				(9.10)			(2.41)				(11.79)		
TB6	−0.73		0.93						0.06		0.99	0.79	98.8	11.47
	(2.16)		(36.16)						(2.28)			(9.62)		
TB6	−3.78			1.09					0.14		0.97	0.91	127.9	10.62
	(3.70)			(16.58)					(2.66)			(15.91)		
TB6	−5.14				1.20				0.14		0.95	0.91	139.7	22.58

	(3.77)				(12.69)	(2.11)			(15.68)		
TB12	1.62	0.85					−0.02	0.99	0.80	83.3	19.62
	(5.55)	(35.72)					(0.70)		(10.30)		
TB12	0.78		0.83				0.01	0.9998	0.65	83.8	23.52
	(20.80)		(241.60)				(2.87)		(6.32)		
TB12	−2.27			1.02			0.08	0.99	0.96	110.7	21.36
	(3.28)			(29.61)			(2.42)		(25.69)		
TB12	−3.85				1.14		0.07	0.98	0.95	123.5	30.47
	(4.05)				(20.14)		(1.61)		(22.66)		

Notes: COF is the FHLBB monthly median cost of funds; MR is the FHLBB mortgage contract rate; TB3 is the 3-month Treasury bill rate; TB6 is the 6-month Treasury bill rate; TB12 is the 12-month Treasury bill rate; TY1 is the 1-year Treasury security yield; TY3 is the 3-year Treasury security yield; TY5 is the 5-year Treasury security yield; The numbers in parentheses represent the absolute values of the t-statistic.

* The adjustability coefficient and the duration are calculated assuming a 12 percent mortgage, amortized monthly over 30 years.

IV. Conclusions

The price sensitivity of an ARM to changes in the market interest rate can be characterized by its duration. This paper demonstrated that the duration of an ARM depends critically on the index type chosen to adjust the mortgage rate. An ARM indexed to something other than the market interest rate can be viewed partially as an ARM that fully adjusts to the market on its adjustment date and partially as a fixed-rate level payment mortgage. The proportion of each is determined mainly by an adjustability coefficient that is the weighted average of the changes in the index for each adjustment period relative to a change in the market interest rate. The weights are the present value of each period as a percentage of the total present values of all the periods, and interestingly, parallel with those of the Macaulay duration measure.

Examining the different indexes reveals that a 1-year ARM could have a duration ranging between 3.8 months using a 3-month Treasury bill rate and 4.0 years using the cost-of-funds index. Thrift institutions viewing all 1-year ARMs as though they fully adjust to the market on the adjustment date may not find the interest rate sensitivity in this mortgage type that they anticipated if a large change in the market interest rate occurred. For thrift institutions that are trying to reduce the duration of their assets, ARMs tied to a cost-of-funds type index would be the least effective. ARMs using a market rate index with a term less than or equal to their adjustment frequency will typically have the greatest effect on reducing the overall duration of a thrift institution's assets.

Although the index type is shown to be a major concern when examining the interest rate sensitivity of ARMs, there are other important characteristics. Rate caps and prepayment options are among the most important since they are included in nearly all the ARMs being made. This paper provides a framework for analyzing the effect of these other options on the price sensitivity of an ARM. Although including these options will change the impact of the index type on the duration of an ARM, the index type will continue to be an essential characteristic in assessing the interest rate sensitivity of this mortgage type.

Appendix

PROPOSITION. *As the term to maturity approaches infinity, the duration of a fully adjusted* ARM *is given by* $(D_{ARM}) = (1 - (1 + r)^{-j}(1 + r)/r$.

Proof: From Equation (6′), the duration of a fully adjusted ARM can be expressed as:

$$D_{ARM} = m'\left(\sum_{t=1}^{j} (1 + r)^{-t} \Big/ \sum_{t=1}^{n} (1 + r)^{-t}\right). \tag{6′}$$

Since the $\sum_{t=1}^{i} (1 + r)^{-t} = (1 - (1 + r)^{-i})/r$, Equation (6′) can be written more simply as:

$$\mathrm{D}_{ARM} = \mathrm{m}'(1 - (1 + r)^{-j})/(1 - (1 + r)^{-n}). \tag{6a}$$

Since the limit of the product of two expressions is the product of the limits of the two expressions, the limit of D_{ARM} is divided into the limit of the duration of a fixed-rate mortgage and the limit of the ratio of the two present values:[6]

$$\lim_{n\to\infty} m' = \lim_{n\to\infty}((1+r)/r - (n(1+r)^{-n})/(1-(1+r)^{-n}))$$
$$= (1+r)/r, \qquad (6b)$$

which is the duration of a perpetuity or consol, and

$$\lim_{n\to\infty}(1 - (1+r)^{-j})/(1 - (1+r)^{-n})$$
$$= (1 - (1+\mathrm{r})^{-j})/\lim_{n\to\infty}(1 - (1+r)^{-n})$$
$$= (1 - (1+r)^{-j})/(1 - (1+r)^{-\infty})$$
$$= (1 - (1+r)^{-j}). \qquad (6c)$$

The product of Equations (6b) and (6c) gives the desired result.

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[6] See Chiang [3] for an exposition on calculating limits.