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The Pricing of Futures and Options Contracts on the Value Line Index

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ABSTRACT

This paper considers the problems peculiar to the Value Line Index, because of its use of geometric averaging, as regards the pricing of options and futures on that index. The Value Line Composite Index (VLCI) is an equally weighted geometric average index of nearly 1700 stocks. The VLCI futures market has existed since 1982 while the VLCI options market was established in 1985. This paper provides valuation formulas and analyzes the economic properties of these contracts. Because of the geometric averaging in the VLCI, its contingent claims have special properties. For example, the futures price may fall short of the spot price and the value of a VLCI call option may decline when the volatility of the index is increased. VLCI futures are shown to provide a direct means for duplicating an equally weighted portfolio of the underlying stocks.

THE VALUE LINE COMPOSITE Index (VLCI) is an unweighted geometric average of stock prices expressed in index form. The index on June 30, 1961, was set at 100, and subsequent values are compared against it. Futures contracts on the VLCI are traded on the Kansas City Board of Trade (KCBT). They were the first stock index futures to be traded in the United States, starting on February 24, 1982. Unlike commodity futures contracts, no delivery ever takes place. Instead, any open contracts on expiration day are settled in cash according to the actual VLCI value.[1] More recent market innovations are options on the VLCI that have been traded on the Philadelphia Stock Exchange (PSE) since January 11, 1985.[2]

There are currently two other futures contracts trading on broad-based stock indices: the Standard & Poor's 500 Index futures and futures on the New York Stock Exchange Composite Index. These two indices are value-weighted, that is, each stock enters the index with a weight proportional to its aggregate market

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[1] The VLCI futures contracts are traded in a three-month delivery cycle in March, June, September, and December. The delivery is at the first business day following the last trading day of the contract month. Futures contracts are valued at either $500 times the VLCI (MAXI) or at $100 times the VLCI (MINI). There are no daily price limits. The minimum tick is $25 or $5 for the MAXI and MINI contracts, respectively. Commissions are approximately $100 or less on a round trip (see, e.g., Figlewski [8]).

[2] The VLCI options are available in unit trading of $100 times the index with five expiration cycles in February, March, April, June, and September. Expiration occurs on the Saturday following the third Friday of the expiration month. The aggregate exercise price is found by multiplying the exercise price by $100. In the case of exercise, a cash settlement is used, based on the difference between the closing VLCI value and the option exercise price times $100 (Figlewski [9]).

value. Therefore, the performance of the S&P 500 and the NYSE Composite indices can be replicated by a self-financing portfolio wherein the fraction of wealth invested in each of the underlying stocks is equal to its weight in the index.

What makes the analysis of contingent claims written on the VLCI interesting is the fact that the VLCI, being a geometric average, cannot be duplicated by a self-financing portfolio. This implies that the index dynamics are not the same as the dynamics of an existing financial asset. Therefore, the expected growth rate of the VLCI is not equal to the expected rate of return commensurate with the risk of the index. Since the VLCI does not behave like a financial asset, its contingent claims have nonintuitive properties. For example, a standard result is that in a perfect capital market the forward price should be higher than the spot index by an amount equal to the difference between the risk-free rate of interest and the dividend yield on the index portfolio. This is not necessarily the case with forward prices of the VLCI. Similarly, the value of the call option on a financial asset will increase when the volatility of the asset's rate of return is increased. However, it will be shown that an increase in the volatility of the VLCI growth rate may *reduce* the value of the VLCI call option.

Discussion of the properties of the VLCI contingent claims in the literature is scarce. Cornell and French [4] and Figlewski [9] recognized that VLCI futures require special treatment but chose not to pursue it. Modest and Sundaresan [17] examined the relationship between pricing of stock index futures and their corresponding spot indices. Using arbitrage arguments, they proved that, in a perfect market setting without taxes and in the absence of dividends, the discounted forward price of the *geometric index* may fall *short* of the spot value, as opposed to an *arithmetic index*, for which the discounted forward price must *equal* the current spot price to prevent arbitrage. The introduction of dividends to the analysis was shown to introduce a further wedge between the discounted forward price and the spot value. Since Modest and Sundaresan [17] have not derived exact valuation formulas, their arbitrage framework does not allow for comparative statics. The objective of this paper is to provide valuation formulas for VLCI options and futures contracts and to analyze their characteristics. Itô's calculus is used to derive the dynamics of the assets.

