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Source: *The Journal of Finance*, Vol. 41, No. 3, Papers and Proceedings of the Forty-Fourth Annual Meeting of the America Finance Association, New York, New York, December 28-30, 1985 (Jul., 1986), pp. 545-557
Published by: Blackwell Publishing for the American Finance Association
Stable URL: http://www.jstor.org/stable/2328482
Accessed: 26/11/2009 08:29

THE JOURNAL OF FINANCE • VOL. XLI, NO. 3 • JULY 1986

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ABSTRACT

This paper derives a framework for arbitrage models in markets with frictions. It generalizes the existence of a valuation operator to such markets. As in perfect markets, the valuation operator is a linear operator and its existence is implied by the no-arbitrage condition. In imperfect markets the valuation operator is individual-specific and depends on the agent's position in the market. The methodology employed in the paper is duality in convex programming.

IN SHARP CONTRAST to the vast body of literature on perfect markets, little is known about markets with frictions. Of special interest are the implications of no arbitrage opportunities in such markets. Models based on arbitrage arguments have considerably improved our understanding of pricing and valuation of risky and riskless securities, e.g., bond pricing, option pricing, and the effects of capital structure.

The importance of the valuation operator in a perfect market is explained in Ross [8]. "Arbitrage theory can be used to value . . . assets in a simple and straightforward fashion that makes use only of information available in the market." Its crucial role in perfect markets suggests that it will be important in markets with frictions.

In a market with frictions, the valuation operator is individual-specific. It may depend on the position of the individual in the market. But the valuation operator of any individual can be used to value assets. However, the implications of no-arbitrage arguments to economies with frictions have received little attention.

In a perfect market, these implications are derived from either Farkas' Lemma, or from duality in linear programming, as in Ross [7]. In a market with frictions these arguments would not work, unless it is assumed that the frictions are linear. (See Garman and Ohlson [2]).

Recently Prisman [6] showed how the theory of duality in convex programming can be used to build arbitrage models in markets with frictions. That paper refers only to riskless bonds and focuses on an estimator of the term structure which minimizes the maximum arbitrage profit in markets with frictions. However, the analysis applies also to risky securities if time and the number of time periods

* Arizona State University. The author acknowledges Stephen Ross, the participants of the ORSA/TIMS meeting (November 1985) for helpful comments, and Jaime Dermody for helping improve the exposition of the paper. This paper was inspired from a paper by Garman and Ohlson [2], in particular from the footnote, "A treatment of the relaxed assumption (linearity of transaction costs) would lead into nonlinear duality theory . . . ". The research was supported in part by the Council 100 grant from Arizona State University.

are replaced by a time-state dimension and the number of time-state pairs, respectively.

This paper derives a general framework for arbitrage models in economies with frictions. Attention is directed to the effects of the absence of arbitrage opportunities on the valuation of risky securities in economies with frictions. (Some of our results were independently obtained by Ross [9], who used the theorem of Fan et al. [1]). In order to focus attention on the main issues the model has a single period; time in the riskless security case is replaced with the state dimension in the case of risky securities. The generalization to a multiperiod model is deferred to a later paper. The proposed framework is a natural extension of the arbitrage models in a perfect market. To demonstrate this we review the perfect market case.

I. No Arbitrage in Frictionless Economies

Let a_{ij}, $i = 1 \cdots n, j = 1 \cdots t$, be the return from security i if the state of nature is j, P_i the price of the i'th security, and x_i the number of units of security i purchased. Consider the following pair of primal-dual linear programming problems, where 0 is the vector of zeros and $P = (P_1 \cdots P_n)$.

$$\text{ML: } \min x^T P \text{ s.t. } x^T A \geq 0$$

$$\text{DL: } \min 0^T d \text{ s.t. } Ad = P,\ d \geq 0$$

The problem ML describes the process of identifying the portfolio which minimizes the initial investment needed to purchase a portfolio that generates non-negative payoff in every state of nature. The no-arbitrage condition is defined in terms of problem ML.

DEFINITION 1. *No Arbitrage condition* (NA, *weak form*): There exist no arbitrage opportunities if the optimal value of ML is zero.
NA, *strong form*: There exist no arbitrage opportunities if the optimal value of ML is zero and all the optimal solutions satisfy the constraints as equalities (e.g., $x^T A \geq 0$ as $x^T A = 0$). The strong form means no arbitrage profit in any future state.

The duality theory of linear programming ensures that the optimal value of ML is zero if, and only if, the optimal value of its dual problem DL is zero. If a feasible solution to problem DL exists, obviously its optimal value is zero, otherwise the optimal value of DL is $+\infty$ (the $+\infty$ is obtained by minimizing over the empty set). We thus obtained the following result.

