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A Discrete Time Option Model Dependent on Expected Return: A Note

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BRENNAN [3], USING THE SINGLE-PERIOD, discrete time "law of one price" valuation approach of Beja [1] and Rubinstein [4], proved the necessity and sufficiency of a lognormally-distributed market factor for "risk-neutral" option valuation, at least for options on lognormally and normally distributed underlying assets. The lognormality of the market factor resulted from Brennan's theorizing the market factor to be the marginal utility of wealth of a representative investor and then combining either (a) constant proportional risk-averse (CPRA) utility with lognormally distributed wealth (following Rubinstein), or (b) exponential utility with normally distributed wealth; either way, the market factor, interpreted as marginal utility, is lognormally distributed.

Since Brennan's framework established the necessity of a lognormally distributed market factor for "risk-neutral" pricing of options on lognormal assets, the assumption of a non-lognormal distribution for the market factor must result in option valuation that is not "risk neutral." This note examines the appropriate valuation equation for an option on a lognormal asset when the market factor is normally distributed.[1]

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[1] Technically, the market factor in the law of one price valuation approach can take on no negative values, and the assumption here of a normally distributed market factor violates that condition. In principle, this situation could be corrected by considering the market factor to follow a truncated normal distribution. However, the resulting option expressions become very complex and require numerical integration. In the interest of a simplified expression of option values, the normal market factor assumption is retained here as an approximation. If one wishes to follow Rubinstein and Brennan and characterize the market factor as the marginal utility of wealth of a representative investor, then the point can be made in economic terms. The assumption of a normal distribution for marginal utility would technically allow for the possibility of negative marginal utility corresponding to extreme levels of aggregate wealth. In practice, the possibility that aggregate wealth would fall anywhere near zero or be high enough to induce negative marginal utility is negligible. Thus, the tail of the normal curve representing negative values for marginal utility is too insignificant to consider, particularly for the very short life span that characterizes most options.

If the market factor is to be characterized as the marginal utility of wealth of a representative investor, then normality will follow from either a normal market portfolio with quadratic utility or a lognormal market portfolio with utility of the form $U(z) = z\ln z$. While the representative investor's marginal utility characterization contains significant economic insight and while the Rubinstein and Brennan combinations may appear more defensible on utility-theoretic grounds, one has to accept that the assumptions required to justify the representative investor device (i.e., for aggregation and time-additive tastes) are very restrictive. In the general case of no aggregation to a representative investor, the market factor could be some very complex combination of variables, and under suitable conditions, its approximate normality could follow as a result of the Central Limit Theorem.

The option value in this equation depends directly upon the underlying asset's equilibrium expected rate of return as an independent pricing factor. However, the option's required return does not appear as an explicit pricing variable. Thus, in the discrete time, normal market factor framework, the option valuation would represent a possible way to obtain implied expectations in actual option prices. The value of such implied expectations was recognized previously by Boness [2] and Sprenkle [5]. Unfortunately, Boness and Sprenkle were frustrated by option models which depended on both the expected return of the asset and the option, instead of on only the expected return of the asset.

I. Derivation of the Call Option Model

Investors are assumed to have identical single-period discrete time horizons that correspond to the expiration date of the option. There are no taxes or transactions fees. The Rubinstein/Beja law of one price valuation model is assumed to hold for all assets and securities, regardless of probability distribution. The options considered here are European options on nondividend-paying assets.

Let X represent the current price of the underlying asset, $\tilde{X}$ its random end-of-period (option expiration time) price, W the current call option price, $\tilde{W}$ the call option's random end-of-period (expiration time) price, and K the exercise price of the option. Let $\tilde{Y}$ represent the random end-of-period (expiration time) value of the "factor" commonly related to the returns of all securities and assets.