This paper is organized as follows: Section I presents and interprets the continuous time dynamics of the VLCI and the fundamental differential equation for all VLCI contingent claims. Section II derives the VLCI futures price formula and examines its properties. Section III presents the pricing of VLCI options and examines their characteristics. The last section contains conclusions.

I. The Basic Model

The continuous time model presented here is reduced to the minimal necessary complexity for purposes of easy exposition, without sacrifice of generality. The VLCI is an equally weighted geometric average of the n stock prices $P_1, P_2, \cdots, P_n$ as of period t and is denoted by I_t,

$$I_t = A\left(\prod_{j=1}^{n} P_j\right)^{1/n} \tag{1}$$

where A is a constant equal to 100 divided by geometric price average of 1961.

The following assumptions are made:

(A1) Markets are "frictionless," i.e., there are no transactions costs, no taxes, and no restrictions on short sales. Borrowing and lending rates are equal.
(A2) Trading takes place continuously.
(A3) The stock price dynamics satisfy the stochastic differential equations:

$$\frac{dP_j}{P_j} = (\mu_j - d_j)dt + \sigma_j d\tilde{Z}_j, \quad j = 1, 2, \cdots, n. \tag{2}$$

It is assumed that σ_j^2, which is the instantaneous variance of the rate of return on security j, is constant. No such assumption is made with respect to μ_j and d_j, which are, respectively, the instantaneous expected rate of return and the dividend yield. In particular, the continuous dividend yield, d_j, will be treated explicitly as time dependent to reflect the seasonality of dividends. The $d\tilde{Z}_j$'s are standard Wiener process increments whose instantaneous coefficients of correlation are denoted by ρ_{ij}, $i, j = 1, 2, \cdots, n$.
(A4) The risk-free interest rate r is assumed to be time invariant.

Some of these assumptions could be relaxed. In particular, it is not difficult to introduce stochastic interest rates as in Merton [16], but it will add no significant insight. The stipulation of nonstochastic interest rates makes forward and futures prices equal.[3] Empirical evidence suggests that deviations of forward prices from futures prices are sufficiently small to warrant this assumption.[4] Although the issue of tax effects is unresolved, it appears that the introduction of personal differential taxes will reduce futures prices (see Constantinides [3] and Cornell and French [4]).

The assumption of continuous but nonstochastic dividend yields is common to the literature (see, for example, Samuelson [18], Samuelson and Merton [19], and Merton [16]). In this paper there is stronger justification for this behavior since the pricing of the contingent claim is affected by the average dividend of the nearly 1700 securities included in the VLCI and, therefore, one would expect the introduction of stochastic dividend yields to have only a marginal effect. While allowing the μ_j's to be time dependent is not material, since they do not enter any of the final formulas, the assumption of time-variant d_j's is important since it affects pricing, and dividend yields exhibit a strong seasonal pattern.[5]

A. VLCI Dynamics

Using Itô's Lemma and (2), the index dynamics are given by:

$$dI_t = \left[\sum_{j=1}^n \frac{\partial I_t}{\partial P_j} P_j(\mu_j - d_j) + \frac{1}{2}\sum_{j=1}^n \sum_{i=1}^n \frac{\partial^2 I_t}{\partial P_i \partial P_j} P_i P_j \sigma_i \sigma_j \rho_{ij}\right]dt + \sum_{j=1}^n \frac{\partial I_t}{\partial P_j} P_j \sigma_j d\tilde{Z}_j. \tag{3}$$

[3] See, e.g., Cox, Ingersoll, and Ross [7] and French [11].
[4] See Cornell and Reinganum [5] and French [11].
[5] See, e.g., data provided in Cornell and French [4].

