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The Relationship between Arbitrage and First Order Stochastic Dominance

ROBERT JARROW*

ABSTRACT

This paper joins together two fields of research in financial economics. The first field studies stochastic dominance, while the second field studies arbitrage pricing. The two fields are linked together through the derivation and the proof of a characterization theorem. The characterization theorem gives necessary and sufficient conditions for the existence of arbitrage opportunities in terms of the existence of two assets, one of which first order stochastically dominates the other and the price of a particular contingent claim. Examples are provided to demonstrate the theorem's content.

THIS PAPER JOINS TOGETHER two fields of research in financial economics. The first field studies portfolio choice from the perspective of stochastic dominance (see Hadar and Russell [5, 6], Hanoch and Levy [7], Levy and Kroll [13]). The second field studies arbitrage pricing, its characterization and implications (see Ross [15], Kreps [12], Harrison and Kreps [8], Harrison and Pliska [9]). In these fields, it is well known that the existence of arbitrage opportunities implies first order stochastic dominance and that the converse does not hold. This paper provides a set of conditions under which the converse is true. Given "complete" markets, the existence of an arbitrage opportunity is characterized in terms of the existence of two traded assets, one of which stochastically dominates the other, along with one additional condition. The additional condition is roughly that state-contingent prices must be zero over the states where the first asset dominates the second. The theorem has a qualification, however. It is applicable only in markets which are "complete," not in the Arrow-Debreu sense, but in a weaker sense—in the sense that all contingent claims on the primary assets must trade. In many applications, especially with respect to continuous trading models, this condition is usually satisfied (see Duffie [2] and Harrison and Pliska [9]).

In the case of stochastically dominated assets, this paper shows that the relationship of the assets to the state-contingent prices is important in identifying arbitrage opportunities. In this sense, this paper is related to a recent paper of Jagannathan [10]. Jagannathan studies the relationship of second order stochastic dominance (not first order) to call option valuation.

This characterization theorem relating arbitrage opportunities to stochastic dominance should provide a valuable tool. In the area of stochastic dominance, this theorem specifies conditions under which two assets can be compared using first order stochastic dominance, on an *individual* basis, for inclusion in the

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larger portfolio. The existing theorems in the area of stochastic dominance usually pertain only to comparisons among *final* portfolios. In the area of arbitrage pricing, this theorem provides an alternative approach for guaranteeing the nonexistence of arbitrage opportunities in an economy. The existing theorems (see Harrison and Kreps [8]) require the existence of a unique “equivalent martingale probability measure.” The alternative characterization provided here, in terms of stochastic dominance, may prove useful in certain applications.

The remainder of this paper is organized as follows: Section I presents the model, Section II presents the theorem, while Section III concludes the paper. All proofs are in the Appendix.

I. The Model

This section of the paper presents the details of the model, i.e., the notation, the assumptions, and the definitions in the context of a finite state space economy.

We consider a single period economy starting at time 0 and ending at time 1. At time 1, there are a finite number of states of nature denoted by the set $\Omega = \{1, 2, \dots, S\}$. We assume that all investors agree that each state occurs with a strictly positive probability; however, they need not agree on the absolute magnitudes of the probabilities.

Assets are identified by their state-contingent cash flows at time 1. For example, if x is an asset, then $x(\omega)$ gives the cash flow of asset x at time 1 under state $\omega \in \Omega$. The set of assets traded or marketed at time 0 is denoted by the set M . We assume that the marketed assets trade in a frictionless market where all investors act as price takers. We let M consist of all portfolios generated by m primary assets, $\{x_1, \dots, x_m\}$; formally, $M = \text{span}\{x_1, \dots, x_m\}$. Let the market price of one unit of asset x_i at time 0 be given by π_i . So, a portfolio consisting of n_i shares of asset x_i for $i = 1, \dots, m$ has a time 1 cash flow of $\sum_{i=1}^m n_i x_i$ and a time 0 price of $\sum_{i=1}^m n_i \pi_i$.