The law of one price equation (Rubinstein's Theorem 1) applied to the call option and its underlying asset yields

$$W = (1 + R_f)^{-1}[W_e + (1/Y_e)\mathrm{cov}(\tilde{W}, \tilde{Y})] \tag{1}$$

and

$$X = (1 + R_f)^{-1}[X_e + (1/Y_e)\mathrm{cov}(\tilde{X}, \tilde{Y})], \tag{2}$$

where

R_f = the risk-free rate, and
W_e, X_e, Y_e = the expected values of $\tilde{W}$, $\tilde{X}$, and $\tilde{Y}$, respectively.

Equations (1) and (2) are straightforward single-period certainty-equivalent valuation models.

If the underlying asset is distributed lognormally with mean X_e ($= Xe^{\rho t}$, where t is the time until the horizon) and standard deviation of the logarithm of $\tilde{X}$ given by σ_x, then the future option price is distributed according to a truncated lognormal distribution, with truncation at K, and with mean

$$W_e = Xe^{\rho t}N(d_1) - KN(d_2), \tag{3}$$

where

$d_1 = [\ln(X/K) + (\rho + (1/2)\sigma_x^2)t]/\sigma_x\sqrt{t}$,
$d_2 = d_1 - \sigma_x\sqrt{t}$, and
$N(\cdot)$ = the cumulative unit normal distribution function value.

The covariance of a lognormal asset with a normal market factor is

$$\operatorname{cov}(\tilde{X}, \tilde{Y}) = X_e \operatorname{cov}(\ln \tilde{X}, \tilde{Y}), \tag{4}$$

and the covariance of the truncated (at K) lognormal call option with the normal market factor can be written as

$$\operatorname{cov}(\tilde{W}, \tilde{Y}) = \operatorname{cov}(\ln \tilde{X}, \tilde{Y})\{Xe^{\rho t}N(d_1)\}, \tag{5}$$

Equation (5) is derived in the Appendix.

To derive the new equation for a call option on a lognormal asset with a normal market factor, first substitute $e^{rt} = 1 + R_f$ and then Equation (4) into Equation (2) and simplify to obtain that $(1/Y_e)\operatorname{cov}(\ln \tilde{X}, \tilde{Y}) = (e^{rt} - e^{\rho t})/e^{\rho t}$. This last relationship combined with Equations (5) and (3) in Equation (1) leads to the following call option pricing formula after some rearranging

$$W = XN(d_1) - Ke^{-rt}N(d_2). \tag{6}$$

The call option value, W, depends upon the expected appreciation rate of the underlying asset, ρ, in addition to the five other factors that are well known to affect theoretical option values when the underlying asset is lognormally distributed. Note that the two terms in Equation (6) are similar to the Rubinstein discrete time equivalent to the Black-Scholes equation, except that ρ is in d_1 and d_2 here instead of r. Thus, $N(d_2)$ represents the probability of exercise.

II. Discussion of the Call Option Pricing Model

The first derivative of the option value in Equation (6) with respect to X (the current value of the underlying asset) is positive as in other established equilibrium option models. Another standard result is that the first derivative with respect to the standard deviation variable, σ_x, is also positive. The first derivatives of the call price with respect to the exercise price (negative), the risk-free rate (positive), and the time to expiration (positive), are also as expected. This section will focus on the novel relationship between the option price and the expected rate of return of the underlying asset.

The value of the option increases with increases in ρ, the expected appreciation rate of the underlying asset, as long as ρ is less than the risk-free rate. However, if $\rho > r$, then the call value decreases with increases in ρ. If $\rho = r$, then the model yields Rubinstein's option equation. This last result should be expected since ρ will equal r under risk neutrality, and Rubinstein's formula is consistent with risk neutrality. This role of the asset expectation in the new option equation requires some explanation:

Note that, in the law of one price valuation model if expected asset return increases while $1/Y_e$ remains constant, the asset price, X, will remain fixed only if the covariance is reduced by an amount proportional to the increase in X_e. The effects of these simultaneous changes in X_e and $\operatorname{cov}(\tilde{X}, \tilde{Y})$ on the expected price of the option, W_e, and on the covariance of the option with the market factor, $\operatorname{cov}(\tilde{W}, \tilde{Y})$, do *not* have the same proportion, however. For values of ρ which are greater (less) than r, offsetting changes in X_e and $\mathrm{cov}(\tilde{X}, \tilde{Y})$, the latter logically being negative (positive) if $\rho > r$ ($\rho < r$), cause W_e to increase less (more) than proportionately to the decrease in $\mathrm{cov}(\tilde{W}, \tilde{Y})$. Hence, W decreases (increases) as ρ increases if $\rho > r$ ($\rho < r$).

