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Source: *The Journal of Finance*, Vol. 41, No. 4 (Sep., 1986), pp. 857-870
Published by: Blackwell Publishing for the American Finance Association
Stable URL: http://www.jstor.org/stable/2328233
Accessed: 26/11/2009 08:33

THE JOURNAL OF FINANCE • VOL. XLI, NO. 4 • SEPTEMBER 1986

Futures Options and the Volatility of Futures Prices

CLIFFORD A. BALL and WALTER N. TOROUS*

ABSTRACT

Assuming nonstochastic interest rates, European futures options are shown to be European options written on a particular asset referred to as a futures bond. Consequently, standard option pricing results may be invoked and standard option pricing techniques may be employed in the case of European futures options. Additional arbitrage restrictions on American futures options are derived. The efficiency of a number of futures option markets is examined. Assuming that at-the-money American futures options are priced accurately by Black's European futures option pricing model, the relationship between market participants' ex ante assessment of futures price volatility and the term to maturity of the underlying futures contract is also investigated empirically.

FUTURES OPTIONS ARE OPTION contracts written on the futures price of a commodity or financial instrument. In particular, a futures option provides its owner the right to assume a long position, in the case of a call, or a short position, in the case of a put, in a given futures contract at a fixed futures price termed the exercise price. Since their introduction in October 1982, futures option markets have experienced significant growth, both in terms of available contracts and trading volume. The prior analyses of Black [3], Brenner, Courtadon, and Subrahmanyam [5], and Ramaswamy and Sundaresan [12] have investigated the equilibrium valuation of futures options. The purpose of this paper is to clarify further our understanding of futures options and to examine the efficiency of a number of futures option markets. Moreover, futures option prices allow us to imply market participants' ex ante assessment of futures price volatility, thereby providing additional insight into the futures price process.

A futures price is simply that delivery price for which the value of the futures contract is zero. A futures price, therefore, does not correspond to the value of a futures contract. However, assuming frictionless markets in futures contracts and default-free pure discount bonds of all maturities, Cox, Ingersoll, and Ross [7] establish that a futures price does indeed correspond to the value of a particular asset. In the absence of arbitrage, the equilibrium value of an option

* The University of Michigan, and The University of Michigan and University of California-Los Angeles, respectively. We thank Fischer Black, Bradford Cornell, David Emanuel, Ronald Masulis, Mark Rubinstein, and an anonymous referee for helpful comments, and Thomas Thompson of the Chicago Board of Trade for useful discussions. Navin Chopra provided research assistance. Remote Computing Division of Hale Systems, Inc., supplied the data. This work was partially supported by research grants from the Graduate School of Business Administration at The University of Michigan and the Center for the Study of Futures Markets at Columbia University. Any remaining errors are the authors' responsibility.

written on this asset provides the equilibrium value of a futures option. From this perspective, the equilibrium pricing of futures options does not differ fundamentally from the equilibrium pricing of options on common stock. The advantage to this approach is that standard option pricing techniques may be applied. Previously derived option pricing results may be translated simply to the case of futures options.

We investigate the efficiency of futures option markets by examining violations of arbitrage restrictions on the equilibrium values of American futures options. Three futures option contracts are considered—the Deutsche mark futures option contract traded on the Chicago Mercantile Exchange, the gold futures option contract traded on the Commodity Exchange of New York, and the sugar futures option contract traded on New York's Coffee, Sugar, and Cocoa Exchange. By examining various contracts traded on different exchanges, we minimize the possibility that our results are particular to a specific contract or exchange.

The introduction of futures options provides a means of measuring market participants' ex ante assessment of futures price volatility. The relationship between the volatility of futures prices and the time to maturity of the underlying futures contract will provide insights into the stochastic properties of futures prices. For example, Samuelson [13, 14] argues that the volatility of futures prices increases monotonically as the time to maturity of the futures contract diminishes. If the volatility of futures prices varies deterministically as the futures contract nears maturity, implications follow for optimal hedging strategies and the setting of margin by clearing firms. Since equilibrium futures option pricing models allow the underlying futures price volatility to be nonstochastic and, at most, a known function of time, corresponding implied volatilities will provide additional evidence on Samuelson's hypothesis. In this paper, we examine empirically the relationship between market participants' ex ante assessment of futures price volatility and time to maturity for the Deutsche mark, gold, and sugar futures contracts.