Important in the next section will be the concept of a “complete” market. Since $M = \text{span}\{x_1, \dots, x_m\}$, without loss of generality, we can assume that $\{x_1, \dots, x_m\}$ are linearly independent. Therefore, the number of linearly independent primary assets, m , is less than or equal to the number of states, S . If $m = S$, then the marketed assets are complete in the Arrow-Debreu sense. If $m < S$, the marketed assets are incomplete in the Arrow-Debreu sense. In this case, the asset cash flows are not sufficient to create Arrow-Debreu securities paying one unit under each state and zero units otherwise. Under this condition, we search for a weaker concept of “completeness.”

In general, the traded primary assets $\{x_1, \dots, x_m\}$ will generate a partition of Ω into subsets, each subset of which is identifiable at time 1 using only the asset cash flows. Let us denote this disjoint partitioning of Ω by $\bar{\Omega} = \{\bar{1}, \bar{2}, \dots, \bar{N}\}$ where each $\bar{j} \in \bar{\Omega}$ consists of a finite number of elements from Ω and $\bar{N} \leq S$. A market M is called $\bar{\Omega}$ -complete if $\chi_{\bar{j}}$ trades for all $\bar{j} \in \bar{\Omega}$ where the asset $\chi_{\bar{j}}$ by definition pays one unit if the event $\bar{j}$ occurs, and zero units otherwise. The asset $\chi_{\bar{j}}$ is said to *trade* if it can be obtained as a portfolio of assets $\{x_1, \dots, x_m\}$. This definition is adopted from Green and Jarrow [3]. It is similar to Arrow-Debreu

completeness, but weaker. In more abstract economics, it can be shown to be equivalent to the statement that all possible contingent claims on the primary assets trade. If the partition and the state space coincide, $\Omega = \bar{\Omega}$, then $\bar{\Omega}$ -completeness is equivalent to Arrow-Debreu completeness.

Necessary and sufficient conditions on the traded assets such that the market is $\bar{\Omega}$ -complete have been studied extensively (see Ross [14], Arditti and John [1] or Green and Jarrow [3]). Essentially, the space of marketed assets M must contain all call options on $\{x_1, \dots, x_m\}$. For finite state spaces, however, there exist efficient funds such that traded call options on these funds $\bar{\Omega}$ -complete the market.

We can now define the concepts of an arbitrage opportunity and first order stochastic dominance.

Definition 1: An Arbitrage Opportunity. A portfolio $(n_1, \dots, n_m)$ is an *arbitrage opportunity* if (i) $\sum_{i=1}^m n_i x_i(\omega) \geq 0$ for all $\omega \in \Omega$, (ii) $\sum_{i=1}^m n_i x_i(\omega) > 0$ for some $\omega \in \Omega$, and (iii) $\sum_{i=1}^m n_i \pi_i \leq 0$.

An arbitrage opportunity is defined to be a portfolio which has (i) non-negative cash flows across all states at time 1, (ii) strictly positive cash flows under some state at time 1, and (iii) non-negative cash flows at time 0. This definition is standard (see Ross [15] or Harrison and Kreps [8]).

$\bar{\Omega}$ -complete markets possess a special property with respect to the identification of arbitrage opportunities.

LEMMA: *Given M is $\bar{\Omega}$ -complete, there are no arbitrage opportunities in M if and only if the price of $\chi_{\bar{j}}$ is strictly positive for all $\bar{j}$.*

This well-known lemma provides a convenient method to check for the existence of arbitrage opportunities in $\bar{\Omega}$ -complete markets; for a proof see Green and Srivastava [4]. The recipe is to examine the prices of the traded assets $\{\chi_{\bar{j}}\}$. If they are all positive there are no arbitrage opportunities. If one is zero or negative, it represents an arbitrage opportunity. This lemma is utilized in the next section. It is the key step in proving the characterization theorem. We now define first order stochastic dominance.

Definition 2: (Strong) First Order Stochastic Dominance with Respect to the i th Investor. Given the assets $x, y \in M$, asset x *stochastically dominates* asset y for the i th investor if: (i) $P_i(x > \alpha) \geq P_i(y > \alpha)$ for all $\alpha \in (-\infty, +\infty)$ and (ii) $P_i(x > \alpha) > P_i(y > \alpha)$ for some $\alpha \in (-\infty, +\infty)$ where $P_i(\cdot)$ is the i th investor's subjective probability belief over Ω .