THEOREM 1. *The* NA (weak form) *condition holds if, and only if, there exists a semipositive linear valuation operator d, such that $Ad = P$. The strong form holds if, and only if, the valuation operator is positive.*

Proof: The strong form is proved utilizing a separation theorem in Klee [4], and the complementary slackness condition.

The j'th component of the vector d is interpreted as the implicit price of state j. It is the price of \$1 in state j (the price of the state j Arrow-Debreu security).

The vector d is a linear valuation operator that maps the state-contingent payoff into equilibrium prices.

A. The Indirect Arbitrage Function, $\pi(d)$

Problem ML states the minimum investment needed to purchase a portfolio generating a non-negative payoff in every state of nature. It is expressed in terms of the portfolio x. The problem DL states this in a dual way, in terms of the vector d. That is, the maximum arbitrage profit (which is $-x^TP$) admitted by d is the function $\pi(d)$ defined by:

$$\pi(d) = \begin{cases} 0, & \text{if} \quad d \geq 0 \text{ and } Ad = P \\ +\infty, & \text{otherwise.} \end{cases}$$

The function π is the indirect arbitrage function (à la the terminology of an indirect utility function). It characterizes the valuation operators that preclude arbitrage profit, i.e., the vectors $\bar{d}$ such that $\min\{\pi(d) \mid d\} = \pi(\bar{d}) = 0$.

Since the *maximum* arbitrage profit allowed by $\bar{d}$ is zero, if $\bar{d}$ is the valuation operator there are no opportunities for arbitrage. We will say that $\bar{d}$ admits no arbitrage if the maximum arbitrage profit allowed by $\bar{d}$ is zero. Theorem 1 is extended to a market with frictions in the same manner, i.e., via the indirect arbitrage function.

B. NA *and the Expectation Operator*

It is possible to define a no-arbitrage condition in terms of a risk-free rate. This approach is related to that of Harrison and Kreps [3]. The valuation operator derived from such an approach is an expectation operator. It is induced by a probability distribution over the states of nature.

If a risk-free rate exists, and the NA holds, no risky portfolio dominates the risk-free rate. There exists no portfolio x such that

$$(x^TP)(1 + r) < \min_j\{x^TA_j\} \tag{1}$$

where A_j is the j'th column of A. If such an x exists, investors would invest $-x^TP$ in the risk-free rate and purchase the portfolio, x. From (1) it follows that investors realize arbitrage profits in every state of nature. Consider the following primal optimization problem:

$$\text{MLP: } \inf_x \; (x^TP)(1 + r) - \min_j \; \{x^TA_j\}$$

An x satisfying (1) does not exist if, and only if, the optimal value of MLP is zero.

PROPOSITION 1. *The* NA (Definition 1) *holds if, and only if, there exists a finite positive r, such that the optimal value of* MLP *is zero.*

Proof: The "if" part follows immediately by contradiction. Assume that NA holds and that there exists no $r > 0$ such that the optimal value of MLP is zero. Hence, there exists no x such that $x^TP < 0$ and $x^TA \geq 0$. Furthermore, for every

$r > 0$ there exists an x_r such that

(a) $x_r^T P < 0$ and $\min_j \{x_r^T A\} < 0$, or

(b) $x_r^T P > 0$ and $x_r^T A > 0$, and in both cases,
$(x_r^T P)(1 + r) - \min\{x_r^T A_j\} < 0$

Without loss of generality, assume that (a) holds. This implies the existence of a path, x_r, along which the marginal payoff from x_r is infinite. That is, the "derivative" of $\min\{x_r^T A_j \mid j\}$ approaches infinity. Since the function,

$$\min\{x_r^T A_j \mid j\}$$

is piecewise linear and concave, this is impossible. Hence, by contradiction, the "only if" part holds. Q.E.D.

Note that the "only if" part of the proof depends on the linearity of $x^T A_j$. It would not hold, in general, in a market with frictions (compare this to the discussion after the proof of Lemma 2). It is possible to prove the proposition using a duality argument. Since the feasibility of problems DL and DLP (to be defined later) are equivalent, the result follows immediately.

Proposition 1 proves that, in a perfect market, NA holds if, and only if, it is possible to introduce a risk-free rate that does not create arbitrage opportunities. Therefore, an equivalent definition of the no-arbitrage condition can be stated in terms of r and the problem MLP.