The derivatives, from (1), are:

$$\frac{\partial I_t}{\partial P_j} = \frac{I_t}{nP_j}, \qquad j = 1, 2, \cdots, n$$

$$\frac{\partial^2 I_t}{\partial P_j^2} = \frac{(1-n)I_t}{(nP_j)^2}, \qquad j = 1, 2, \cdots, n$$

$$\frac{\partial^2 I_t}{\partial P_i \partial P_j} = \frac{I_t}{n^2 P_i P_j}, \quad i \neq j, \quad i,j = 1, 2, \cdots, n.$$

Substituting them into (3) and rearranging we obtain,

$$\frac{dI_t}{I_t} = \frac{1}{n} \sum_{j=1}^{n} (\mu_j - d_j)dt - \gamma dt + \frac{1}{n} \sum_{j=1}^{n} \sigma_j d\tilde{Z}_j \tag{4}$$

where

$$\gamma = \frac{1}{2}\left(\frac{1}{n} \sum_{j=1}^{n} \sigma_j^2 - \sigma_e^2\right) \tag{4a}$$

$$\sigma_e^2 = \frac{1}{n^2} \sum_{j=1}^{n} \sum_{i=1}^{n} \sigma_i \sigma_j \rho_{ij}. \tag{4b}$$

To interpret these dynamics consider a portfolio involving equal investment in each of the n stocks, and denote it by e. The "ex-dividend" rate of return dynamics of this portfolio is given by the sum of the first and third terms of equation (4). As for the second term of this equation, one can easily show that except for the uninteresting case of all n stocks being perfectly correlated and possessing identical volatilities, γ is strictly positive (the variance of the diversified portfolio is smaller than the average variance of the assets in the portfolio). Hence one can think of the VLCI dynamics as equivalent to those of an equally weighted portfolio composed of the underlying stocks, the instantaneous dividend yield of which has been augmented by γ. We will refer to γ as the *concavity adjustment factor*, which reflects the effect of Jensen's inequality on the geometric average, in the continuous time model.

Note now that σ_e^2 is the instantaneous variance of the rate of return from the equally weighted portfolio and let

$$\mu_e = \frac{1}{n} \sum_{j=1}^{n} \mu_j \tag{5}$$

$$d_e = \frac{1}{n} \sum_{j=1}^{n} d_j \tag{5a}$$

$d\tilde{Z}_e$ = the increment of a standard Wiener process for portfolio e, (5b)

ρ_{je} = the instantaneous correlation coefficient between $d\tilde{Z}_e$ and $d\tilde{Z}_j$. (5c)

The VLCI dynamics can now be conveniently expressed as:

$$\frac{dI_t}{I_t} = (\mu_e - d_e - \gamma)dt + \sigma_e d\tilde{Z}_e. \tag{6}$$

Note that γ enters (6) in the same way that a continuous dividend yield does.

σ_e can also be expressed in a form more convenient for comparative statics. From equation (4b), we obtain

$$\begin{aligned}
\sigma_e^2 dt &= \frac{1}{n^2} \textstyle\sum_{j=1}^n \sum_{i=1}^n \sigma_i \sigma_j \rho_{ij} dt \\
&= \frac{1}{n} \textstyle\sum_{j=1}^n \operatorname{cov}\left(\sigma_j d\tilde{Z}_j, \frac{1}{n} \sum_{i=1}^n \sigma_i d\tilde{Z}_i\right) \\
&= \frac{1}{n} \textstyle\sum_{j=1}^n \operatorname{cov}(\sigma_j d\tilde{Z}_j, \sigma_e d\tilde{Z}_e) = \frac{1}{n} \sum_{j=1}^n \sigma_j \sigma_e \rho_{je} dt.
\end{aligned}$$

Therefore,

$$\sigma_e = \frac{1}{n} \textstyle\sum_{j=1}^n \rho_{je}\sigma_j. \tag{7}$$

Finally, note that the *concavity adjustment factor*, γ, can be interpreted as a measure of diversification for portfolio e; from (4a) it is clear that γ is one-half the difference between the instantaneous variance of the rates of return on the average stock in the portfolio and the portfolio itself. Since γ enters the futures price formula, VLCI futures provide a direct measure of the market assessment of risk reduction potential in a large, well-diversified portfolio.