The plan of this paper is as follows. In Section I, assuming nonstochastic interest rates, we introduce a contract referred to as a futures bond, which, when initiated today, guarantees the prevailing futures price at a specified later point in time. Invoking the law of one price, we establish that the current value of a European futures option must equal the current value of a corresponding European option written on the futures bond. Arbitrage restrictions on European futures options and their equilibrium valuation follow simply as a consequence of earlier option pricing results. Section II considers briefly the properties and pricing of American futures options. As established by Brenner, Courtadon, and Subrahmanyam [5] and Ramaswamy and Sunderasan [12], it may be rational to prematurely exercise American futures options even in the absence of payouts to the underlying spot commodity or financial instrument. Hence, the possibility of premature exercise must be taken into account explicitly when deriving arbitrage restrictions on American futures options and when investigating their equilibrium valuation. We demonstrate that uncertain payouts to the underlying spot commodity or financial instrument have a minimal effect on the pricing of American futures options. The concept of a futures bond also allows a simple derivation of arbitrage restrictions on American futures options. Distinct from previous research, in Section III we examine the efficiency of futures option markets. Given the volume of trading in futures options, an empirical investigation into the efficiency of a number of these markets is warranted. Violations of previously established arbitrage restrictions on American futures options are examined for the Chicago Mercantile Exchange's Deutsche mark futures option market, the Commodity Exchange's gold futures option market, and the Coffee, Sugar, and Cocoa Exchange's sugar futures option market. Section IV provides statistical evidence consistent with a significant linear relationship between market participants' ex ante assessment of futures price volatility and time to maturity for the Deutsche mark, gold, and sugar futures contracts. Section V provides our summary and conclusions.

I. European Futures Options

A European option expiring at time τ written on a futures contract maturing at t^*, $t^* \geq \tau$, gives the holder the right at τ to assume a long position, in the case of a call, or a short position, in the case of a put, in the underlying futures contract with the futures price equal to the option's exercise price. We let $F(\tau, t^*)$ denote the corresponding futures price at τ, and K the option's exercise price. Given that the futures contract is marked-to-market, the payoff at expiration to a European futures call option is

$$\max[F(\tau, t^*) - K, 0],$$

while the payoff at expiration to a European futures put option is

$$\max[K - F(\tau, t^*), 0].$$

A difficulty in valuing futures options is that the futures price does not correspond to the value of the futures contract. The futures price is a parameter value which renders the value of a futures contract identically zero (Black [3]). However, the futures price is equal to the value of a particular asset (Cox, Ingersoll, and Ross [7]). We may therefore value a futures option by valuing an option written on this asset. Standard option pricing arguments may be employed. Furthermore, other option pricing results may be translated simply to the case of futures options.

We begin by characterizing equilibrium futures prices. The following assumptions are maintained throughout:

- (A1) *Investor Preferences.* Investors are rational and prefer more wealth to less.
- (A2) *Frictionless Markets.* Perfect frictionless markets for futures contracts and default-free pure discount bonds of all maturities are assumed.
- (A3) *Nonstochastic Interest Rates.* Spot rates of interest of all maturities are assumed to be deterministic and equal to r per unit time.

We posit a flat term structure of interest rates in order to simplify exposition. The subsequent analysis holds if spot rates of interest are deterministic. Under assumption (A3), futures prices will equal forward prices (Cox, Ingersoll, and

Ross [7]). Empirical evidence is consistent with the equality of futures and forward prices (French [8]).

We let $A(t^*)$ denote the price at t^* of the spot commodity or financial instrument upon which the futures contract is written. Of course, convergence of spot and futures prices gives $F(t^*, t^*) = A(t^*)$.

PROPOSITION (Cox, Ingersoll, and Ross [7]): *The current time t futures price, $F(t, t^*)$, $t \le \tau \le t^*$, is the value today of a contract which will pay at time t^* the amount*

$$A(t^*) \exp(r(t^* - t)).$$

Proof: (See Cox, Ingersoll, and Ross [7], p. 324).

We now value a contract today which will be worth the prevailing futures price, $F(\tau, t^*)$, at time τ.