Condition (i) is the standard condition (see Ross [16, p. 153]). It says that the probability that asset x 's cash flows are larger than any constant α , is larger than the probability that asset y 's cash flows exceed the same α . Condition (ii) is included so that x does not stochastically dominate itself. Condition (ii) does not change the economic content of stochastic dominance. For brevity we write $x \text{ dom}_i y$ to mean x stochastically dominates y with respect to the i th investor. The condition that $x \text{ dom}_i y$ can alternatively be written as: $G_i(\alpha) \geq F_i(\alpha)$ for all $\alpha \in (-\infty, +\infty)$ and $G_i(\alpha) > F_i(\alpha)$ for some $\alpha \in (-\infty, +\infty)$ where $F_i(\alpha) = P_i(x \leq \alpha)$, and $G_i(\alpha) = P_i(y \leq \alpha)$.

II. The Theorem

This section of the paper presents the theorem relating arbitrage opportunities to first order stochastic dominance. This theorem is the major contribution of this paper. To motivate the theorem's hypotheses, we first present two examples constructed to highlight the issues involved. The first example illustrates the fact that the existence of an arbitrage opportunity implies the existence of two assets, one of which stochastically dominates the other. This fact is well known.

Example 1: This example constructs an economy with two assets trading, x and y . The assets are constructed such that $(x - y)$ is an arbitrage opportunity. Suppose the economy consists of two states, $\Omega = \{1, 2\}$, and the i th investor sees both states as equally likely. Let $M = \text{span}(x, y)$ where $x = (x(1), x(2)) = (2, 2)$ and $y = (y(1), y(2)) = (1, 2)$. Since x and y are linearly independent, M is Arrow-Debreu complete. Set the price of asset x to be $\pi_x = 1$ and asset y to be $\pi_y = 1$.

There exist arbitrage opportunities in this economy. Asset x is underpriced relative to y , so an arbitrage opportunity is the asset $(x - y)$. To prove this statement, simply observe that the asset's cash flows are always non-negative and strictly positive under state 2. However, the price of this portfolio is zero, and the definition of an arbitrage opportunity is satisfied.

Given this result, it is obvious that asset x stochastically dominates asset y for any investor. To verify this fact, a straightforward calculation shows that

$$F_i(\alpha) = P_i(x \leq \alpha)$$

$$= \begin{cases} 0 & \text{if } \alpha \in (-\infty, 2) \\ 1 & \text{if } \alpha \in [2, \infty) \end{cases} \leq \begin{cases} 0 & \text{if } \alpha \in (-\infty, 1) \\ 1/2 & \text{if } \alpha \in [1, 2) \\ 1 & \text{if } \alpha \in [2, \infty) \end{cases} = P_i(y \leq \alpha) = G_i(\alpha)$$

for all α with strict inequality for $\alpha \in [1, 2)$.

To gain deeper insight into the structure of this economy, consider the assets $G_i(y) = (G_i(y(1)), G_i(y(2))) = (1/2, 1)$ and $F_i(y) = (F_i(y(1)), F_i(y(2))) = (0, 1)$. These trade because $G_i(y)$ represents $(1/2)y$, and $F_i(y)$ represents $-(1/2)x + y$. The price of $G_i(y)$ is therefore $(1/2)\pi_y = 1/2$ and the price of $F_i(y)$ is $-(1/2)\pi_x + \pi_y = 1/2$. The portfolio $(G_i(y) - F_i(y))$ is an arbitrage opportunity. To prove this note that by $x \text{ dom}_i y$, $G_i(y(\omega)) - F_i(y(\omega)) \geq 0$ for all $\omega \in \{1, 2\}$ and $G_i(y(\omega)) - F_i(y(\omega)) > 0$ for $\omega = 1$. So, the portfolio's cash flows are always non-negative and strictly positive under state 1. But, the price of this portfolio is $1/2 - 1/2$, or zero. The key observations about this arbitrage opportunity are that it utilizes only the distributions of the two underlying assets through the price of a particular traded asset, $[G_i(y) - F_i(y)]$, and the fact that $x \text{ dom}_i y$.