B. The Fundamental Differential Equation of VLCI Contingent Claims

Let $W = W(I_t, t, T)$ denote the price of a VLCI contingent claim with expiration date T, at time t. Applying Itô's Lemma and equation (6), the dynamics of W are expressed by

$$d\tilde{W} = \left[\frac{\partial W}{\partial I_t} I_t(\mu_e - d_e - \gamma) + \frac{1}{2}\frac{\partial^2 W}{\partial I_t^2} I_t^2 \sigma_e^2 + \frac{\partial W}{\partial t}\right]dt + \frac{\partial W}{\partial I_t} I_t \sigma_e d\tilde{Z}_e. \tag{8}$$

Consider now a continuously rebalanced portfolio, involving one contingent claim short, and a total sum of $\frac{\partial W}{\partial I_t} I_t$ spread equally into n long positions in the VLCI underlying stocks to create an equally weighted portfolio. The investment in this portfolio is $\theta = \frac{\partial W}{\partial I_t} I_t - W$. Since the holder of this portfolio is entitled to the actual dividends,

$$d\theta = \frac{\partial W}{\partial I_t} I_t(\mu_e dt + \sigma_e d\tilde{Z}_e) - d\tilde{W}.$$

Substitute for dW from (8) to obtain

$$d\theta = \left[\frac{\partial W}{\partial I_t}(d_e + \gamma)I_t - \frac{\partial W}{\partial t} - \frac{1}{2}\frac{\partial^2 W}{\partial I_t^2}\sigma_e^2 I_t^2\right]dt. \tag{9}$$

Since $d\theta$ is nonstochastic, the instantaneous return on the hedged portfolio must be the instantaneous risk-free rate; hence,

$$d\theta = \theta r dt = \left(\frac{\partial W}{\partial I_t} I_t - W\right) r dt. \tag{10}$$

From (9) and (10) we obtain the differential equation:

$$\frac{\partial W}{\partial t} + \frac{1}{2}\frac{\partial^2 W}{\partial I_t^2} I_t^2 \sigma_e^2 + \frac{\partial W}{\partial I_t} I_t(r - d_e - \gamma) - Wr = 0. \tag{11}$$

All VLCI contingent claims must obey this equation. However, different boundary conditions, resulting from the specific properties of the various claims, will bring about different solutions to (11).

The partial differential equation (11) resembles the one obtained in the constant dividend yield models of Samuelson [18], Samuelson and Merton [19], and Merton [16]. It is different in two respects: the dividend yield has a time-dependent component d_e, and the constant component, γ, is not arbitrarily assumed, as in their models, but rather stipulated by the properties of the VLCI. It is clear from (11) that, if the underlying stocks do not pay dividends, the pricing of VLCI contingent claims will be identical to the pricing of contingent claims on a stock with hypothetical constant dividend yield, γ.

Denoting the payoff function at T by $H(I_t)$, one can rigorously show that the solution to (11) is given by:

$$W = e^{-r(T-t)} E_t[H(\hat{I}_T)] \tag{12}$$

where E_t is the expectations operator at time t, and $\hat{I}_t$ is the time T value of the random variable satisfying $\hat{I}_t = I_t$ and with dynamics given by

$$\frac{d\hat{I}_\tau}{\hat{I}_\tau} = (r - d_e - \gamma)d\tau + \sigma_e d\tilde{Z}_e, \quad \tau \geq t. \tag{12a}$$

Equation (12a) describes the dynamics of the random variable:

$$\hat{I}_\tau = I_t \exp\left[(r - \gamma - \tfrac{1}{2}\sigma_e^2)(\tau - t) - \int_t^\tau d_e(s)\, ds + \sigma_e \tilde{Z}_e(\tau - t)\right] \tag{12b}$$

where $\tilde{Z}_e(\tau - t)$ is normally distributed with zero mean and variance $\tau - t$.