LEMMA: *The value today, time t, of a contract which will pay at time t^* the amount*

$$F(\tau, t^*)$$

is

$$H(t, \tau; t^*) = F(t, t^*) \exp(-r(\tau - t)).$$

Proof: We consider the following strategy. At time t, lend the amount $F(t, t^*) \exp(-r(\tau - t))$ for $\tau - t$ periods at the spot rate of r per unit time. At time $\tau - 1$ take a long position in the futures contract, and at each time k, $k = t, t + 1, \cdots, \tau - 2$, take a long position in $\exp(-r(\tau - k))$ futures contracts, liquidating each contract after one period and reinvesting the proceeds and accumulated interest in one-period bonds until time τ. The current investment required for this strategy is $F(t, t^*) \exp(-r(\tau - t))$. The payoff at time τ is

$$F(t, t^*) + \textstyle\sum_{k=t}^{\tau-1} \{F(k + 1, t^*) - F(k, t^*)\} = F(\tau, t^*).$$

When convenient, this contract will be referred to as a futures bond; lending the requisite sum today and maintaining the appropriate positions in futures contracts guarantees receipt of the prevailing futures price at time τ.

Consider a European option written on the contract outlined in the Lemma. As before, the exercise price is K with expiration date $\tau \le t^*$. Notice that

$$H(\tau, \tau; t^*) \equiv F(\tau, t^*).$$

Therefore, in the case of a European call option, the payoff at expiration is

$$\max[F(\tau, t^*) - K, 0],$$

while the payoff at expiration to a European put option is given by

$$\max[K - F(\tau, t^*), 0].$$

The payoff at expiration to a European option written on the futures bond equals the payoff at expiration to a European futures option. Appealing to assumption (A1), the law of one price requires that the current value of a European futures option equals the current value of a European option written on the futures bond.

Having identified the European futures option's underlying security as the futures bond with current value $H(t, \tau; t^*) = F(t, t^*) \exp(-r(\tau - t))$, arbitrage restrictions on the value of a European futures option follow simply as a consequence of the earlier work of Merton [9]. Significantly, we require no assumptions regarding the dynamics of the futures price over time. Letting c and p denote, respectively, the values of European futures call and put options, we have

$$\max[0, (F(t, t^*) - K) \exp(-r(\tau - t))] \leq c \leq F(t, t^*) \exp(-r(\tau - t)), \quad (1)$$

$$\max[0, (K - F(t, t^*)) \exp(-r(\tau - t))] \leq p \leq K \exp(-r(\tau - t)), \text{ and} \quad (2)$$

$$p = c - F(t, t^*) \exp(-r(\tau - t)) + K \exp(-r(\tau - t)). \quad (3)$$

Expression (1) provides lower and upper boundaries on the value of the European futures call option, while expression (2) provides lower and upper boundaries on the value of the European futures put option. Expression (3) is the put-call parity relationship for European futures options.

Expressions (1) and (2) also give information about the extreme values of European futures options. As $F(t, t^*) \to 0$, then $c \to 0$ and $p \to K \exp(-r(\tau - t))$. Also, as $K \to 0$, then $c \to F(t, t^*) \exp(-r(\tau - t))$ and $p \to 0$. Notice that we may identify the futures bond as a zero-exercise price European futures call option. Interestingly, as $\tau \to \infty$, then both $c \to 0$ and $p \to 0$. The current value of the underlying security, the futures bond, diminishes with increasing term to maturity, thereby diminishing the current value of a European futures call option. The present discounted value of the exercise price also diminishes with increasing term to maturity, thereby diminishing the current value of a European futures put option.

It is well known that under the additional assumption that futures price dynamics are modelled by a geometric Brownian motion process

$$dF(t, t^*) = \alpha\, F(t, t^*)dt + \sigma\, F(t, t^*)dZ(t),$$

where α and σ denote, respectively, the instantaneous drift and standard deviation of the proportionate change in futures prices and where $Z(t)$ is standard Brownian motion, then Black's [3] formula applies to European futures options. By Itô's Lemma, the price dynamics of the futures bond are given by

$$dH(t, \tau; t^*) = (\alpha + r)H(t, \tau; t^*)dt + \sigma H(t, \tau; t^*)dZ(t).$$

From Black and Scholes [4], we may determine immediately the equilibrium value of a European call written on the futures bond and, as such, the equilibrium value of a European futures call option given by Black [3].

If uncertain cash dividends will be paid to the underlying spot commodity or financial instrument, then simple cost-of-carry models of futures prices will not obtain (Cornell and French [6]). However, if investors' expectations of these future dividends change sufficiently smoothly with the receipt of new information, then futures prices will follow a geometric Brownian motion process (Merton [11]). Hence, Black's formula holds even if simple cost-of-carry models of futures prices, as assumed by Brenner, Courtadon, and Subrahmanyam [5] and Ramaswamy and Sundaresan [12], do not obtain.