The next example shows that the existence of stochastically dominated assets does not imply the existence of an arbitrage opportunity.

Example 2: Consider the same economy as in example 1, except we change the prices of assets x and y . Let the price of asset x , $\pi_x = 2$, and let the price of asset y , $\pi_y = 3/2$. We now show that there are no arbitrage opportunities in this modified economy. To do this, we construct the assets χ_1 and χ_2 and examine their prices. Note that $\chi_1 = (1, 0) = x - y$, so the price of χ_1 is $1/2 > 0$. Similarly, $\chi_2 = (0, 1) =$

$-(1/2)x + y$, so the price of χ_2 is $1/2 > 0$. By the lemma, there are no arbitrage opportunities in this economy. Yet, from example 1, we know that $x \text{ dom}_i y$. So the existence of stochastically dominated assets does not imply the existence of arbitrage opportunities.

What differs (in comparison to example 1) is clarified by considering the asset $[G_i(y) - F_i(y)]$. It trades since the market is Arrow-Debreu complete. It has $G_i(y(1)) - F_i(y(1)) = 1/2 > 0$ and $G_i(y(2)) - F_i(y(2)) = 0$ for its cash flows under state 1 and state 2. This time, however, the price of this claim $[G_i(y) - F_i(y)]$ is $1/4 > 0$ and this asset does not represent an arbitrage opportunity. This claim $(G_i(y) - F_i(y))$ isolates the differences between examples 1 and 2.

These two examples foreshadow the following theorem.

THEOREM (Arbitrage versus Stochastic Dominance): *Given M is $\bar{\Omega}$ -complete, there exists an arbitrage opportunity if and only if there exists $x, y \in M$ such that for some $i \in I$, (i) $x \text{ dom}_i y$ and (ii) the price of $[G_i(y) - F_i(y)]$ is nonpositive where $F_i(\alpha) = P_i(x \leq \alpha)$ and $G_i(\alpha) = P_i(y \leq \alpha)$ for all $\alpha \in (-\infty, +\infty)$.*

Given $\bar{\Omega}$ -complete markets, this theorem relates the existence of arbitrage opportunities to the existence of stochastically dominated assets. Given that $x \text{ dom}_i y$, condition (ii) of the theorem is sufficient to guarantee the existence of an arbitrage opportunity. The arbitrage opportunity identified is *not* $x - y \in M$, but the contingent claim $G_i(y) - F_i(y) \in M$. Since $x \text{ dom}_i y$, we know that $G_i(\alpha) \geq F_i(\alpha)$ for all α . The probabilistic difference between the outcomes of x and y occur on a set where y has an outcome with positive probability. Condition (i), therefore, guarantees that the cash flows to the claim $G_i(y) - F_i(y)$ are always non-negative and strictly positive for some states. Condition (ii) states that the price of the contingent claim $G_i(y) - F_i(y)$ is nonpositive. This claim is an arbitrage opportunity. Perhaps surprising is the fact that these conditions are also necessary.

A limitation in using the above theorem is the requirement that the marketed assets are $\bar{\Omega}$ -complete and that the state space is finite. The restriction that the state space is finite is easily relaxed (see Jarrow [11]). The restriction that the marketed assets are $\bar{\Omega}$ -complete cannot be completely eliminated. The reason for this is contained in the proof of the theorem. The arbitrage opportunity identified using $x \text{ dom}_i y$ is the contingent claim $G_i(y) - F_i(y)$. To be an arbitrage opportunity, the claims $G_i(y)$ and $F_i(y)$ must trade for "arbitrary" distributions. This essentially requires $\bar{\Omega}$ -complete markets.