It is possible, however, to obtain (12) through the following economic argument (The Risk-Neutral Valuation Relationships): note that (11) does not depend on the instantaneous expected return to portfolio e. This means that W does not depend on investors' risk preferences. It is therefore possible to assume a risk-neutral world. In such a world the expected rate of return to all financial assets is r, and the dynamics of VLCI is given by the dynamics of $\hat{I}$. The value of a contingent claim in this world is consequently equal to the expected payoff at maturity discounted at the interest rate r (see, e.g., Smith [20] and Cox and Ross [6]).

II. Futures Prices and Their Properties

The VLCI futures price, $f(t, T)$, is the price at which one has an obligation to buy (or sell) the index value at expiration time, T, although no money is invested at the contract initiation time t.

Since no money is paid at the start, $f(t, T)$ is such that the value of the contract is zero, i.e., $W = 0$. From equation (12) it follows that

$$E_t[H(\hat{I}_T)] = 0. \tag{13}$$

When interest rates are constant the futures and the forward prices are the same; thus it is possible to ignore the marking to market characteristic of futures contracts and specify the payoff as that for a forward contract:

$$H(\hat{I}_T) = \hat{I}_T - f(t, T). \tag{13a}$$

From (13) and (13a) we have

$$f(t, T) = E_t(\hat{I}_T). \tag{13b}$$

Calculation of this expected value from (12b) yields[6]:

$$f(t, T) = I_t\, e^{[r-\gamma-D_e(t,T)](T-t)} \tag{14}$$

where $D_e(t, T)$ is the time average dividend yield of the equally weighted portfolio e and is given by

$$D_e(t, T) = \frac{1}{T-t}\int_t^T d_e(s)\, ds. \tag{14a}$$

A. Discussion

The fundamental difference between the VLCI futures prices and futures prices on dividend-paying financial assets is the appearance of the concavity adjustment factor, γ, in (14). One can think of the VLCI futures price as being the futures price of the portfolio e (which equals $I_t \exp[(r - D_e(t, T))(T - t)]$) multiplied by the adjustment factor $\exp[-\gamma(T - t)]$.

The futures price *adjusted for dividends* is defined by

$$f^d(t, T) = f(t, T)\exp[D_e(t, T)(T - t)]. \tag{15}$$

[6] The expected value of $\hat{I}_T$ is calculated as follows:

$$E(\hat{I}_T) = I_t \exp\left\{(r-\gamma)(T-t) - \int_t^T d_e(s)\, ds\right\}$$

$$\left\{(2\pi(T-t))^{-1/2}\int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\sigma_e^2(T-t) + \sigma_e Z - \frac{1}{2}\left(\frac{Z}{\sqrt{(T-t)}}\right)^2\right] dZ\right\}$$

$$= I_t \exp\left\{(r-\gamma)(T-t) - \int_t^T d_e(s)\, ds\right\}$$

$$\left\{\frac{1}{\sqrt{2\pi(T-t)}}\int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\left(\frac{Z-\sigma_e\sqrt{T-t}}{\sqrt{T-t}}\right)^2\right] dZ\right\}$$

which yields equation (14).

It is a well-known result that the futures price (adjusted for dividend) of a financial asset must command a premium over the spot price of the asset, reflecting the time value of money. By contrast, with the VLCI,

$$f^d(t, T) = I_t \exp[(r - \gamma)(T - t)], \tag{16}$$

which may represent a premium or a discount (if $\gamma > r$) over I_t. This unique property of the VLCI futures emanates from the fact that the VLCI's instantaneous growth departs by γ from the instantaneous expected rate of return associated with the risk of the index.

Modest and Sundaresan [17], using an arbitrage argument, showed that the discounted-adjusted futures price never exceeds the spot VLCI:

$$f^d(t, T)B(t, T) \leq I_t \tag{17}$$

where $B(t, T)$ is the price, as of period t, of a risk-free pure discount bond paying one dollar at T. While this result is consistent with (16), it is clear from our previous discussion that, since γ is strictly positive, a strict inequality must hold in (17). Furthermore, since γ is directly related to the degree of diversification in an equally weighted portfolio of the underlying securities, the departure from equality in (17) is larger if this portfolio is more diversified.