DEFINITION 2. *No arbitrage condition* (NAP *weak form*): There exist no opportunities for arbitrage if it is possible to introduce a risk-free rate, r, to the economy such that $1 + r > 0$ and the optimal value of MLP is zero.
NAP (*strong form*): There exist no opportunities for arbitrage if it is possible to introduce a risk-free rate, r, to the economy such that $1 + r > 0$, the optimal value of MLP is zero, and all the optimal solutions satisfy $(x^T P)(1 + r) = x^T A_j$ for every j.

The weak form does not preclude the existence of a state k, such that $(x^T P)(1 + r) < x^T A_k$, while the strong form does. The weak form avoids the arbitrage profit in every state of nature, i.e., avoids realizing it with probability one. The strong form avoids such in any state of nature, i.e., avoids realizing it with a positive probability. The last assertion is based on the assumption that each of the states specified in A may occur.

The problem MLP is equivalent to the following linear programming problem,

$$\inf_{x,\alpha}(x^T P)(1 + r) + \alpha \quad \text{s.t.} \quad \alpha + x^T A_{j0} \geq 0 \quad j = 1, \ldots, t.$$

The dual of this problem is problem DLP:

$$\text{DLP: } \min 0^T \lambda \quad \text{s.t. } A\lambda = (1 + r)P, \quad \textstyle\sum_{j=1}^{t} \lambda_j = 1, \quad \lambda \geq 0$$

Using duality theory in linear programming, we conclude that the optimal value of MLP is zero if, and only if, the optimal value of DLP is zero. Thus, we obtained the following result.

THEOREM 2. *The* NAP (weak form) *condition holds if and only if there exists a particular probability distribution over the states of nature. Such a distribution induces an expectation operator which makes the expected payoff from each security equal its price times* $(1 + r)$. *The strong form holds if, and only if, every state has a positive probability.*

Proof: The strong form is proved with the same arguments used for the strong form in Theorem 1.

The valuation operator in this case is the vector $\lambda = (\lambda_1 \cdots \lambda_t)$. The j'th component is the value at the end of the period of a claim that pays \$1 in state j. Its value is the probability of realizing state j and its present value is $\lambda_j/(1 + r)$. Thus, d_j is equal to $\lambda_j/(1 + r)$. The indirect arbitrage function is defined analogously to π.

In a perfect market the approaches based on the ML and MLP problems are equivalent. In a market with frictions it is advantageous to use the latter. Throughout the paper we adopt the former and assume the weak form of the NA condition. In a market with frictions, the strong form of the NAP condition may not imply $\lambda_j > 0$ for every j. After extending the result to a market with frictions, we discuss the advantage of using the latter approach and the implication of the strong form.

II. No Arbitrage in Economies with Frictions

In this section the existence of a valuation operator is extended to a market with taxes. We shall formulate the no-arbitrage condition as the optimal value of an optimization problem, like problem ML, being zero and then derive the indirect arbitrage function from it. A valuation operator, if it exists, is a vector at which the indirect arbitrage function vanishes. The primal-dual relation between the optimization problem and the indirect arbitrage function provides the existence result and the properties of the operator. Our derivation will highlight both the similarities and differences between a frictionless market and one with frictions.

In a frictionless market, violation of the no-arbitrage condition creates an opportunity for infinite arbitrage profit. Investors would have long positions in one set of assets and short positions in another, to the point that their demands blow up, irrespective of their initial positions Thus, in equilibrium there will be no opportunities for arbitrage from any position.

Equilibrium in a market with non-linear taxes exhibits different characteristics than those with linear taxes or no taxes. It is possible that from a position held by no investor, an arbitrage opportunity exists. Depending on the structure of the tax code, the arbitrage opportunity may be either finite or infinite. The finite case occurs where the marginal tax rate eliminates any marginal profit. The infinite case is somewhat more complicated.

Consider an investor holding a portfolio, y, purchased for intertemporal smoothing or other reasons. An opportunity for arbitrage exists if a portfolio change, x, exists, such that $x + y$ satisfies the investor's needs and provides an arbitrage profit. A rational investor, i.e., one with monotonic preference order, would exploit such an opportunity to its limit. In the finite case there exists an x which maximizes the arbitrage profit. In the infinite case, for every x that provides an arbitrage profit, there is an x' that provides more, i.e., $y + x'$ dominates $y + x$. Eventually, the investor's demand will blow up to infinity.

In both cases, when all investors are rational, equilibrium implies no arbitrage opportunity only for those positions held by investors. Thus, the valuation operator is individual-specific. It depends on the investor's position. Nevertheless, a valuation operator of any agent in the economy may be used to value assets.