III. Conclusions

In the context of a finite state space economy, this paper provides a set of conditions under which first order stochastic dominance implies the existence of an arbitrage opportunity. The condition is that the price of a particular contingent claim, defined in terms of the distributions involving the stochastically dominated assets, is nonpositive. In fact, these conditions are both necessary and sufficient, in "complete" markets, for the existence of an arbitrage opportunity. By "com-

plete," we mean an economy where all contingent claims on the primary assets trade.

The restriction to a finite state space can be easily relaxed; however, market completeness is essential. For many applications, especially continuous trading models, this completeness condition is already satisfied and the theorem can be applied at no additional cost. This theorem should, therefore, provide a valuable tool in the continuing study of arbitrage opportunities and the continuing study of portfolio analysis through the use of stochastic dominance.

Appendix

Proof of the Theorem

Step 1: Consider $x, y \in M$ where $x \text{ dom}_i y$ for some i and the price of $G_i(y) - F_i(y)$ is nonpositive. We claim that $G_i(y) - F_i(y)$ is an arbitrage opportunity. This follows from (a), (b), and (c) below. (a) Since $G_i(y)$ and $F_i(y)$ are constants on the partition $\bar{\Omega}$ and M is $\bar{\Omega}$ -complete, $G_i(y) - F_i(y) \in M$, (b) $G_i(y(\omega)) - F_i(y(\omega)) \geq 0$ for all $\omega \in \Omega$ since $G_i(\alpha) - F_i(\alpha) \geq 0$ for all α , and (c) Given that $\lim_{\alpha \rightarrow -\infty} G_i(\alpha) = \lim_{\alpha \rightarrow -\infty} F_i(\alpha) = 0$, let $\alpha_0 = \inf \{\alpha: G_i(\alpha) > F_i(\alpha)\}$. By hypothesis α_0 exists. G_i must have a point of increase at α_0 , so $P_i(y = \alpha_0) = G_i(\alpha_0) - G_i(\lim_{\epsilon \rightarrow 0} \alpha_0 - \epsilon) > 0$. This says that for some $\omega \in \Omega$, $G_i(y(\omega)) - F_i(y(\omega)) = G_i(\alpha_0) - F_i(\alpha_0) > 0$.

Step 2: Assume there exists an arbitrage opportunity. For convenience, denote the price of $\chi_{\bar{j}}$ by $\pi_{\bar{j}}$ for $\bar{j} = \bar{1}, \dots, \bar{N}$. By the lemma, there exists a $\bar{j}_0 \in \{\bar{1}, \dots, \bar{N}\}$ such that $\pi_{\bar{j}_0} \leq 0$. Consider $0, \chi_{\bar{j}_0} \in M$. We claim that $0 \text{ dom}_i -\chi_{\bar{j}_0}$, and that the price of $G_i(-\chi_{\bar{j}_0}) - F_i(-\chi_{\bar{j}_0})$ is nonpositive. This follows from (a) and (b) below. (a) $G_i(\alpha) = P_i(-\chi_{\bar{j}_0} \leq \alpha) = \{0 \text{ if } \alpha < -1, P_i(\bar{j}_0) \text{ if } -1 \leq \alpha < 0, \text{ and } 1 \text{ if } 0 \leq \alpha\}$. $F_i(\alpha) = P_i(0 \leq \alpha) = \{0 \text{ if } \alpha < 0, \text{ and } 1 \text{ if } 0 \leq \alpha\}$. So $G_i(\alpha) \geq F_i(\alpha)$ for all α with strict inequality on $[-1, 0)$. (b) $G_i(-\chi_{\bar{j}_0}) = \{P_i(\bar{j}_0) \text{ if } \omega \in \bar{j}_0, \text{ and } 1 \text{ if } \omega \notin \bar{j}_0\}$ and $F_i(-\chi_{\bar{j}_0}) = \{0 \text{ if } \omega \in \bar{j}_0, \text{ and } 1 \text{ if } \omega \notin \bar{j}_0\}$. Hence $G_i(-\chi_{\bar{j}_0}) - F_i(-\chi_{\bar{j}_0}) = P_i(\bar{j}_0)\chi_{\bar{j}_0}$, the price of the asset is $P_i(\bar{j}_0)\pi_{\bar{j}_0} \leq 0$.

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