II. American Futures Options

An American futures option may be exercised at any time up to and including its expiration date. Therefore, the holder of an American futures option may prematurely assume a long position in the case of a call, or a short position in the case of a put, in the underlying futures contract at the predetermined futures price given by the option's exercise price.

As established by Brenner, Courtadon, and Subrahmanyam [5] and Ramaswamy and Sundaresan [12], it may be rational to prematurely exercise American futures call and put options even in the absence of payouts to the underlying spot commodity or financial instrument. This result is evident in viewing futures options as options written on a particular asset. Cox, Ingersoll, and Ross [7], especially in Proposition 7, demonstrate that the futures price $F(t, t^*)$, for all $t \leq t^*$, corresponds to the value of an asset which provides a continuous dividend yield of r for all s, $t < s < t^*$, and the amount $A(t^*)$ at t^*. However, Merton [9] shows that for this proportional dividend policy, there is always a positive probability of premature exercise.

The payment of certain dividends to the underlying spot commodity or financial instrument will not play a role in the premature exercise of American futures options. In this case, the futures price remains continuous through the corresponding ex-dividend date (Ramaswamy and Sundaresan [12]). However, unexpected dividends may impart discontinuity to the futures price. This follows since unexpected dividends may prompt investors to revise discontinuously their expectations of future dividends. To the extent that these payouts are not only random in their size but also random in their timing, as in a convenience yield to a spot commodity, futures price dynamics may be modelled more accurately as a jump-diffusion process, as opposed to a diffusion process. Concomitantly, futures option prices may be modelled more accurately by Merton's [10] option pricing model, as opposed to Black's option pricing model. While Ball and Torous [2] do provide statistical evidence consistent with the presence of jumps in a majority of a random sample of NYSE listed common stock issues, they find no operationally significant differences between corresponding Merton model call prices and Black-Scholes model call prices. Therefore, uncertain payouts to the underlying spot commodity or financial instrument are expected to be of minimal operational consequence in valuing futures options.

The possibility of premature exercise must be taken into account explicitly when deriving arbitrage restrictions on American futures options. We let C and P denote, respectively, the values of American futures call and put options written on a futures contract maturing at t^*, both with exercise price K and expiration date τ, $\tau \leq t^*$. Ramaswamy and Sundaresan [12] obtain a number of arbitrage restrictions on American futures options:

$$\max[0, F(s, t^*) - K] \leq C \leq F(s, t^*) \text{ for all } s \text{ such that } t \leq s < \tau, \tag{4}$$

$$\max[0, K - F(s, t^*)] \leq P \leq K \text{ for all } s \text{ such that } t \leq s < \tau, \tag{5}$$

$$\text{and } P \geq C - F(s, t^*) + Kb(s, t) \text{ for all } s \text{ such that } t \leq s < \tau,$$

where $b(s, \tau)$ represents the price at time s of a pure discount bond with unit face value maturing at time τ. These restrictions are valid regardless of futures price dynamics and whether interest rates are stochastic or nonstochastic. However, as noted by Ramaswamy and Sundaresan [12], further characterization of the relationship between American futures call and put options requires that we impose additional restrictions on spot rates of interest.

If we assume that interest rates are deterministic and, without loss of generality, constant, then the following arbitrage restriction must hold:

$$P \leq C - F(s, t^*) \exp(-r(\tau - s)) + K \text{ for all } s \text{ such that } t \leq s < \tau.$$

The concept of a futures bond allows a simple derivation of this arbitrage restriction and the preceding arbitrage restrictions obtained initially by Ramaswamy and Sundaresan [12]. The current value of a futures bond corresponds to the current investment required to implement a specific dynamic strategy in available futures contracts and default-free pure discount bonds. This particular strategy will then allow us to exploit any available arbitrage opportunities in futures option markets.