The existence of markets in VLCI futures provides investors with an alternative means of duplicating the performance of an equally weighted portfolio, e. To see this, apply Itô's Lemma to (14) to obtain the instantaneous change in the futures price:

$$df(t, T) = f(t, T)(\mu_e - r)dt + f(t, T)\sigma_e d\tilde{Z}_e. \tag{18}$$

The return dynamics of a strategy involving a \$$G$ long position in default-free pure discount bonds and a $G/f(t, T)$ long position in futures (continuously adjusted) is

$$dG = G\mu_e dt + G\sigma_e d\tilde{Z}e, \tag{18a}$$

which is equivalent to that of investing \$$G$ in portfolio e.

B. Comparative Statics

To investigate the qualitative impact of changes in the underlying parameters on the VLCI futures price, it is useful to incorporate equation (7) into (4a) to obtain

$$\gamma = \frac{1}{2}\left(\frac{1}{n}\sum_{j=1}^{n}\sigma_j^2 - \sigma_e^2\right) = \frac{1}{2n}\sum_{j=1}^{n}\sigma_j\left(\sigma_j - \frac{\rho_{j,e}}{n}\sum_{k=1}^{n}\rho_{ke}\sigma_k\right). \tag{19}$$

Changes in the concavity adjustment factor, γ, may result either from variations in the volatility of individual securities or from variations in their correlations with the index, or both. When the volatilities are held constant we have:

$$\frac{\partial\gamma}{\partial\sigma_e} < 0, \quad \frac{\partial\gamma}{\partial\rho_{je}} < 0, \quad j = 1, 2, \cdots, n \tag{20}$$

and, correspondingly,

$$\frac{\partial f(t, T)}{\partial \sigma_e} > 0, \quad \frac{\partial f(t, T)}{\partial \rho_{je}} > 0, \quad j = 1, 2, \cdots, n. \tag{20a}$$

Thus, an increase in either the VLCI volatility or the correlation of an individual stock j with the index will, ceteris paribus, increase the VLCI futures prices.

When the correlation structure is held constant we have:

$$\frac{\partial \gamma}{\partial \sigma_j} = \frac{1}{n}(\sigma_j - \rho_{je}\sigma_e) \gtreqless 0, \quad j = 1, 2, \cdots, n \tag{21}$$

and, therefore, the sign of the *volatility derivative* is ambiguous:

$$\frac{\partial f(t, T)}{\partial \sigma_j} \gtreqless 0, \quad j = 1, 2, \cdots, n. \tag{21a}$$

In particular, an increase in the volatility of those stocks with low volatility but high correlation with the index is more likely to reduce the concavity adjustment factor and, therefore, increase the futures price.

The effect of a change in the maturity date is also ambiguous, since the *time derivative* is:

$$\frac{\partial f(t, T)}{\partial (T)} = [r - \gamma - d_e(T)]f(t, T) \gtreqless 0. \tag{22}$$

Thus, in contrast with futures prices of financial assets which are, in the absence of dividends, always higher for longer maturities, VLCI futures prices may be lower if the concavity adjustment factor is sufficiently high.

The effects of changes in r and $D(t, T)$ are not ambiguous and are given by:

$$\frac{\partial f(t, T)}{\partial r} > 0, \quad \frac{\partial f(t, T)}{\partial D_e(t, T)} < 0. \tag{23}$$

Thus, a higher interest rate would increase the VLCI futures price, whereas a higher dividend yield would reduce the futures price, ceteris paribus.

III. VLCI Options and Their Characteristics

While actual VLCI options are American, we will price only European options. Analytical solutions to pricing American options can be obtained only in the form of transcendental equations, which do not yield themselves to easy interpretation (see Johnson [15] and Geske and Johnson [13]). Going through these exercises will not add much insight to this paper.