DEFINITION 3. *No arbitrage condition in a market with frictions* (NAF): Let y be the position of an investor. There are no arbitrage opportunities at y if the optimal value of problem MF, defined below, is zero.

$$\text{MF: } \inf\{x^T P\} \text{ s.t. } (x + y)^T A_j - g_j(x + y) \geq y^T A_j - g_j(y) \quad j = 1 \cdots t.$$

The function g_j is the tax paid on the portfolio in state j. Taxes are modeled here in a general form, and are dependent on the state of nature. Later we discuss the case where $g_j(x) = g(x^T A_j)$, e.g., tax is a function of the payoff. Rearranging the constraints in MF yields:

$$\text{MF: } \inf\{x^T P\} \text{ s.t. } x^T A_j - [g_j(x + y) - g_j(y)] \geq 0, \quad j = 1, \ldots, t.$$

This form demonstrates that in a market with no taxes, i.e., where $g \equiv 0$, the NAF condition is independent of y. This also holds where the tax code is linear. In what follows we assume that the tax code is captured by a convex function, that is, g_j is a convex function. For ease of exposition we define the convex function, $f_j(x) = g_j(x + y) - g_j(y)$, and thus, problem MF becomes:

$$\text{MF: } \inf\{x^T P\}, \text{ s.t. } x^T A_j - f_j(x) \geq 0 \quad j = 1 \cdots t.$$

Consider a change, x, from the initial position y to $y + x$, such that in each state j, the after tax return from $y + x$ exceeds the after tax return from y by at least u_j. That is, the change x satisfies

$$x^T A_j - f_j(x) \geq u_j \quad j = 1, \ldots, t. \tag{2}$$

Let the function $q(u)$, where $u = (u_1, \ldots, u_t)$, be the minimal investment needed for x satisfying (2). Mathematically, q is defined by $q(u) = \inf\{x^T P \mid x$ satisfies (2)$\}$, and $+\infty$ if there exists no x satisfying (2).

The domain of the definition of q is the set $\text{dom}(q) = \{u \mid q(u) < \infty\}$.

LEMMA 1. *The function q is a convex function.*

Proof: Follows from Theorem 5.7 in Rockafellar [10].

The value of q at the origin is the optimal value of problem MF. Thus, the no-arbitrage condition (Definition 3) can be restated in terms of q.

DEFINITION 4. There exists no arbitrage opportunity at y if $q(0) = 0$.

The necessary and sufficient condition will be established for a particular valuation operator to exist. Consider a valuation operator, d, the j'th component of which is the price of \$1 after tax in state j. This operator values the change x. By definition, the additional after-tax return from x is u_j in state j. Hence the value of the change in positon from y to $y + x$ is $u^T d$.

DEFINITION 5. The operator d admits no arbitrage opportunities if the minimal investment needed for every change defined by (2) is greater than or equal to the value of the change. That is, d satisfies

$$q(u) \geq u^T d \quad \text{for every } u. \tag{3}$$

Definition 5 can be motivated in a slightly different way. To reveal the value of implicit state prices (if they exist), consider the following argument. Assume that the investor can buy or sell \$1 in state j for \$$d_j$. What property should the vector $d = (d_1, \ldots, d_t)$ satisfy to avoid arbitrage profits? Now, the investor considers all portfolios x and calculates the value of $x^T A_j - f_j(x) = u_j$ for every j. If u_j is negative, then the investor pays \$$u_j d_j$ now and buys \$$u_j$ in state j. If state j is realized, then he would have to pay \$$u_j$ but would also get \$$u_j$. If $x^T A_j - f_j(x) = u_j$ is positive, then he would sell \$$u_j$ in state j and be paid \$$u_j d_j$ now. If state j is realized, he would have to pay \$$u_j$ which is provided by $x^T A_j - f_j(x)$. By buying and selling state j securities, the investor ensures that he will be able to fulfill his obligations in any state. The minimum cost of such a portfolio (including the costs of the state j securities) is $q(u) - u^T d$.

Thus, the investor realizes arbitrage profit if there exists a u such that $q(u) - u^T d < 0$. Therefore, d admits no arbitrage if (3) holds.

Every convex function is associated with another convex function called the conjugate function. The next theorem proves that the indirect arbitrage profit is the conjugate of q and explores the relation between the notion "d admits no arbitrage" and NAF, but first we need the following definition.

DEFINITION 6. Let $h(x)$ be a convex function. The convex conjugate of h, denoted $h^*(z)$, is $h^*(z) = \sup_x\{x^T z - h(x)\}$. The geometric interpretation of the conjugacy operator is explained in Rockafellar [10]. The problem dual to MF is

$$\text{DF: } \inf_d q^*(d).$$

In a perfect market the NA condition guarantees the existence of an optimal solution to the dual problem with an optimal value of zero. In a market with frictions the NAF condition implies only the non-negativity of the optimal value of DF. In what follows, we discuss the nature of this difference and the condition under which the NAF condition holds if, and only if, there exists a valuation operator. The properties of the operator are investigated after its existence is established.