Assume that $P > C - F(s, t^*) \exp(-r(\tau - s)) + K$ or, in terms of the futures bond, $P > C - H(s, \tau; t^*) + K$. Writing the put, purchasing the call, selling a futures bond, and lending the exercise price K will generate a positive cash inflow today. If the written put is not prematurely exercised, holding the call until expiration will guarantee a positive cash inflow at that time:

	Payoff at τ	
	$F(\tau, t^*) < K$	$F(\tau, t^*) \geq K$
Write Put	$F(\tau, t^*) - K$	0
Buy Call	0	$F(\tau, t^*) - K$
Sell Futures Bond	$-F(\tau, t^*)$	$-F(\tau, t^*)$
Lend K	$K \exp(r(\tau - s))$	$K \exp(r(\tau - s))$
Total	$K(\exp(r(\tau - s)) - 1) > 0$	$K(\exp(r(\tau - s) - 1) > 0$

Alternatively, if the written put is prematurely exercised at time t', $s \leq t' < \tau$, the position will experience a positive cash inflow at time t':

	Payoff at t'
	$F(t', t^*) < K$
Write Put	$F(t', t^*) - K$
Buy Call	≥ 0
Sell Futures Bond	$-H(t', \tau; t^*) = -F(t', t^*) \exp(-r(\tau - t'))$
Lend K	$K \exp(r(t' - s))$
Total	$\geq F(t', t^*)(1 - \exp(-r(\tau - t')))$ $+ K(\exp(r(t' - s)) - 1) > 0$

Hence, we have the following arbitrage relationship between American futures call and put options:

$$\mathrm{C} - \mathrm{F}(\mathrm{s}, \mathrm{t}^*) + \mathrm{K} \exp(-r(\tau - s)) \leq P \leq C - F(s, t^*) \exp(-r(\tau - s)) + K \tag{6}$$

for all s such that $t \leq s < \tau$, given the additional assumption of a deterministic and, without loss of generality, flat term structure of interest rates.

The possibility of premature exercise must also be explicitly taken into account when valuing American futures options. Assuming cost-of-carry models for futures prices, both Brenner, Courtadon, and Subrahmanyam [5] and Ramaswamy and Sunderasan [12] provide perspicuous analyses of the valuation of American futures options and the impact of dividends upon their optimal exercise. Alternatively, given assumptions (A1), (A2), and (A3) and merely positing that futures prices follow a geometric Brownian motion process, we may value American futures options by solving the corresponding Black-Scholes partial differential equation subject to boundary conditions incorporating the possibility of premature exercise. For example, to value an American futures call, we must solve the Black-Scholes partial differential equation

$$\tfrac{1}{2} \frac{\partial^2 C}{\partial H^2} \sigma^2 H^2 + \frac{\partial C}{\partial t} + rH \frac{\partial C}{\partial H} - rC = 0 \tag{7}$$

subject to the following boundary conditions:

$$C(\tau, H(\tau, \tau; t^*)) = \max[H(\tau, t; t^*) - K, 0]$$

and $C(s, H(s, \tau; t^*)) \geq \max[H(s, \tau; t^*) \exp(r(\tau - s)) - K, 0]$

$$\text{for all } s \text{ such that } t < s < \tau. \tag{8}$$

III. Violations of Arbitrage Restrictions

Traded futures options are American options rather than European options. To investigate the efficiency of futures option markets, we investigate empirically the relevant arbitrage restrictions given by expressions (4), (5), and (6). Three distinct futures option contracts, traded on three separate exchanges—the Commodity Exchange's gold futures option contract, the Chicago Mercantile Exchange's Deutsche mark (DM) futures option contract, and the Coffee, Sugar, and Cocoa Exchange's world sugar futures option contract—are considered. By representing a metal, a financial instrument, and an agricultural commodity, we sample a spectrum of the prevailing futures option contracts. Further, the empirical results obtained will not be restricted to a particular underlying futures contract or a specific exchange.

Transaction costs to trading in futures contracts and futures option contracts are minimal. For floor traders in a particular contract, round trip transaction costs to a position in that instrument are on the order of 24¢ per contract. These costs include a service charge of 2¢ per contract per side payable to the exchange, and a clearing charge of 10¢ per contract per side payable to the clearing firm. However, for floor traders in a particular contract taking a position in a different instrument, say, a futures option trader taking a position in the underlying futures contract, an additional round trip service charge of approximately $3 per contract is levied.

The Commodity Exchange's gold futures option is written on the futures price of its contract for 100 troy ounces of gold. There are two distinct groups of contract months—Group A (February, June, and October) and Group B (April, August, and December)—with trading taking place in the nearest month from Group A plus every month from Group B through the next calendar year. The last trading day is the second Friday of the month prior to the expiration of the underlying futures contract. Our data include closing option prices and corresponding settlement futures prices over the period January 3, 1983 to June 19, 1984.