Let $C = C(I_t, t, T, X)$ be the price at t of a European call option maturing at T, with exercise price X. The payoff to the call at maturity is given by

$$H(I_T) = \max[I_T - X, 0]. \tag{24}$$

The call must also satisfy

$$C(0, t, T, X) = 0. \tag{24a}$$

Solving the differential equation (11) subject to the boundary conditions (24) and (24a) by means of (12) and (12a) yields:

$$C = I_t\, e^{-[\gamma+D_e(t,T)](T-t)} N(k_1) - e^{-r(T-t)} XN(k_2) \tag{25}$$

where

$$k_1 = \frac{\ln\left(\frac{I_t}{X}\right) + \left(r + \sigma_e^2 - D_e(t, T) - \frac{1}{2n} \sum_{j=1}^{n} \sigma_j^2\right)(T - t)}{\sigma_e \sqrt{T - t}} \tag{25a}$$

$$k_2 = k_1 - \sigma_e \sqrt{T - t} \tag{25b}$$

and $N(\cdot)$ is cumulative standard normal distribution.

This pricing equation can also be expressed in terms of the futures price by substituting from (4a) and (14) into (25), (25a), and (25b):

$$C = e^{-r(T-t)}[f(t, T)N(k_1) - XN(k_2)] \tag{26}$$

where

$$k_1 = \frac{\ln\left(\frac{f(t, T)}{X}\right) + \frac{\sigma_e^2}{2}(T - t)}{\sigma_e \sqrt{T - t}} \tag{26a}$$

$$k_2 = k_1 - \sigma_e \sqrt{T - t}, \tag{26b}$$

which is precisely Black's [1] equation (16) for pricing European options on futures contracts and on spot commodities. Equation (26) is identical to the price of a call option on a hypothetical security with a continuous dividend yield r, which is not surprising since the dynamics of the futures price given in (18) can be thought of as the dynamics of such a security.

The pricing of the VLCI European put option follows from the call option price via the following put-call parity: the payoff to the put option at maturity is $\max[X - I_T; 0]$. Buying one call and a pure discount default-free bond with face value X and selling a put, all with the same maturity, will produce a maturity payoff identical to that resulting from holding a call with $X = 0$. Hence,

$$C(I_t, t, T, X) + Xe^{-r(T-t)} - P(I_t, t, T, X) = C(I_t, t, T, 0)$$

and, using (25),

$$P(\cdot) = e^{-r(T-t)}X[1 - N(k_2)] - I_t\, e^{-[\gamma+D_e(t,T)](T-t)}[1 - N(k_1)]. \tag{27}$$

A. Interpretation of Results

In some respects, the comparative statics results are standards:[7]

$$\frac{\partial C}{\partial I_t} > 0; \quad \frac{\partial C}{\partial f(t, T)} > 0; \quad \frac{\partial C}{\partial X} < 0; \quad \frac{\partial C}{\partial r} > 0; \quad \frac{\partial C}{\partial D_e(t, T)} < 0 \tag{28}$$

[7] We restrict our attention to call options. Comparative statics results for put options can be derived similarly.

and their intuitive interpretations are similar to those given for a stock call option (see, e.g., Smith [20]).

The VLCI call has, however, several unique properties that are diametrically opposed to those of options on financial assets.[8] The sign of the *time derivative* is ambiguous,

$$\frac{\partial C}{\partial (T-t)} \gtreqless 0. \tag{29}$$

It should be emphasized that, in general, European call options on *dividend-paying stocks* may have negative time derivatives (e.g., Merton [16]). In our case, even if the VLCI stocks do not pay dividends (i.e., $D_e(t, T) = 0$), the sign of the volatility derivative is ambiguous.[9] This is attributable to the appearance of the concavity adjustment factor, γ, in (25). In-the-money VLCI call prices will have negative time derivatives when the time to expiration is long and when the concavity adjustment factor γ is larger than the risk-free rate of interest, r.