THEOREM 3. *The indirect arbitrage function is $q^*(d)$, and d admits no arbitrage if, and only if, $q^*(d) = 0$. For every d, $q^*(d) + q(0) \geq 0$. If the* NAF *condition holds, i.e., if $q(0) = 0$, then for every d, $q^*(d) \geq 0$.*

Proof: From (3) above, it follows that d admits arbitrage if there exists a u such that $u^T d - q(u) \geq 0$. The maximum arbitrage profit admitted by d is therefore

$$\sup\{u^T d - q(u) \mid u\} = q^*(d).$$

Hence, d admits no arbitrage if $q^*(d) = 0$. For every d, $u^T d - q(u)$ evaluated at zero is $-q(0)$. Consequently, for every d, we have $q^*(d) + q(0) \geq 0$. Thus, if $q(0) = 0$, then for every d, $q^*(d) \geq 0$. Q.E.D.

LEMMA 2. *The problem* DF *possesses an optimal solution* (*i.e., the infimum in* DF *is attained at some* d), *if, and only if, there exists a* d, *such that for every* u,

$$q(0) + u^T d \leq q(u) \tag{4}$$

If $\bar{d}$ *satisfies this condition, then* $q^*(\bar{d}) = -q(0)$.

Proof: The vector $\bar{d}$ satisfies condition (4) in the Lemma if, and only if, q^* evaluated at $\bar{d}$ satisfies:

$$q^*(\bar{d}) = \sup\{u^T \bar{d} - q(u) \mid u\} \leq -q(0)$$

From Theorem 3, $q^*(d)$ is greater than or equal to $-q(0)$ for every d. Thus $\bar{d}$ is the optimal solution to DF and $q^*(\bar{d}) = -q(0)$. Q.E.D.

Now, assume that NAF holds, i.e., $q(0) = 0$, and that condition (4) in Lemma 2 does not hold. Then for *every vector* of implicit state prices that are finite, d, there exists a u such that $u^T d - q(u) > 0 = -q(0)$. Thus, in spite of the fact that for every d, $u^T d - q(u) = 0$ at $u = 0$, we have $q^*(d) > q(0)$, for every d. This investor will pay any price, d, as long as it is finite, for a state j Arrow-Debreu security, and then form an arbitrage position. Therefore, even if NAF holds, every d may admit arbitrage.

This phenomenon is a result of the non-linearity of the tax code. Although $q(0) = 0$, the slope of q at zero is infinite, and no finite linear pricing can compensate for an infinite marginal improvement in the minimum investment. (As an example, consider $q(u) = -\sqrt{u}$ as $u \rightarrow 0$ from the right). This phenomena could not occur in a perfect market where $q(u)$ is convex and piecewise linear. Even in a market with frictions, this situation is rare. It is characterized below.

The left hand side of the inequality in condition (4) defines the linear function, $k(u) = q(0) + u^T d$. The slope of k is d, and $k(0) = q(0)$. If the condition does not hold, there exists no linear function, k, such that $k(0) = q(0)$ and the graph of the function q is always above that of k (e.g., for every u, $k(u) \leq q(u)$).

DEFINITION 7. A vector d is said to be a subgradient of q at 0 if, for every u, $k(u) = q(0) + u^T d \leq q(u)$. The set of subgradients of q at a 0 is denoted $\partial q(0)$.

LEMMA 3. *The set of vectors,* d, *that satisfy the condition in Lemma* 2 *is* $\partial q(0)$. *This is the set of optimal solutions to problem* DF.

Proof: Follows immediately from Lemma 2.

Since q is a convex function we have the following lemma.

LEMMA 4. *Assume that the* NAF *condition holds. The following are true.*

(a) Every d admits arbitrage if, and only if, a particular condition holds for every u in the relative interior of dom(q), [ri dom(q)]. This condition is that the two-sided directional derivatives of q at 0 in the direction u exist and equal $-\infty$.
(b) The set of valuation operators that admit no arbitrage is $\partial q(0)$.
(c) The valuation operator is unique if, and only if, q is differentiable at zero. If 0ϵ ri dom(q) then we have the following result.
(d) The NAF holds if, and only if, there exists a valuation operator d that admits no arbitrage.

Proof: Parts (a) and (c) follow from Theorems 23.3 and 23.4 in Rockafellar [10]. Parts (b) and (d) follow from Theorem 3 and Lemma 2.