The Chicago Mercantile Exchange's DM futures option is written on the futures price of its contract for 125,000 DMs. There is only one cycle of contract months—March, June, September, and December—with trading taking place in all contract months through the upcoming calendar year. The last trading day is two Fridays before the third Wednesday of the contract month. Our data include closing option prices and corresponding settlement futures prices over the period January 24, 1984 to March 1, 1985.

The Coffee, Sugar, and Cocoa Exchange's sugar futures option is written on the futures price of its contract for 112,000 pounds of world sugar. There also is one cycle of contract months—March, July, and October—with trading taking place in the upcoming four contracts. The last trading day is the second Friday of the month prior to the expiration of the underlying futures contract. Our data include closing option prices and corresponding settlement futures prices over the period October 1, 1982 to March 1, 1985.

Table I examines violations of the previously derived arbitrage restrictions on American futures call and put options. The lower call test considers violations of the arbitrage restriction that $C \geq \max[0, F(s, t^*) - K]$, while the lower put test considers violations of the arbitrage restriction that $P \geq \max[0, K - F(s, t^*)]$. In no case did we detect violations of the upper call and upper put arbitrage restrictions, $C \leq F(s, t^*)$ and $P \leq K$, respectively. The lower put-call parity test examines violations of the arbitrage restriction that $P \geq C - F(s, t^*) -$

Table I

Violations of Rational Boundary Conditions

	Fraction[a]	Percentage
Lower call		
Gold	317/8,643	3.67
Deutsche marks	108/3,996	2.70
Sugar	47/9,112	0.52
Lower put		
Gold	554/7,327	7.56
Deutsche marks	261/3,688	7.08
Sugar	229/7,928	2.89
Lower put-call parity		
Gold	134/5,976	2.24
Deutsche marks	18/3,130	0.58
Sugar	352/5,838	6.03
Upper put-call parity		
Gold	151/5,976	2.53
Deutsche marks	23/3,130	0.73
Sugar	251/5,838	4.30

[a] Fraction = number of violations/number of tests.

$K \exp(-r(\tau - s))$, while the upper put-call parity test examines violations of the arbitrage restriction that $P \leq C - F(s, t^*) \exp(-r(\tau - s)) + K$.

Across all contracts and all restrictions, the percentage of violations, defined by $100 \times$ (number of violations/number of tests), are minimal. Sugar futures options display the smallest percentage of violations of the lower call and lower put restrictions, while DM futures options display the smallest percentage of violations of the put-call parity restrictions. For all contracts, the percentage of violations of the lower put restriction exceeds the percentage of violations of the lower call restriction. For gold and DM futures options, the percentage of violations of the upper put-call parity restriction exceeds the percentage of violations of the lower put-call parity restriction.

Recall that we assumed nonstochastic interest rates in order to derive the upper put-call parity arbitrage restriction. By contrast, all of the other tested arbitrage restrictions obtain regardless of whether interest rates are stochastic or nonstochastic. However, a comparison of upper put-call parity violations with lower put-call parity violations reveals no significant differences in either their absolute or relative occurrences. Hence, it would not appear that violations of the upper put-call parity arbitrage restriction can be interpreted simply as a rejection of the assumption of nonstochastic interest rates.

IV. Volatility of Futures Prices and the Maturity of Futures Contracts

The relationship between the volatility of futures prices and the time to maturity of the underlying futures contract is of both theoretical and practical importance. The nature of this relationship will provide insights into the statistical distribution of futures price changes, permitting a better understanding of the stochastic properties of futures prices. From a practical perspective, given a statistically significant association between futures price volatility and the underlying futures contract's time to maturity, participants in futures markets should systematically alter hedge ratios as a function of time to maturity to improve upon hedging effectiveness. Likewise, to the extent that the volatility of futures prices impacts upon the setting of margin requirements, clearing firms designing an optimal margin policy should systematically alter margin requirements as a function of the futures contract's time to maturity. A margin requirement established at the introduction of a contract may be inappropriate near the contract's maturity.

Prominent among the theories of futures price volatility is Samuelson's [13, 14] hypothesis that the variance per unit time of futures prices increases monotonically as the time to maturity of the underlying futures contract decreases. This theoretical result follows from the assumption that the commodity's spot price follows a stationary first-order autoregressive process, together with the assumption that current futures prices are unbiased estimators of future spot prices. Intuitively, the current arrival of unanticipated information has greater impact on futures contracts nearer to maturity. Current information has lesser impact on futures contracts maturing further in the future.