For changes in σ_e resulting only from changes in the correlation structure of the underlying securities, we have

$$\frac{\partial C}{\partial \sigma_e} \gtreqless 0; \quad \frac{\partial C}{\partial \rho_{je}} > 0, \quad j = 1, 2, \cdots, n. \tag{30}$$

The intuitive interpretation for the sign ambiguity in the volatility derivative is that an increase in volatility increases the concavity adjustment factor, γ (see equations (19) and (20)), and this may be sufficient to offset the usual effect of the increase in the expected exercise value. Increase in the correlation coefficient of an underlying security with the index will decrease γ and, therefore, has the nonambiguous effect of increasing the option value.

When the correlations ρ_{je} are held constant, we have

$$\frac{\partial C}{\partial \sigma_j} \gtreqless 0; \quad \frac{\partial C}{\partial \sigma_e} \gtreqless 0. \tag{31}$$

An increase in σ_j may either increase or decrease σ_e depending on whether ρ_{je} is positive or negative, respectively. From (21) it is clear that the change in γ may be positive or negative also; as a result of these two effects the expected exercise value may or may not be increased. An increase in σ_e may result from an increase in the volatility of stocks positively correlated with the index, from a decrease in the volatility of stocks negatively correlated with the index, or both, and, as a

[8] In Black [1], commodity options are modeled as options on financial assets but expressed in the form of (26). To the extent that the underlying commodity price dynamics deviates from that of a financial asset, similar unusual results may be obtained in the comparative statics of his model.

[9] Assuming $D_e(t, T) = 0$, we have,

$$\frac{\partial C}{\partial (T-t)} = -\gamma e^{-\gamma(T-t)} I_t N(k_1) + re^{-r(T-t)} XN(k_2)$$

$$+ re^{-r(T-t)} \sigma_e X \frac{\partial N(k_2)}{\partial k_2} \Big/ 2\sqrt{T-t} \gtreqless 0.$$

result, γ may increase or decrease. If γ increases sufficiently, the value of the option will decline.

IV. Summary and Conclusions

This paper considers the problems peculiar to the Value Line Index, because of its use of geometric averaging, as regards the pricing of options and futures on that index. The geometric VLCI is significantly different from other market indices since its expected growth rate is not commensurate with the index risk. This observation is independent of the definition of risk, since the stochastic component of the VLCI growth rate is identical to the stochastic component of the rate of return on an equally weighted, continuously rebalanced portfolio of the securities included in the VLCI, but the expected growth rate is lower than the portfolio's expected return. VLCI contingent claims were shown to be equivalent to contingent claims on a hypothetical, equally weighted portfolio of the underlying securities the instantaneous dividend yield of which was augmented by a constant γ. This *concavity adjustment factor* depends on the properties of the covariance matrix of the underlying securities and equals one-half the difference between the variances of an average VLCI stock and the variance of the equally weighted portfolio. The effect of the *concavity adjustment factor*, as well as the *dividend yield*, is captured in the futures price, and, therefore, the pricing of VLCI options, when expressed in terms of the futures prices, is identical to Black's [1] valuation formula for European options on futures contracts and on spot commodities.

Unlike futures contracts on value-weighted arithmetic indices (such as S&P 500 and the NYSE Composite Index), the VLCI futures price may fall short of the spot index, and the futures price (adjusted for dividends) discounted at the risk-free rate is strictly less than the spot index.

Some comparative statics properties of the VLCI option are different from the standard Black-Scholes [2] results for stock options; the value of the call option may decline as the volatility of the index is increased, and a longer time to maturity may imply a lower value for the European call option, even if the underlying stocks do not pay dividends. Similarly, futures prices exhibit the following unusual properties: they depend on the volatilities of the underlying stocks and may decline when the volatility of the index is increased. An increase in time to maturity may reduce the futures price even in the absence of dividends.

It was also shown that by taking positions in the bond and VLCI futures markets one can duplicate the effect of owning an equally weighted portfolio of the (nearly 1700) stocks included in the VLCI. Since there is currently no mutual fund that duplicates this large and well-diversified portfolio, VLCI futures provide a unique hedging opportunity, which may explain the popularity of these instruments.

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