In what follows we assume that there exist an x such that the after tax return from $x + y$ is greater than that from y.
The next theorem extends the existence of a valuation operator to markets with frictions.

THEOREM 4. *There exists no arbitrage opportunities if and only if there exists a valuation operator that admits no arbitrage, i.e., there exists a* $\bar{d}$ *such that the infimum in problem* DF *is attained at* $\bar{d}$ *and the optimal value is zero.*

Proof: By assumption, 0ϵ ri dom(q), and thus the result follows from part (d) of Lemma 4. Q.E.D.

When the NAF is stated in terms of the risk free rate, as in Definition 2, Theorem 4 holds without the assumption. We shall come back to this point after we investigate some properties of d.

A. Interpretations of the Valuation Operator

Of special interest are the relation between Ad, P, and the tax code. Recall from Definition 6 that f_j^* is the convex conjugate of f_j.

THEOREM 5. *Each of the following conditions is equivalent to the* NAF *condition. Each of the optimization problems below is a dual problem to* MF.

DT_1: *There exists a vector* $\bar{d} \geq 0$ *that solves*

$$\inf\{(\textstyle\sum_{j=1}^{t} d_j f_j)^*(\epsilon) \mid Ad - \epsilon = P,\ d \geq 0\} = 0$$

DT_2: *There exist vectors* $\bar{d} \geq 0$ *and* b_j *that solve*

$$\inf\{\textstyle\sum_{j=1}^{t} d_j f_j^*(b_j) \mid [A - B]d = P,\ d \geq 0\} = 0$$

where B *is an* $n \times t$ *matrix whose columns are* $b_j, j = 1, \ldots, t$.

DT_3: *There exists a vector* $\bar{d} \geq 0$ *that solves*

$$\sup_{d \geq 0} \inf_x x^T P - \textstyle\sum_{j=1}^{t} [x^T A_j - f_j(x)] d_j = 0$$

Proof: The proof is omitted.

The condition DT_1 demonstrates the relation between Ad and P. In an economy with no frictions, NA is equivalent to the existence of an operator $d \geq 0$ such that $Ad = P$. Any deviation of Ad from P would cause opportunities for arbitrage. In an economy with frictions the NAF is equivalent to the existence of an operator $d \geq 0$ such that some "measure" of the deviation, $Ad - P = \epsilon$, is zero. This measure is the convex conjugate of the function $(\sum d_j f_j)$. The objective function in DT_1 is the indirect arbitrage function which is a function of $Ad - P$.

The condition DT_2 identifies the valuation operator in terms of the subgradients of the functions f_j, at the optimal solution(s) of MF. The arguments of f_j^* are these subgradients. Since $x = 0$ is an optimal solution to MF, the vector b_j is an element of $\partial f_j(0)$.

The objective function in DT_2 is the indirect arbitrage function, which expresses the maximum arbitrage profit in terms of these subgradients. The i'th element of b_j is interpreted as the marginal tax rate on the i'th security in state j. This interpretation will be clearer when we assume that $f_j(x) = f(x^T A_j)$. The meaning of DT_2 is that there is no arbitrage if and only if P equals a positive linear combination of the column vectors in $[A - B]$, i.e., $[A - B]d = P$, where $d \geq 0$. The coefficients in the linear combination are the components of the vector d. The maximum arbitrage profit is measured in terms of the subgradients of f_j at the optimal solution(s) of MF.

Equation (3) could also be derived from DT_3 by applying Theorem 28.3 in Rockafellar. Condition DT_3 reveals another interpretation of the NAF. This is summarized in the next proposition.

PROPOSITION 2. There exists no arbitrage opportunity if and only if the convex-concave function $L(x, d)$, defined below, possesses a saddle point at $x = 0$ and $d = \bar{d}$, and $L(0, \bar{d}) = 0$. That is,

$$\sup_{d \geq 0} \inf_x L(x, d) = L(0, \bar{d}) = \inf_x \sup_{d \geq 0} L(x, d)$$

where

$$L(x, d) = x^T P - \sum_{j=1}^{t} [x^T A_j - f_j(x)] d_j$$

Proof: From DT_3 in Theorem 5, NAF holds if and only if $\sup_{d \geq 0} \inf_x L(x, d) = L(0, \bar{d}) = 0$. The value of $\inf_x \sup_{d \geq 0} L(x, d)$ is always equal to the optimal value of MF, which is zero if NAF holds. Q.E.D.