Previous empirical analyses of the Samuelson hypothesis (see, for example, Anderson [1]) have not investigated the variability of market participants' ex ante assessment of futures price volatility through a particular contract's maturity date. Market participants may anticipate increased futures price volatility as the maturity date of a contract approaches even if, ex post, the volatility of realized futures prices has remained unchanged. Therefore, to provide further insight into the Samuelson hypothesis, we focus on the relationship between market participants' ex ante measure of futures price volatility and the time to maturity of the underlying futures contract. The introduction of futures options provides a means of measuring market participants' ex ante assessment of futures price volatility. By equating the market price of a futures option with Black's European futures option pricing model, we may imply uniquely the variance per unit time of the futures price. The implied variance, so calculated, is the market's ex ante appraisal of the variance per unit time of the futures price, given the appropriateness of the futures option pricing model.

Traded futures options are American options and, as established earlier, it may be rational to prematurely exercise American futures calls and puts. Treating American futures options as European futures options introduces upward bias in the resultant implied variances; this bias diminishes as the option's term to expiration decreases. However, simulation analysis (Ramaswamy and Sundaresan [12]) confirms minimal bias in Black's European futures option pricing model for pricing at-the-money American futures calls and puts. Hence, variances of futures prices implied from Black's European futures option pricing model, given market prices of nearest-the-money American futures options, will be biased minimally.

A Newton-Raphson iterative procedure is implemented to imply the variance of futures prices on a daily basis. We take special care in the selection of an initial volatility estimate, $\hat{\sigma}_0^2$, so as to reduce significantly the number of iterations required for convergence. Given that the futures option is at-the-money, equating the market price of the call, C_m, with a first-order Maclaurin series expansion of Black's European futures call pricing formula gives

$$\hat{\sigma}_0^2 = 2\pi C_m^2 \exp(2r(\tau - t))/F(t, t^*)^2 (\tau - t). \tag{9}$$

For options which are not at-the-money, replacing C_m with $(C_m + P_m)/2$, where P_m denotes the market price of an American futures put option, and employing $(F(t, t^*) + K)/2$ rather than $F(t, t^*)$ in expression (9) provides an appropriate initial estimate. With these starting values, in many cases convergence of the Newton-Raphson procedure was achieved in one iteration.

In deriving the European futures option pricing model, we require that the instantaneous volatility of futures prices be at most a nonstochastic function of time. Given a time dependent instantaneous variance of futures prices, $\sigma^2(s)$ denoting the instantaneous variance at time s, the European futures option pricing model then simply requires input of the corresponding average instantaneous variance, $v^2(t)$, where $v^2(t) = (\tau - t)^{-1} \int_t^\tau \sigma^2(s)\, ds$ (see Merton [9], especially pp. 162–167). Therefore, given daily futures option data, we actually impute market participants' ex ante assessment of the average variance of futures prices on a daily basis. However, notice that a linear trend in instantaneous variance translates directly to a linear trend in average instantaneous variance. That is,

$$\sigma^2(s) = \gamma_0 + \gamma_1 s \qquad \text{for all } s \in [t, \tau]$$

gives

$$v^2(t) = (\tau - t)^{-1} \int_t^\tau (\gamma_0 + \gamma_1 s)\, ds = \left(\gamma_0 + \frac{\gamma_1}{2}\tau\right) + \frac{\gamma_1}{2} t.$$

Samuelson's hypothesis posits that the instantaneous variance of the futures price increases monotonically as the time to maturity of the underlying futures contract decreases. Therefore, we investigate the Samuelson hypothesis by considering the following ordinary least squares model:

$$v^2(t) = \beta_0 + \beta_1 (\tau - t) + \varepsilon,$$

where β_0 and β_1 are regression coefficients, $\tau - t$ is the futures option's time to expiration, and ε represents the model's residual. Assuming independent, identically distributed normal residuals, a standard t-test allows us to detect a linear trend in market participants' daily ex ante assessment of futures price volatility.

Table II presents the results for gold, DM, and sugar futures options, respectively. We only consider contracts for which data are available through a corresponding expiration date. Nearest-the-money American futures call options were employed to imply the variance of the underlying futures price. Consistent results were obtained with nearest-the-money American futures put options. The final five days' data on each available futures option contract were discarded, given the erratic behavior of the Black-Scholes option pricing model near expiration.