The remarkable aspect of the law of one price model with a *lognormal* market factor, at least in the Rubinstein/Brennan cases for a lognormal or normal asset price, is that in equilibrium when a change in X_e is offset by a change in $\mathrm{cov}(\tilde{X}, \tilde{Y})$ so as to leave the current asset price unchanged, the resultant change in W_e and $\mathrm{cov}(\tilde{W}, \tilde{Y})$ also offset each other to leave the current option price unchanged. For this reason, neither ρ nor covariance enters explicity into the option valuation equation, and the equation is regarded as a "risk-neutral" valuation relationship. Thus, the properties of the joint lognormality of $\tilde{X}$ and $\tilde{Y}$, when coupled with the fact that the "slope of the risk/return trade-off" is measured by $1/Y_e$ in the law of one price model, create the single-period discrete time "risk-neutral" option pricing results of Rubinstein and Brennan.

It is possible to identify a value for ρ that will result in a negative price for some out-of-the-money options. When this happens, it merely means the choice for ρ is unreasonable in the sense that the option's "risk factor" is too great relative to its expected future price. This is a potential problem for any security in certainty-equivalent valuation models; with too great a covariance (risk) term, a nonsensical negative security price results.

III. Put Pricing

At expiration time, the price of a put option can be expressed as $\tilde{U} = K - \tilde{X} + \tilde{W}$, where $\tilde{U}$ and $\tilde{W}$ are the expiration-time put and call option values, assuming both have exercise price K. This relationship can be used with the law of one price valuation model to verify that that model generally supports the put-call parity relationship. This exercise is straightforward and is left to the reader. This result logically guarantees put-call parity for any choice of distributions for $\tilde{X}$ and $\tilde{Y}$, and put values could be derived this way. However, just as a check, put values are derived here directly from the law of one price formula applied to the put.

In this case, the expected price of a put on a lognormal asset is $U_e = KN(-d_2) - Xe^{\rho t}N(-d_1)$, and the covariance of the put with a normal market factor can be written as

$$\mathrm{cov}(\tilde{U}, \tilde{Y}) = \mathrm{cov}(\ln \tilde{X}, \tilde{Y})\{-Xe^{\rho t}N(-d_1)\}.$$

It is easily verified that the equilibrium put price from these relationships and the law of one price is

$$U = Ke^{-rt}N(-d_2) - XN(-d_1). \tag{7}$$

The verification of Equations (6) and 7) with put-call parity is straightforward.

IV. Summary

This note has derived and discussed a formula for the value of a European option on a lognormal asset in the single-period, discrete time law of one price framework with the assumption of a normally distributed market factor. As Brennan's results predicted, the resulting formula is not a risk-neutral valuation relation. Whether the lognormal or normal assumption for the market factor is better is left to future research.

Appendix

Derivation of Equation (5)

Let $\tilde{x} = \ln(\tilde{X}|\tilde{X})$ with mean μ_x and standard deviation σ_x. Note that $\rho = \mu_x + (1/2)\sigma_x^2$. Let f_x, f_{XY}, f_{xY}, and $f_{Y|x}$ be the density of $\tilde{x}$, the joint density of $\tilde{X}$ and $\tilde{Y}$, the joint density of $\tilde{x}$ and Y (joint normal), and the conditional density of $\tilde{Y}$ on $\tilde{x}$, respectively. From the definition of $\mathrm{cov}(\tilde{W}, \tilde{Y}) = \int_K^\infty \int_{-\infty}^\infty (\tilde{X} - K)\tilde{Y} f_{XY}\, d\tilde{Y}\, d\tilde{X} - W_e Y_e$, we get that $\mathrm{cov}(\tilde{W}, \tilde{Y}) = X \int_{\ln K}^\infty d\tilde{x} \int_{-\infty}^\infty e^{\tilde{x}}\tilde{Y} f_{xY}\, d\tilde{Y} - K \int_{\ln K}^\infty d\tilde{x} \int_{-\infty}^\infty \tilde{Y} f_{xY}\, d\tilde{Y} - W_e Y_e$, which is