In solving for the no-arbitrage condition, a min-max strategy is used, i.e., an immunization strategy is employed (see Prisman [5]). Note that $L(x, d)$ is the current value of the minimal investment. It also captures the situation where the initial investment is positive, i.e., $x^T p > 0$, but the current value of a positive payoff in the future makes the current value of the total investment negative. That is, $\sum d_j (x^T A_j - f_j(x))$ is subtracted from the initial investment.

In game theoretic terminology, the condition in Proposition 1 describes a two-person zero-sum game of the investor against the "market". The investor chooses the x that maximizes the minimum arbitrage profit, minimizes the maximal investment, and solves $\inf_x \sup_{d \geq 0} L(x, d)$. The problem, $\sup_{d \geq 0} \inf_x L(x, d)$ describes the process by which the "market" counteracts the goal of the investor by choosing the feasible implicit state prices, i.e., $d \geq 0$ that maximizes the minimum investment.

Proposition 1 states that each of the players has an optimal strategy and the game has a value of zero.

Until now we have suppressed the dependence of d on the initial position of the investor. However, investors with different initial position might have different valuation operators. The vector d is actually a function of y, in the following manner. Recall from DT_2 in Theorem 4 that the matrix B is formed by the elements of the sets of subgradients of f_j at $x = 0$. From the definition of f_j, it is easy to verify that those are elements of the subgradients of g_j at y.

When the functions, g_j, are differentiable at y we have the following.

PROPOSITION 3. There exists no arbitrage opportunity if and only if there exists a vector d such that $P = [A - J_y]d_y$ where the matrix J_y is formed by the gradients of g_j at y, i.e., $J_y = [\nabla g_1(y), \nabla g_2(y), \ldots, \nabla g_j(y)]$.

Proof: When g_j is differentiable the only element in the subgradient set is the gradient. See Rockafellar [10], Theorem 25.1. Q.E.D.

We now consider the case where $f_j(x) = f(x^T A_j)$, and f is a function defined on the real line. In this case, the tax due in state j is a function of the payoff in state j. Applying Theorem 16.3 in Rockafellar [10] to the objective function, $(\sum d_j f_j)^*(Ad - P)$, in DT_1 yields the following optimization problem.

$$DT_1^*\text{: inf } \textstyle\sum_{j=1}^{n} d_j f^*(z_j) \text{ s.t. } Av = P,\ d \geq 0,\ v_j = (1 - z_j)d_j \quad j = 1, \ldots, t.$$

As before, the z_j is the first derivative of g_j at the point $y^T A_j$. Thus, z_j is the marginal tax rate in state j. The problem DT_1^* expresses the maximum arbitrage profit (the indirect arbitrage function) as a function of the marginal tax rate in each state of the economy. Thus, there will be no arbitrage opportunities if and only if there exists a v that solve DT_1^* and its optimal value is zero. Note that the operator v reflects the effect of taxes and NAF means no deviation of P from Av in this case.

The dual problems DT_1, DT_2, DT_3, and DT_1^* express the maximum arbitrage profit in terms of the convex conjugate f^* of f. The convex conjugate is defined (under some mild regularity condition) on the range of the subdifferential mapping. Thus the arguments of the convex conjugates are subgradients. Each of the above-mentioned problems identifies all the feasible valuation operators. We have interpreted the results in terms of the subgradients of f at 0. However, if the primal problem has more than one solution, then the valuation operator can be defined in terms of the subgradients at these solutions. This will be the case, if there is an x such that y and $x + y$ generate the same payoff in each period and $x^T P$ is zero. That of course depends on whether or not the market is complete. However, in problem DT_1^* such a situation would not change the equation for the valuation operator since everything depends on the payoff in state j.

B. Expectation Operator with Frictions

All the results obtained so far can be restated in terms of a probability distribution instead of a valuation operator. This would be accomplished by stating the NAF in terms of the risk free rate as in Definition 2.

DEFINITION 8. There exists no arbitrage opportunities if and only if it is possible to introduce a risk-free rate r to the economy such that $1 + r > 0$ and

$$\inf_x (x^T P)(1 + r) - \min_j \{x^T A_j - f_j(x)\} = 0$$

This approach obviates the need for the assumption about the existence of a portfolio whose payoff exceed the payoff from y in every state. Proceeding as before, we define the function

$$G(u) = \inf_x (x^T P)(1 + r) - \min_j \{x^T A_j - f_j(x) - u_j\}$$

The NAF in terms of G is $G(0) = 0$. This G is a convex function and $\mathrm{dom}(G) = R^T$; thus, $0 \in \mathrm{ri}\ \mathrm{dom}(G)$ which implies that $\partial G(0) \neq \varnothing$. Thus, the problem of minimizing the value of the indirect arbitrage function is $\inf G^*(d)$.
By Lemma 4, it possesses an optimal solution and the optimal value is zero.