In all but one of the futures option contracts investigated, there exists a statistically significant association between market participants' daily ex ante assessment of average futures price variance and time to maturity of the underlying futures contract. However, the direction of this association varies from contract to contract. For gold futures (Table IIA), seven of the contracts are consistent with implied average futures price variance decreasing with decreasing futures contract maturity, while two of the contracts are consistent with implied

Table IIA

Regression Results on Implied Variance: Gold

Model: $v^2(t) = \beta_0 + \beta_1(\tau - t) + \varepsilon$

Expiration month	Sample size	Simple correlation[a]	t-Statistic[b]	P-Value[c]
04/83	40	0.018	0.112	0.912
06/83	75	0.538	5.464	<0.01
08/83	115	0.564	7.262	<0.01
10/83	69	0.855	13.866	<0.01
12/83	155	0.853	20.232	<0.01
04/84	156	0.406	5.501	<0.01
06/84	76	0.359	3.418	<0.01
08/84	150	0.688	11.723	<0.01
10/84	82	−0.744	−9.968	<0.01
12/84	163	−0.515	−7.622	<0.01

[a] Simple correlation between $v^2(t)$ and $(\tau - t)$.
[b] t-Statistic for testing the null hypothesis $H_0\colon \beta_1 = 0$.
[c] Observed significance level for the two-sided t-test.

Table IIB

Regression Results on Implied Variance: Deutsche Marks

Model: $v^2(t) = \beta_0 + \beta_1(\tau - t) + \varepsilon$

Expiration month	Sample size	Simple correlation[a]	t-Statistic[b]	P-Value[c]
03/84	19	−0.700	−4.037	<0.01
06/84	77	−0.421	−4.024	<0.01
09/84	114	−0.414	−4.837	<0.01
12/84	118	−0.708	−10.807	<0.01
03/85	113	0.833	15.890	<0.01

[a] Simple correlation between $v^2(t)$ and $(\tau - t)$.
[b] t-Statistic for testing the null hypothesis $H_0: \beta_1 = 0$.
[c] Observed significance level for the two-sided t-test.

Table IIC

Regression Results on Implied Variance: Sugar

Model: $v^2(t) = \beta_0 + \beta_1(\tau - t) + \varepsilon$

Expiration month	Sample size	Simple correlation[a]	t-Statistic[b]	P-Value[c]
03/84	326	−0.790	−23.184	<0.01
07/84	296	−0.638	−14.220	<0.01
10/84	299	−0.593	−12.727	<0.01
03/85	330	−0.307	−5.610	<0.01

[a] Simple correlation between $v^2(t)$ and $(\tau - t)$.
[b] t-Statistic for testing the null hypothesis $H_0: \beta_1 = 0$.
[c] Observed significance level for the two-sided t-test.

average futures price variance increasing with decreasing futures contract maturity. Notice that the last two contracts provide evidence consistent with the Samuelson hypothesis. This suggests that, at least for gold futures, the nature of this relationship may not be stationary. For DM and sugar futures (Tables IIB and IIC), all but one of the contracts are consistent with implied average futures price variance increasing with decreasing futures contract maturity. The results for sugar are particularly interesting, given the sufficiently long time series of data available.

Taken together, our analysis suggests that systematic changes in market participants' ex ante assessment of futures price volatility occur as the futures contract nears maturity. The DM and sugar futures results are consistent with the Samuelson hypothesis. However, the gold futures results suggest that the relationship between futures price volatility and the term to maturity of the underlying futures contract may itself vary over time.

VI. Summary and Conclusions

This paper has investigated the equilibrium valuation of futures options. Assuming nonstochastic interest rates, we establish that there exists a contract, referred to as a futures bond, which, when initiated today, guarantees the prevailing futures price at a specified later point in time. In the absence of arbitrage, the current value of a European futures option must equal the current value of a corresponding European option written on the futures bond. Arbitrage restrictions on European futures options and their equilibrium valuation follow as a consequence of earlier option pricing results. Regardless of whether payouts are made to the underlying spot commodity or financial instrument, it may be rational to prematurely exercise both American futures call and put options. Various arbitrage restrictions on American futures options are empirically tested. The results provide little evidence to support the possibility of riskless arbitrage opportunities in futures options markets.

We also take advantage of the informativeness of traded futures option prices to investigate the relationship between the volatility of futures prices and the time to maturity of the underlying futures contract. Market participants' ex ante assessments of futures price volatility implicit in the market prices of gold, DM, and sugar futures option prices tend to vary systematically with the remaining term to maturity of the underlying futures contract. These empirical results then have implications for hedging strategies and margin policy in futures markets.

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