$$\mathrm{cov}(\tilde{W}, \tilde{Y}) = X \int_{\ln K}^\infty e^{\tilde{x}} f_x\, d\tilde{x} \int_{-\infty}^\infty \tilde{Y} f_{Y|x}\, d\tilde{Y} - K \int_{\ln K}^\infty f_x\, d\tilde{x} \int_{-\infty}^\infty \tilde{Y} f_{Y|x}\, d\tilde{Y} - W_e Y_e. \quad \text{(A1)}$$

Let D stand for $\int_{-\infty}^\infty \tilde{Y} f_{Y|x}\, d\tilde{Y}$ [$= Y_e + (\kappa\sigma_Y/\sigma_x)(\tilde{x} - \mu_x)$ under joint normality, where κ is the correlation coefficient between $\tilde{x}$ and $\tilde{Y}$]. Substitute the normal density for f_x with $a = (\sigma_x\sqrt{2\pi})^{-1}$ and $b = (1/2)((\tilde{x} - \mu_x)/\sigma_x)^2$ into Equation (A1) to obtain

$$\mathrm{cov}(\tilde{W}, \tilde{Y}) = Xa \int_{\ln K}^\infty De^{\tilde{x}-b}\, d\tilde{x} - Ka \int_{\ln K}^\infty De^{-b}\, d\tilde{x} - W_e Y_e. \quad \text{(A2)}$$

Let c stand for $(1/2)((\tilde{x} - (\mu_x + \sigma_x^2))/\sigma_x^2)$ so that the exponent in the first integral of Equation (A2) can be rewritten $e^\rho \cdot e^{-c}$. Now substitute $Y_e + (\kappa\sigma_Y/\sigma_x) \cdot (\tilde{x} - \mu_x)$ for D and then separate terms to get

$$\begin{aligned}\mathrm{cov}(\tilde{W}, \tilde{Y}) = {} & Y_e X_e a \int_{\ln K}^\infty e^{-c}\, d\tilde{x} + (\kappa\sigma_Y/\sigma_x) X_e a \int_{\ln K}^\infty \tilde{x} e^{-c}\, d\tilde{x} \\ & - (\kappa\sigma_Y/\sigma_x) X_e \mu_x a \int_{\ln K}^\infty e^{-c}\, d\tilde{x} - KY_e a \int_{\ln K}^\infty e^{-b}\, d\tilde{x} \\ & - K(\kappa\sigma_Y/\sigma_x) a \int_{\ln K}^\infty \tilde{x} e^{-b}\, d\tilde{x} + K(\kappa\sigma_Y/\sigma_x)\mu_x a \int_{\ln K}^\infty e^{-b}\, d\tilde{x} - W_e Y_e. \quad \text{(A3)}\end{aligned}$$

Note that $N(d_1) = a \int_{\ln K}^\infty \tilde{x} e^{-c}\, d\tilde{x}$, $N(d_2) = a \int_{\ln K}^\infty \tilde{x} e^{-b}\, d\tilde{x}$, $n(d_1) = a \int_{\ln K}^\infty e^{-c}\, d\tilde{x}$, and $n(d_2) = a \int_{\ln K}^\infty e^{-b}\, d\tilde{x}$. Also, note that $\kappa\sigma_x\sigma_Y = \mathrm{cov}(\tilde{x}, \tilde{Y})$ and remember $\rho =$

$\mu_x + (1/2)\sigma_x^2$. If one makes these substitutions and uses Equation (3) for W_e in Equation (A3), simplifies, and inserts the time dimension, Equation (5) will result. The mathematics of Rubinstein's appendix was helpful in the derivation.

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