THEOREM 6. *Each of the following conditions is equivalent to the no-arbitrage condition in* Definition 8. *Each of the conditions is given in terms of the existence of a probability distribution over the states of nature.*

DP_1: There exists $\bar{\lambda}$ that solves

$$\inf\{(\textstyle\sum_{j=1}^t \lambda_j f)^*(\epsilon) \mid A\lambda - \epsilon = P(1 + r),\ \sum_{j=1}^t \lambda_j = 1,\ \lambda \geq 0\} = 0$$

DP_2: There exists $\bar{\lambda}$ and $\bar{b}_j$ that solve

$$\inf\{\textstyle\sum_{j=1}^t \lambda_j f_j^*(b_j) \mid [A - B]\lambda = P(1 + r),\ \sum_{j=1}^t \lambda_j = 1,\ \lambda \geq 0\} = 0$$

DP_3: There exists $\bar{\lambda} \in \Lambda = \{\lambda \mid \sum_{j=1}^t \lambda_j = 1,\ \lambda \geq 0\}$ that solves

$$\sup_{\lambda \in \Lambda} \inf_x (x^T P)(1 + r) - \textstyle\sum_{j=1}^t [x_j^T A - f_j(x)]\lambda_j = 0$$

DP_4: There exist $\bar{\lambda}$ and $\bar{\sigma}$ that solve

$$\inf\{\textstyle\sum_{j=1}^t \lambda_j f(z_j) \mid A^T\sigma = P(1 + r),\quad \lambda_j(1 - z_j) = \sigma_j,\quad \sum_{j=1}^t \lambda_j = 1, \lambda_j \geq 0\} = 0$$

Proof: The proof is omitted.

Basically these conditions state that NAF is equivalent to the existence of a particular probability distribution. This distribution induces an expected after-tax payoff from security i equal to the future value of P_i, i.e., equal to $P_i(1 + r)$. In game theoretic terminology, DP_3 implies a game of the investor against nature. Nature assigns probabilities to the state of the economy while the investor composes portfolios. As before, NAF means the value of the game is zero. It follows that the probability distribution used to value assets according to its expected payoff is that which minimizes the maximum expected arbitrage profit.

C. *Weak Versus Strong* NAF

We conclude the paper with a discussion of the strong and weak forms of NAF. Theorems 5 and 6 are based on the weak form and the following is done with respect to Theorem 5. The analysis of Theorem 6 is similar. The optimal solution(s) to each of the problems in Theorem 5 is the semi-positive vector d, which is a subgradient of q at zero. Since q is a convex function, the set $\mathrm{epi}(q)$

$= \{u \in R^t, \mu \in R \mid q(u) \leq \mu\}$ is a convex set. Its lower envelope is the graph of q. The optimal d is a subgradient of q at zero. Thus, the hyperplane $(d, 1)(u, \mu)^T = 0$ separates the set epi(q) from the set $\{(u, \mu) \mid u \geq 0, \mu = 0\} = C$. That is, $(d, 1)(u, \mu)^T \leq 0$ for every $(u, \mu) \in$ epi(h) and $(d, 1)(u, \mu)^T \geq 0$ for every $(u, \mu) \in C$.

Under the strong form of NAF, the set $D = \{(u, \mu) \mid u \geq 0, u \neq 0, \mu = 0\}$ and epi(q) are disjoint, since $q(u) > 0$ for $u \geq 0$ where $u \neq 0$. However, this is not sufficient to prove that the second inequality is strict when C is replaced by D. If such a separation is possible, then obviously $d > 0$. Note that epi(q) is the projection of the closed set $\{(x, u, \mu) \mid x^TP \leq \mu$ and $x^TA_j - f_j(x) \geq u_j$, for every $j\}$ on the (u, μ) coordinates, but epi(q) is not necessarily closed. The weakest possible conditions for such a separation are given in Klee [4]. It is easy to show that if the g_j are piecewise linear and convex, then strict separation is possible.

If strict separation is impossible, then every subgradient of q at zero would have at least one component which is zero. There exists no valuation operator, d, which admits no arbitrage in the strong form with $d > 0$. In this economy, an investor would have no arbitrage opportunities if and only if \$1 after tax is worthless in some state j. In terms of the expectation operator, the investor would have no arbitrage opportunities if and only if this state j would occur with probability zero. That suggests that the NAF strong form would not ensure the existence of a valuation operator if and only if the state of nature were misspecified.

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DISCUSSION

EHUD I. RONN*: Eli Prisman's paper applies linear and non-linear programming (LP/NLP) and duality theory to the valuation of risky assets. The analogy of LP

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